Annales Mathematicae et Informaticae
Accepted manuscript
DOI: 10.33039/ami.2026.10.002
URL: https://ami.uni-eszterhazy.hu

A note on modular group algebras with metabelian unit groups in characteristic two*

István Mozgovoj

Department of Mathematics an Informatics,
Ferenc Rakoczi II Transcarpathian Hungarian University
brainhowl@gmail.com

Abstract. Let FG denote the group algebra of a group G over a field F of characteristic 2, and let $U(FG)$ denote its group of units. The classification of finite groups G for which $U(FG)$ is metabelian is known, while the corresponding problem for infinite groups is considerably more delicate. We first complete the classification for torsion groups by removing the nilpotency assumption from an earlier result of Catino and Spinelli. We prove that, for a non-abelian torsion group G, the unit group $U(FG)$ is metabelian if and only if either G' is a central elementary abelian group of order 2 or 4, or F is the field of two elements and G is an extension of an elementary abelian 3-group by a group of order 2 acting by inversion. Thus, in the torsion case, the classification coincides with that known for finite groups. We also investigate non-torsion groups and prove that if FG is modular and $U(FG)$ is metabelian, then G must be nilpotent. This substantially restricts the possible new cases in the remaining non-torsion setting.

Keywords: group rings, unit group, derived length

2010 *Mathematics Subject Classification:* 16S34, 16U60

1. Introduction

Denote by FG the group algebra of a group G over a field F of characteristic $p > 0$, and by $U(FG)$ its group of units. A classical line of research in group theory is the study of groups satisfying certain group identities. Such identities are useful,

*This work was supported by the Collegium Talentum Programme of Hungary.

Submitted: September 21, 2026
Accepted: October 13, 2026
Published online: October 23, 2026

for example, in characterizing solvability and the metabelian property. Indeed, a group is metabelian if and only if it satisfies the identity

$$((x_1, x_2), (x_3, x_4)) = 1,$$

where $(x_i, x_j) = x_i^{-1} x_j^{-1} x_i x_j$ denotes the commutator of the group elements x_i and x_j. Over the past decades, many authors have studied the solvability of $U(FG)$. The most general result was obtained by Bovdi [1], who classified the group algebras of arbitrary groups having a solvable group of units in the modular case, that is, when G contains an element of order p. For the sufficiency, the usual assumption that G modulo its torsion subgroup is a u.p. group must be imposed.

A natural problem following the characterization of solvability of $U(FG)$ is to determine its derived length, $dl(U(FG))$. One possible approach is to characterize all groups G for which $dl(U(FG))$ has a prescribed value. Clearly, $dl(U(FG)) = 1$ if and only if G is a nontrivial abelian group. The case $dl(U(FG)) = 2$ leads naturally to the study of group algebras whose group of units is metabelian. The first result in this direction, due to Shalev [16], states that if G is a finite non-abelian group and $p \geq 3$, then $U(FG)$ is metabelian if and only if $p = 3$ and G' is central of order 3. The cases where the characteristic of the field is 2 or the group G is infinite remained open. Shalev already pointed out, however, that the case $p = 2$ is more delicate and would require a different treatment. Kurdics [10] and, independently, Coleman and Sandling [3] confirmed this expectation. They proved that if G is a finite non-abelian group and $p = 2$, then $U(FG)$ is metabelian if and only if either G' is a central elementary abelian group of order 2 or 4, or G belongs to a certain class of non-nilpotent groups.

It is important to note that in all of the metabelianity results mentioned so far the group G is assumed to be finite. On the other hand, the solvability problem for $U(FG)$ has also been studied for infinite groups, which naturally raises the question of whether the above classifications can be extended beyond the finite case. The first step toward solving the general problem was taken by Catino and Spinelli [2]. They proved that Shalev's result remains valid when G is an arbitrary torsion group. For $p = 2$, they obtained the corresponding result under the additional assumption that G is nilpotent. The most general result for odd characteristic is due to Juhász and Spinelli [9]. They proved that if G is a non-abelian group and FG is modular, then $U(FG)$ is metabelian if and only if G is nilpotent of class 2 and one of the following conditions holds:

- $p = 3$ and G' has order p;
- $p = 3$ and $G' = \mathrm{Syl}_p(G)$ is elementary abelian of order p^2;
- $p = 5$ and $G' = \mathrm{Syl}_p(G)$ has order p.

