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A Hosszú type functional equation with operation $x \circ y = (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c$

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Abstract. The aim of this paper is to give the general solution of the Hosszú type functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = g(x) + g(y) \quad (x, y \in \mathbb{R}_+)$$

with unknown functions $f, g \colon \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\} \to \mathbb{R}$, and binary operation defined by $x \circ y := (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c$ for all $x, y \in \mathbb{R}_+$ where $c \in \mathbb{R}$, $c \in \{2, 3\}$ is a fixed constant.

Keywords: functional equations, additive functions, Hosszú type functional equations, Hosszú functional equation, Hosszú cycle

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1. Introduction

At the International Symposium on Functional Equation conference held on Zakopane (Poland) in 1967 the functional equation

$$f(x + y - xy) + f(xy) = f(x) + f(y) \tag{1.1}$$

where the unknown function $f \colon \mathbb{R} \to \mathbb{R}$ satisfies the equation for all $x, y \in \mathbb{R}$ was proposed to investigate in the first time by M. Hosszú. This equation is known as Hosszú functional equation.

Z. Daróczy [6, 8] D. Blanusa [4], and H. Swiatak [25, 26] proved that the general solution of equation (1.1) is in the form $f(x) = A(x) + C$ for all $x \in \mathbb{R}$

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where $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function (that is, $A(x+y) = A(x)+A(y)$ is fulfilled for all $x, y \in \mathbb{R}$ [2, 3, 18]), and C is a real constant.

Since 1969, many researchers have investigated the Hosszú equation and its generalizations. In papers [9–12] the equation (1.1) was investigated on various abstract structures. In paper [13] a Pexider version (that is functional equation with more unknown function [1, 3, 15, 18]) of equation (1.1) and its locally integrable function solutions can be found. See also [20, 21].

The general solution of equations

$$f(xy) + g(x + y - xy) = f(x) + f(y) \tag{1.2}$$

$$f(xy) + g(x + y - xy) = h(x) + h(y) \tag{1.3}$$

was given by K. Lajkó [19] in the following cases:

Problem A. If the unknown functions $f, g\colon]0, 1[\to \mathbb{R}$ satisfy the equation (1.2) for all $x, y \in]0, 1[$, then there exist additive functions A_1, A_2, and a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} f(x) &= A_1(x) + A_2(\log x) + C, && (x \in]0, 1[),\\ g(x) &= A_1(x) + C, && (x \in]0, 1[). \end{aligned}$$

(where log denotes the natural logarithm function).

Problem B. If the unknown functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ satisfy the equation (1.3) for all $x, y \in \mathbb{R}$, then there exist an additive function A and constants C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned} f(x) &= A(x) + C_2 && (x \in \mathbb{R}),\\ g(x) &= A(x) + C_3, && (x \in \mathbb{R}),\\ h(x) &= A(x) + C_1, && (x \in \mathbb{R}), \end{aligned}$$

where $2C_1 = C_2 + C_3$. If the unknown functions $f, h\colon \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \to \mathbb{R}$, and $g\colon \mathbb{R} \to \mathbb{R}$ satisfy the equation (1.3) for all $x, y \in \mathbb{R}_0$, then there exist additive functions A_1, A_2 and constants C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned} f(x) &= A_1(x) + A_2(\log |x|) + C_3 && (x \in \mathbb{R}_0),\\ g(x) &= A_1(x) + C_2, && (x \in \mathbb{R}),\\ h(x) &= A_1(x) + A_2(\log |x|) + C_1, && (x \in \mathbb{R}_0). \end{aligned}$$

If the unknown functions $f\colon \mathbb{R} \to \mathbb{R}$, and $g, h\colon \mathbb{R}_1 := \mathbb{R} \setminus \{1\} \to \mathbb{R}$ satisfy the equation (1.3) for all $x, y \in \mathbb{R}_1$, then there exist additive functions A_1, A_2 and constants C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned} f(x) &= A_1(x) + C_3 && (x \in \mathbb{R}),\\ g(x) &= A_1(x) + A_2(\log |1 - x|) + C_2, && (x \in \mathbb{R}_1).\\ h(x) &= A_1(x) + A_2(\log |1 - x|) + C_1, && (x \in \mathbb{R}_1). \end{aligned}$$

In his paper [7] Z. Daróczy investigate the functional equation

$$f(x + y - x \circ y) + f(x \circ y) = f(x) + f(y) \quad (x, y \in \mathbb{R}_+) \tag{1.4}$$

with unknown function $f\colon \mathbb{R} \to \mathbb{R}$ where the binary operation $\circ$ is defined by $x \circ y := \ln(e^x + e^y)$ for all $x, y \in \mathbb{R}$. The general solution of this equation is $f(x) = A(x) + C$ for all $x \in \mathbb{R}$ where A is an additive function, C is a real constant.