Thus, new cases arise once the assumption that G is torsion is removed.

The metabelian case is also the starting point for a more general problem concerning the structure of the commutator subgroup $U(FG)'$. In odd characteristic, this direction has recently been pursued by Juhász and Sahai. In [7], necessary and

sufficient conditions were obtained for the unit group of a modular group algebra to be centrally metabelian. More recently, in [8], the problem was considered under the more general assumption that $U(FG)'$ is nilpotent of class at most 2, and a complete characterization was obtained in odd characteristic. These results suggest that, beyond the metabelian case, the study of the nilpotency class of $U(FG)'$ provides a natural framework for further generalizations of the problem considered here.

In characteristic 2, however, even the metabelian case is not yet completely understood. In this paper, as a first step toward a general classification, we remove the nilpotency assumption from the result of Catino and Spinelli for torsion groups.

Theorem 1.1. *Let F be a field of characteristic 2 and G a non-abelian torsion group. Then $U(FG)$ is metabelian if and only if one of the following conditions holds:*

1. *G' is a central elementary abelian group of order 2 or 4;*
2. *F is the field of two elements, and G is an extension of an elementary abelian 3-group H by a group $\langle b\rangle$ of order 2, with*

$$b^{-1}ab = a^{-1}$$

for every $a \in H$.

Consequently, for torsion groups G, the classification is the same as in the finite case. For non-torsion groups, however, new cases arise. For example, according to [5], if

$$G = \left\langle a, b, c \mid c^4 = 1,\ b^{-1}ab = ac,\ ac = ca,\ bc = cb \right\rangle$$

and $p = 2$, then $dl(U(FG)) = 2$. Thus, when G is non-torsion, the group $U(FG)$ may be metabelian even when G' is cyclic of order 4. Further exceptional cases can be derived from [4], but a complete classification still appears to be out of reach.

Observe that the group G in the preceding example is nilpotent. Our second main result shows that this phenomenon is unavoidable: in the non-torsion case, metabelianity of $U(FG)$ forces G to be nilpotent.

Theorem 1.2. *Let F be a field of characteristic 2 and G a non-torsion group such that FG is modular. If $U(FG)$ is metabelian, then G is nilpotent.*

2. Preliminaries

The aim of this section is to introduce some notation and collect the results needed in the proofs of our main theorems.

For $a, b \in G$, we write $a^b = b^{-1}ab$ for the conjugate of a by b. We also write $\gamma_i(G)$ for the terms of the lower central series of G, with $\gamma_1(G) = G$, and $\zeta(G)$ for the center of G. Our starting point is the following theorem.

Theorem 2.1 (Kurdics [10], Coleman and Sandling [3]). *Let F be a field of characteristic 2 and G a finite non-abelian group. Then $U(FG)$ is metabelian if and only if one of the following conditions holds:*

1. *G' is a central elementary abelian group of order 2 or 4;*
2. *F is the field of two elements, and G is an extension of an elementary abelian 3-group H by a group $\langle b\rangle$ of order 2 with $b^{-1}ab = a^{-1}$ for every $a \in H$.*

The unit group $U(FG)$ can also be studied via the Lie structure of FG. This approach is useful, in particular, for obtaining upper bounds for $dl(U(FG))$.

Let $\delta^{(0)}(FG) = FG$, and for $i \geq 1$, let $\delta^{(i)}(FG)$ denote the associative ideal generated by all Lie commutators $[x, y] = xy - yx$ with $x, y \in \delta^{(i-1)}(FG)$. The algebra FG is said to be strongly Lie solvable if $\delta^{(i)}(FG) = 0$ for some i. The least such i is called the strong Lie derived length of FG and will be denoted by $dl^L(FG)$.

Since $(x, y) = 1 + x^{-1}y^{-1}[x, y]$ for all $x, y \in U(FG)$, we have $U(FG)' \subseteq 1 + \delta^{(1)}(FG)$. An easy induction then yields

$$U(FG)^{(i)} \subseteq 1 + \delta^{(i)}(FG)$$

for every positive integer i, where $U(FG)^{(i)}$ denotes the ith term of the derived series of $U(FG)$. Consequently, if FG is strongly Lie solvable, then $U(FG)$ is solvable and

$$dl(U(FG)) \leq dl^L(FG).$$

The following theorem characterizes strongly Lie solvable group algebras.

Theorem 2.2 (Theorem 5 of [15]). *Let F be a field of characteristic p and G a non-abelian group. Then FG is strongly Lie solvable if and only if $p > 0$ and G' is a finite p-group.*

If N is a normal subgroup of G, let $\Delta(G, N)$ denote the kernel of the natural homomorphism $FG \to F(G/N)$, and put $\Delta(G) = \Delta(G, G)$. Then $\Delta(G, N)$ consists of the F-linear combinations of elements of the form $g(h - 1)$, where $g \in G$ and $h \in N$. It is well known that $\Delta(G)$ is nilpotent if $p > 0$ and G is a finite p-group [11, Lemma 1.1.1]. Let $t(G)$ denote the nilpotency index of $\Delta(G)$. In particular, if G is an abelian group of order p^n, then $1 + n(p - 1) \leq t(G) \leq p^n$. Equality holds on the left if and only if G is elementary abelian, and on the right if and only if G is cyclic.

The following theorem is useful for computing the strong Lie derived length.

Theorem 2.3 (Proposition 1 of [6]). *Let F be a field of characteristic p and G a nilpotent group whose commutator subgroup G' is a finite p-group. If $\gamma_3(G) \subseteq (G')^p$, then*

$$dl^L(FG) = \lceil \log_2(t(G') + 1) \rceil.$$

It follows immediately that if the condition (1) of Theorem 2.1 holds, then $U(FG)$ is metabelian even without assuming that G is finite. Indeed, $t(G') = 2$ if $|G'| = 2$ and $t(G') = 3$ if $|G'| = 4$, so by Theorem 5, the strong Lie derived length of FG is 2, and hence $dl(U(FG)) \leq 2$.

We shall also need some facts concerning dimension subgroups. For their general theory, we refer to Chapters 3 and 11 of [13]. Define

$$D_n(G) = \{g \in G \mid g - 1 \in \Delta^n(G)\}.$$

Clearly, $D_1(G) = G$, and by [13, Theorem 11.1.19],

$$D_n(G) = (D_{n-1}(G), G)D_i(G)^p, \tag{2.1}$$

where i is the smallest integer satisfying $ip \geq n$.

We shall also use the following consequence of Theorem 3.3.7 and Lemma 3.3.5 of [13].

Lemma 2.4. *Let F be a field of characteristic $p > 0$ and G a finite p-group. If $a, b \in G$ are such that $a, b \in D_m(G) \setminus D_{m+1}(G)$ and $aD_{m+1}(G), bD_{m+1}(G)$ are independent in $D_m(G)/D_{m+1}(G)$, then $(a-1)(b-1) \in \Delta^{2m}(G) \setminus \Delta^{2m+1}(G)$.*

Finally, recall that solvability is inherited by subgroups and quotient groups. For unit groups of group algebras, however, the relation between a group and its quotient is more subtle. The following lemma provides the particular implication that we shall need.

Lemma 2.5. *Let F be a field of characteristic $p > 0$ and N a finite normal p-subgroup of G. If $dl(U(FG)) = n$, then $dl(U(F(G/N))) \leq n$.*

Proof. The argument in the proof of [12, Lemma 2.1] applies verbatim. □

3. Proof of the Theorems

We are now ready to prove our main results.

Proof of Theorem 1.1. Suppose first that $U(FG)$ is metabelian. Since G is non-abelian, there exist $g, h \in G$ such that $(g, h) \neq 1$. Since G is torsion and solvable, it is locally finite. Hence $H = \langle g, h \rangle$ is a finite subgroup of G. Moreover, $U(FH)$ is metabelian and therefore, by Theorem 2.1, H contains an element of order 2. Thus FG is modular.

Since $U(FG)$ is solvable, by [1, Theorem 6], G contains a finite normal 2-subgroup N such that G/N contains an abelian subgroup A of index 2 and one of the following conditions holds:

1. G' is a 2-group of bounded exponent.

2. $F = F_2$ is the field of two elements, G/N is an extension of an abelian group A by a group $\langle b\rangle$ of order 2, where $A = B \times W$, B is a bounded abelian 2-group of finite exponent, W is an elementary abelian 3-group, and $b^{-1}ab = a^{-1}$ for every $a \in W$.