The main purpose of our present paper is to give the general solution of the functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = g(x) + g(y) \quad (x, y \in \mathbb{R}_+) \tag{1.5}$$

with unknown functions $f, g\colon \mathbb{R}_+ \to \mathbb{R}$. The binary operation is defined by

$$x \circ y := (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c \quad (x, y \in \mathbb{R}_+), \tag{1.6}$$

where $c \in \mathbb{R} \setminus \{0, 1\}$ is a fixed constant.

We also consider the functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = h(x) + h(y) \quad (x, y \in \mathbb{R}_+) \tag{1.7}$$

with unknown functions $f, g, h\colon \mathbb{R}_+ \to \mathbb{R}$ and binary operation $\circ$ is defined by (1.6). The equation (1.5) is a common generalization of equations (1.2) and (1.4). The equation (1.7) is a common generalization of equations (1.3) and (1.4).

The devices needed to the problems we set out, and to the earlier problems are the theorems giving the solutions of the restricted Pexider additive functional equations (in the rest briefly Additive Extension Theorems) and the application of the Hosszú cycle.

Let $D \subseteq \mathbb{R}^2$ be a non-empty connected set. Define the sets

$$\begin{aligned} D_x &:= \{u \in \mathbb{R} \mid \exists v \in \mathbb{R} : (u, v) \in D\}, \\ D_y &:= \{v \in \mathbb{R} \mid \exists u \in \mathbb{R} : (u, v) \in D\}, \\ D_{x+y} &:= \{z \in \mathbb{R} \mid \exists (u, v) \in D : z = u + v\}. \end{aligned}$$

The functional equation

$$f(x + y) = g(x) + h(y) \quad ((x, y) \in D)$$

with unknown functions $f\colon D_{x+y} \to \mathbb{R}$, $g\colon D_x \to \mathbb{R}$, $h\colon D_y \to \mathbb{R}$ is a restricted Pexider additive functional equation. According to Rimán's Extension Theorem [24], there exist an additive function $A\colon \mathbb{R} \to \mathbb{R}$ and constants C_i $(i = 1, 2)$ such that

$$\begin{aligned} f(u) &= A(u) + C_1 + C_2 && (u \in D_{x+y}), \\ g(v) &= A(v) + C_1 && (v \in D_x), \\ h(z) &= A(z) + C_2 && (z \in D_y). \end{aligned}$$

Concerning the Additive Extension Theorems see also [1, 5, 14, 16, 18, 23].

A Hosszú cycle is a functional equation

$$F(x+y,z)+F(x,y)=F(x,y+z)+F(y,z)\quad (x,y,z\in D)$$

with unknown function $F\colon A^2\to D$, where $A=A(+)$ is a semi-group, $D=D(+)$ is an Abelian semi-group. The first appearance of this equation was in [17], although many researchers use the Hosszú Cycle to solve functional equations for example equations (1.1), and (1.2).

2. The decomposition of equation (1.5) by Hosszú cycle

Theorem 2.1. *If the functions $f,g\colon\mathbb{R}_+\to\mathbb{R}$ satisfy the functional equation (1.5) where the binary operation $\circ$ is defined by (1.6), then f satisfies the functional equation*

$$\begin{aligned} f(|(x+y+z)^c-((x+y)^c+z^c)|)+f(|(x+y)^c-((x^c+y^c)|)\\ =f(|(x+y+z)^c-(x^c+(y+z)^c)|)\\ +f(|(y+z)^c-(y^c+z^c)|)\quad (x,y\in\mathbb{R}_+). \end{aligned}\tag{2.1}$$

Proof. For the proof we shall use the well-known Hosszú Cycle (see [7]). Define the function $F\colon\mathbb{R}_+^2\to\mathbb{R}$ by

$$F(x,y):=g(x)+g(y)-g(x\circ y)\quad (x,y\in\mathbb{R}_+).$$

Since the operation $\circ$ is associated thus we have that

$$F(x\circ y,z)+F(x,y)=F(x,y\circ z)+F(y,z)\quad (x,y,z\in\mathbb{R}_+).\tag{2.2}$$

By equations (1.5) we have that

$$F(x,y)=f(|x+y-(x\circ y)|)\quad (x,y\in\mathbb{R}_+).\tag{2.3}$$

From equations (2.2) and (2.3) we have the equation (2.1). □

The following proposition is well known (and will be used later).