Suppose first that case (1) holds, so that G' is a 2-group of bounded exponent. For arbitrary $g, h \in G$, the subgroup $\langle g, h\rangle$ is finite. Since G' is a 2-group, $\langle g, h\rangle'$ is a 2-group as well, so case (2) of Theorem 2.1 cannot occur; hence, by Theorem 2.1(1), $\langle g, h\rangle'$ is central and therefore $((g, h), h) = 1$. Since g and h were arbitrary, G is a 2-Engel group. By Levi's Theorem [14, p. 12.3.6], $\gamma_3(G)^3 = 1$. On the other hand, $\gamma_3(G) \subseteq G'$, and G' is a 2-group. Consequently, $\gamma_3(G) = 1$ and G' is central.

It remains to determine the structure of G'. Suppose that some commutator (g, h) has order greater than 2. Since G is locally finite, the subgroup $L = \langle g, h\rangle$ is finite. Moreover, $U(FL)$ is metabelian, and L' is a 2-group, so Theorem 2.1(1) applies and L' is elementary abelian, contradicting the assumption that (g, h) has order greater than 2. Consequently, every commutator in G has order at most 2. Since G' is central, it follows that G' has exponent 2. Suppose now that $|G'| > 2$. Then there exist $a_1, a_2, a_3, a_4 \in G$ such that $x = (a_1, a_2) \neq 1$ and $y = (a_3, a_4) \notin \langle x\rangle$. Thus $\langle x, y\rangle \cong C_2 \times C_2$. Let $g, h \in G$ be arbitrary and consider the finite subgroup $H = \langle a_1, a_2, a_3, a_4, g, h\rangle$. The group $U(FH)$ is metabelian, so by Theorem 2.1, H' is a central elementary abelian group of order 2 or 4. Since $x, y \in H'$ are independent, we have $|H'| \geq 4$, and therefore $H' = \langle x, y\rangle$. In particular, $(g, h) \in \langle x, y\rangle$. Since $g, h \in G$ were arbitrary, every commutator of G belongs to $\langle x, y\rangle$. Hence

$$G' = \langle x, y\rangle \cong C_2 \times C_2.$$

Therefore G' is a central elementary abelian group of order 2 or 4.

Conversely, suppose that G' is a central elementary abelian group of order 2 or 4. Then FG is strongly Lie solvable by Theorem 2.2. Since $\gamma_3(G) = 1$ and $t(G') \leq 3$, Theorem 2.3 yields $dl^L(FG) = 2$. Therefore $dl(U(FG)) \leq 2$, that is, $U(FG)$ is metabelian.

Now suppose that case (2) holds. Thus

$$G/N = A \rtimes \langle b\rangle = (B \times W) \rtimes \langle b\rangle.$$

By Lemma 2.5, the metabelianity of $U(FG)$ implies that $U(F(G/N))$ is also metabelian. Choose $1 \neq a \in W$, and let $c \in B$. Consider the subgroup $K = \langle a, b, c\rangle$ of G/N. The group K is finite and $U(FK)$ is metabelian. Since $\langle a, b\rangle' \cong C_3$, the commutator subgroup K' is not a 2-group. Hence case (1) of Theorem 2.1 cannot hold for K and therefore K satisfies case (2). It follows that $c = 1$. Since $c \in B$ was arbitrary, we obtain $B = 1$. Consequently,

$$G/N = W \rtimes \langle b\rangle.$$

Let L be the subgroup of G such that $L/N = \langle a, b\rangle$. Since both $\langle a, b\rangle$ and N are finite, L is finite. Moreover, $U(FL)$ is metabelian. Since $(L/N)' \cong C_3$, case (1) of Theorem 2.1 cannot hold for L. Hence L satisfies case (2). In particular, L has no

nontrivial normal 2-subgroup. Since N is a normal 2-subgroup of L, it follows that $N = 1$. Therefore

$$G \cong W \rtimes \langle b \rangle,$$

as desired.

Conversely, suppose that condition (2) holds. Let K be a finitely generated subgroup of G. Since G is solvable and torsion, so is K, and therefore K is finite. If K is abelian, then FK is commutative, and hence $U(FK)$ is abelian. If K is non-abelian, then

$$K = (K \cap W) \rtimes \langle c \rangle$$

for some involution $c \in K \setminus W$, where $K \cap W$ is an elementary abelian 3-group and c acts on it by inversion. Thus K satisfies condition (2) of Theorem 2.1, and consequently $U(FK)$ is metabelian. Now let $u_1, u_2, u_3, u_4 \in U(FG)$. The supports of these units and their inverses are contained in a finitely generated subgroup K of G. Hence $u_1, u_2, u_3, u_4 \in U(FK)$. By the preceding argument, $U(FK)$ is metabelian. Therefore

$$((u_1, u_2), (u_3, u_4)) = 1.$$

It follows that $U(FG)$ is metabelian. □

Before proving Theorem 1.2 we establish the following lemma.

Lemma 3.1. *Let F be a field of characteristic 2 and G a non-abelian group such that $U(FG)$ is metabelian. If G contains a central element of order 2, then G is nilpotent of class at most 4.*

Proof. Let $g, h \in G$ such that $(g, h) = d \neq 1$, and let $c \in \zeta(G)$ be an element of order 2. We first show that $(d, g) \in \langle c \rangle$.

Since c is central and has order 2, the element $1 + (c+1)g$ is a unit of order 2. Hence

$$\begin{aligned}
(1 + (c+1)g, h) &= (1 + (c+1)g)h^{-1}(1 + (c+1)g)h \\
&= (1 + (c+1)g)(1 + h^{-1}(c+1)gh) \\
&= 1 + h^{-1}(c+1)gh + (c+1)g + (c+1)^2 gh^{-1}gh \\
&= 1 + (c+1)g((g, h) + 1) \\
&= 1 + (c+1)g(d+1).
\end{aligned}$$

Since $U(FG)$ is metabelian, the elements $(1 + (c+1)g, h)$ and $d = (g, h)$ commute. Therefore

$$\begin{aligned}
0 &= [(1 + (c+1)g, h), d] \\
&= [1 + (c+1)g(d+1), d] \\
&= (c+1)[g, d](d+1) \\
&= (c+1)gd((d, g) + 1)(d+1).
\end{aligned}$$

Since gd is a unit, it follows that $(c+1)((d,g)+1)(d+1) = 0$. Expanding this equality gives

$$1 + d + (d,g) + d(d,g) + c + cd + c(d,g) + c(d,g)d = 0.$$

Since the elements of G form an F-basis of FG and $\operatorname{char} F = 2$, every group element occurring in this sum must occur an even number of times. Assume that $(d,g) \neq 1$. Then c must coincide with one of the other terms in the sum. Clearly,

$$c \neq 1, \quad c \neq d, \quad c \neq cd, \quad c \neq c(d,g).$$

If $c = d(d,g)$, then $c = d^g$, which is impossible, because c is central and $d \neq c$. If $c = c(d,g)d$, then $d^g = 1$, which is impossible since $d \neq 1$. Hence the only remaining possibility is $c = (d,g)$. Thus $(d,g) \in \langle c \rangle$.

Now $(d^{-1}, g) = (h, g, g)$ and

$$(d^{-1}, g) = ((d,g)^{-1})^{d^{-1}} = d(d,g)^{-1}d^{-1} = (d,g)^{-1} \in \langle c \rangle,$$

because $(d,g) \in \langle c \rangle \leq \zeta(G)$. Therefore $(h,g,g) \in \langle c \rangle$ for all $g, h \in G$. Consequently $G/\langle c \rangle$ is a 2-Engel group and hence, by Levi's Theorem, is nilpotent of class at most 3. Since $\langle c \rangle$ is central in G, it follows that G is nilpotent of class at most 4. □

Proof of Theorem 1.2. Suppose that $U(FG)$ is metabelian, but G is not nilpotent. Let P be a Sylow 2-subgroup of G. By our modularity assumption, P is nontrivial. We first show that G contains a finite non-trivial normal 2-subgroup H.

Since $U(FG)$ is metabelian, it is solvable. We distinguish two cases according to whether P is finite or infinite. If P is finite, then, by [1, Theorem 4], P is normal in G, and we may take $H = P$.

Suppose now that P is infinite. By [1, Theorem 6], there exists a finite normal 2-subgroup N of G such that G/N contains an abelian subgroup A of index 2. We claim that $N \neq 1$.