Proposition 2.2. *Let $c\in\mathbb{R}\setminus\{0,1\}$ and define the function $\varphi_c\colon\mathbb{R}_+\to\mathbb{R}_+$ by*

$$\varphi_c(x):=x^c\quad (x\in\mathbb{R}_+).$$

a. *If $c>1$, then the function φ_c is strictly superadditive in the sense that*

$$(x+y)^c>x^c+y^c\quad (x,y\in\mathbb{R}_+).\tag{2.4}$$

b. *If* $0 < c < 1$, *or* $c < 0$ *then the function* φ_c *is strictly subadditive in the sense that*

$$x^c + y^c > (x+y)^c \quad (x, y \in \mathbb{R}_+) \tag{2.5}$$

Example 2.3. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ thus by Proposition 2.2 it is easy to see that $e^{x+y} + 1 > e^x + e^y$ for all $x, y \in \mathbb{R}_+$, and $\log(x+1) + \log(y+1) > \log(x+y+1)$ for all $x, y \in \mathbb{R}_+$. It is also easy to see that $\log(x+y) > \log(x) + \log(y)$ for all $x, y \in]0,1[$.

Define the function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ by $\varphi(x) := x + x^2 + x^3$ for all $x \in \mathbb{R}_+$. Then function φ is a strictly superadditive bijection.

Our references concerning subadditive, and superadditive functions is [22].

Corollary 2.4. *Preserving the notations of Theorem 2.1 by Proposition 2.2 it is easy to see that*

- *if* $c > 1$, *then the function* f *satisfies the equation*

$$\begin{aligned} f((x+y+z)^c - ((x+y)^c + z^c)) + f((x+y)^c - ((x^c + y^c)) \\ = f((x+y+z)^c - (x^c + (y+z)^c)) \\ + f((y+z)^c - (y^c + z^c)) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{2.6}$$

- *if* $0 < c < 1$ *or* $c < 0$, *then the function* f *satisfies the equation*

$$\begin{aligned} f((x+y)^c + z^c - (x+y+z)^c) + f((x^c + y^c - (x+y)^c) \\ = f(x^c + (y+z)^c - (x+y+z)^c) \\ + f(y^c + z^c - (y+z)^c) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{2.7}$$

Corollary 2.5. *If the function* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy the functional equation* (2.6) *or equation* (2.7) *for all* $x, y, z \in \mathbb{R}_+$ *thus it is also satisfies the equation*

$$\begin{aligned} f\left(\frac{(x+y+z)^c - ((x+y)^c + z^c)}{(x+y+z)^c - (x^c + (y+z)^c)}\right) + f\left(\frac{(x+y)^c - (x^c + y^c)}{(x+y+z)^c - (x^c + (y+z)^c)}\right) \\ = f(1) + f\left(\frac{(y+z)^c - (y^c + z^c)}{(x+y+z)^c - (x^c + (y+z)^c)}\right) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{2.8}$$

3. On the equation (2.8) in the cases $c = 2, 3$

Proposition 3.1. *If the function* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfies the equation*

$$f(u) + f(v) = f(1) + f(u + v - 1) \quad ((u, v) \in D) \tag{3.1}$$

where the set $D \subseteq \mathbb{R}^2$ *is defined by*

$$D := \{(u, v) \in \mathbb{R}^2 \mid v > -u + 1, v < 1\}, \tag{3.2}$$

then there exists an additive function $A\colon \mathbb{R} \to \mathbb{R}$ *and a constant* $C \in \mathbb{R}$ *such that.*

$$f(x) = A(x) + C \quad (x \in \mathbb{R}_+).$$

Proof. Define the function $g\colon]1,\infty[\to \mathbb{R}$ by

$$g(z) = f(z-1) + f(1) \quad (z \in]1,\infty[).$$

Then the functions f and g satisfy the equation

$$g(u+v) = f(u) + f(v) \quad ((u,v) \in D)$$

where the set D is defined by (3.2). Thus by Rimán's Extension Theorem (see [14, 24]) we obtain that there exist an additive function $A\colon \mathbb{R} \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ with