Suppose, to the contrary, that $N = 1$. Then $G = \langle A, b \rangle$ with $b^2 \in A$, and G is the disjoint union of A and bA. We show that A contains no element of order 2. Suppose that $x \in A$ has order 2. By Lemma 3.1, x cannot be central in G, since otherwise G would be nilpotent. Hence $x^b \neq x$. Since A is abelian, xx^b has order 2. Moreover, xx^b belongs to A and is fixed by conjugation by b:

$$(xx^b)^b = x^b x^{b^2} = x^b x = xx^b,$$

because $b^2 \in A$. Thus xx^b is central in G and has order 2, contradicting Lemma 3.1. Therefore A contains no element of order 2.

It follows that $A \cap P = 1$. Since $G = A \cup bA$, every nonidentity element of P must therefore belong to bA. As P is infinite, we can choose $c, d \in P \setminus \{1\}$ such that $cd \neq 1$. But $c, d \in bA$ implies $cd \in A$. At the same time, $cd \in P$, and hence $cd \in A \cap P = 1$, a contradiction. Thus $N \neq 1$, and we may take $H = N$.

We may further assume that H is abelian. Indeed, since G is not nilpotent, Lemma 3.1 implies that $H \cap \zeta(G) = 1$. If H is non-abelian, choose $1 \neq a \in H$.

Then $a \notin \zeta(G)$, so there exists $g \in G$ such that $(a,g) \neq 1$. Since H is normal in G, $1 \neq (a,g) \in G' \cap H$. The subgroup $G' \cap H$ is finite, nontrivial, and normal in G. Moreover, it is abelian because $G' \leq U(FG)'$ and $U(FG)'$ is abelian. Thus, replacing H by $G' \cap H$ if necessary, we may assume that H is abelian.

Since H is a finite abelian 2-group, it follows from (2.1) that $D_1(H) = H$ and $D_n(H) = H^{2^k}$, whenever $2^{k-1} < n \leq 2^k$. Choose s such that $D_s(H) \neq 1$, but $D_{s+1}(H) = 1$. Then, for some $\alpha \geq 0$, $D_s(H) = H^{2^\alpha} \neq 1$ and $D_{s+1}(H) = H^{2^{\alpha+1}} = 1$. In particular, $D_s(H)$ is an elementary abelian 2-group.

Choose $1 \neq y \in D_s(H)$. Since $H \cap \zeta(G) = 1$, the element y is not central in G. Hence there exists $g \in G$ such that $x = (y,g) \neq 1$. Since $D_s(H)$ is characteristic in H and H is normal in G, the subgroup $D_s(H)$ is normal in G. Therefore $x \in D_s(H)$. Again, $x \notin \zeta(G)$, so there exists $a \in G$ such that $(x,a) \neq 1$. By the normality of $D_s(H)$, $(x,a) \in D_s(H) \setminus \{1\}$.

Consider the unit $v = 1 + a(x+1)$ and put $u = (v,a)$. Then

$$\begin{aligned} u &= 1 + v^{-1}a^{-1}[a(x+1),a] \\ &= 1 + v^{-1}[x,a] \\ &= 1 + v^{-1}ax((x,a)+1). \end{aligned}$$

Since $x + 1 \in \Delta^s(H)$, we have $v \equiv 1 \pmod{\Delta^s(G,H)}$ and hence

$$v^{-1} \equiv 1 \pmod{\Delta^s(G,H)}.$$

Moreover, $x \equiv 1 \pmod{\Delta^s(G,H)}$. Since $(x,a)+1 \in \Delta^s(H)$, it follows that

$$u \equiv 1 + a((x,a)+1) \pmod{\Delta^{s+1}(G,H)}.$$

Thus we may write $u = 1 + a((x,a)+1) + \lambda$, for some $\lambda \in \Delta^{s+1}(G,H)$.

The commutator (x,a) cannot be central in G, because $1 \neq (x,a) \in H$ and $H \cap \zeta(G) = 1$. Hence there exists $b \in G$ such that $(x,a,b) \neq 1$. Thus $(x,a)^b \neq (x,a)$. Since $x \in G'$ and G' is abelian, the commutator identities give $(a,x^b) = (a,x)^b$.

We now compute the Lie commutator of u and x^b. Since $x^b + 1 \in \Delta^s(H)$, we have

$$[\lambda, x^b] = [\lambda, x^b + 1] \in \Delta^{2s+1}(G,H).$$

Therefore

$$\begin{aligned} [u,x^b] &= [a((x,a)+1),x^b] + [\lambda,x^b] \\ &\equiv a[((x,a)+1),x^b] + [a,x^b]((x,a)+1) \\ &= ax^b((a,x^b)+1)((x,a)+1) \\ &= ax^b((a,x)^b+1)((x,a)+1) \qquad (\mathrm{mod}\ \Delta^{2s+1}(G,H)). \end{aligned}$$

Now $(a,x)^b$ and (x,a) are nonidentity elements of $D_s(H)$, whose images are independent in $D_s(H)/D_{s+1}(H) = D_s(H)$. By Lemma 2.4,

$$\big((a,x)^b + 1\big)\big((x,a)+1\big) \notin \Delta^{2s+1}(H).$$

Since $\Delta^{2s+1}(G,H) \cap FH = \Delta^{2s+1}(H)$, it follows that $[u, x^b] \neq 0$, and hence $(u, x^b) \neq 1$. On the other hand, $u = (v, a) \in U(FG)'$ and $x^b \in G' \leq U(FG)'$. Thus two elements of $U(FG)'$ do not commute, contradicting the assumption that $U(FG)$ is metabelian. This contradiction completes the proof. □

References

[1] A. A. Bovdi: *Group algebras with a solvable group of units*, Communications in Algebra 33.10 (2005), pp. 3725–3738, DOI: 10.1080/00927870500243213.

[2] F. Catino, E. Spinelli: *On the derived length of the unit group of a group algebra*, Journal of Group Theory 13.4 (2010), pp. 577–588, DOI: 10.1515/jgt.2010.008.

[3] D. B. Coleman, R. Sandling: *Mod 2 group algebras with metabelian unit groups*, Journal of Pure and Applied Algebra 131.1 (1998), pp. 25–36, DOI: 10.1016/S0022-4049(97)00105-9.

[4] Á. Figula, T. Juhász: *On the derived length of the unit group of group algebras of groups with cyclic commutator subgroup*, Communications in Algebra 51.2 (2023), pp. 633–647, DOI: 10.1080/00927872.2022.2107212.

[5] T. Juhász: *A note on the derived length of the group of units of group algebras of characteristic two*, Annales Mathematicae et Informaticae 46 (2016), pp. 115–120.

[6] T. Juhász: *On the derived length of Lie solvable group algebras*, Publicationes Mathematicae Debrecen 68.1-2 (2006), pp. 243–256, DOI: 10.5486/PMD.2006.3435.

[7] T. Juhász, M. Sahai: *Modular group algebras with centrally metabelian unit groups*, Journal of Algebra 632 (2023), pp. 783–800, DOI: 10.1016/j.jalgebra.2023.06.011.

[8] T. Juhász, M. Sahai: *On the commutator subgroup of the unit group of a modular group algebra*, Journal of Group Theory 29 (2026), pp. 105–132, DOI: 10.1515/jgth-2024-0209.

[9] T. Juhász, E. Spinelli: *Group rings with metabelian unit groups*, Journal of Pure and Applied Algebra 226.6 (2022), p. 106946, DOI: 10.1016/j.jpaa.2021.106946.

[10] J. Kurdics: *On group algebras with metabelian unit groups*, Periodica Mathematica Hungarica 32.1-2 (1996), pp. 57–64, DOI: 10.1007/BF01879732.

[11] G. T. Lee: *Group Identities on Units and Symmetric Units of Group Rings*, vol. 12, Springer, 2010, DOI: 10.1007/978-1-84996-504-0.

[12] D. S. Passman: *Group algebras whose units satisfy a group identity. II*, Proceedings of the American Mathematical Society 125.3 (1997), pp. 657–662, DOI: 10.1090/S0002-9939-97-04024-0.

[13] D. S. Passman: *The Algebraic Structure of Group Rings*, New York: Wiley, 1977.

[14] D. J. S. Robinson: *A Course in the Theory of Groups*, 2nd ed., vol. 80, Graduate Texts in Mathematics, New York: Springer-Verlag, 1995, DOI: 10.1007/978-1-4419-8594-1.

[15] S. K. Sehgal: *Topics in Group Rings*, Marcel Dekker, 1978.

[16] A. Shalev: *Meta-abelian unit groups of group algebras are usually abelian*, Journal of Pure and Applied Algebra 72.3 (1991), pp. 295–302, DOI: 10.1016/0022-4049(91)90067-C.