$$f(u) = A(u) + C \quad (u \in D_x = \mathbb{R}_+).$$ □

Theorem 3.2. *If the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ satisfy equation (2.8) for all $x, y, z \in \mathbb{R}_+$ with constant $c \in \{2, 3\}$, there exist an additive function $A_1\colon \mathbb{R} \to \mathbb{R}$ and a constant $C_1 \in \mathbb{R}$ such that*

$$f(x) = A_1(x) + C_1 \quad (x \in \mathbb{R}_+). \tag{3.3}$$

Proof. **Case 1.** Let $c = 2$. By equation (2.8) the function f satisfies the equation

$$f\left(\frac{(x+y)z}{x(y+z)}\right) + f\left(\frac{y}{y+z}\right) = f(1) + f\left(\frac{yz}{x(y+z)}\right) \quad (x, y, z \in \mathbb{R}_+). \tag{3.4}$$

Take the substitution in equation (3.4)

$$y \longleftarrow \frac{x(u+v-1)}{1-v} \qquad z \longleftarrow \frac{x(u+v-1)}{v}$$

thus we have that the function f satisfies the equation (3.1) where the set $D \subseteq \mathbb{R}^2$ is defined by (3.2). By Proposition 3.1 we have that the function f is in the form of (3.3).

Case 2. Let $c = 3$. By equation (2.8) the function f satisfies the equation

$$\begin{aligned} f\left(\frac{(x+y)z}{x(y+z)}\right) &+ f\left(\frac{y(x+y)}{(y+z)(x+y+z)}\right) \\ &= f(1) + f\left(\frac{yz}{x(x+y+z)}\right) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{3.5}$$

Take the substitution in equation (3.5)

$$y \longleftarrow \frac{\sqrt{uvx^2(u+v-1)} + x(u+v-1)}{1-v}, \qquad z \longleftarrow \frac{\sqrt{uvx^2(u+v-1)}}{v}$$

whence we have that the function f satisfy the equation (3.1) where the set D is defined by (3.2). The rest of the proof of this case is analogous to the case $c = 3$. □

4. Results and open problems

Theorem 4.1. *If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly superadditive bijection, the binary operation $\circ$ is defined by*

$$x \circ y := \varphi(\varphi^{-1}(x) + \varphi^{-1}(y)) \quad (x, y \in \mathbb{R}_+), \tag{4.1}$$

$f\colon \mathbb{R}_+ \to \mathbb{R}$ is a function such that that exist an additive function $A_1\colon \mathbb{R}_+ \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ with

$$f(x) = A_1(x) + C \quad (x \in \mathbb{R}_+), \tag{4.2}$$

$g, h\colon \mathbb{R}_+ \to \mathbb{R}$ are functions such that

$$f(|x + y - (x \circ y)|) + g(x \circ y) = h(x) + h(y) \quad (x, y \in \mathbb{R}_+), \tag{4.3}$$

then there exist an additive function $A_2\colon \mathbb{R} \to \mathbb{R}$ and constant C_2 with

$$\begin{aligned} g(x) &= -A_1(x) + A_2(\varphi^{-1}(x)) + 2C_2 - C_1 && (x \in \mathbb{R}_+), \\ h(x) &= -A_1(x) + A_2(\varphi^{-1}(x)) + C_2 && (x \in \mathbb{R}_+). \end{aligned} \tag{4.4}$$

Proof. During the proof, $\circ$ will denote the usual function composition (for example $(g \circ \varphi)(x) := g(\varphi(x))$). Since $\varphi(x + y) > \varphi(x) + \varphi(y)$ thus from equation (4.3) we have that

$$\begin{aligned} &f(\varphi(x + y) - (\varphi(x) + \varphi(y))) + g(\varphi(x + y)) \\ &\qquad = h(\varphi(x)) + h(\varphi(x)) \quad (x, y \in \mathbb{R}_+). \end{aligned} \tag{4.5}$$

From equations (4.2) and (4.5) we have that

$$\begin{aligned} &((A_1 \circ \varphi) + (g \circ \varphi) + C_1))(x + y) \\ &\quad = ((A_1 \circ \varphi) + (h \circ \varphi))(x) + ((A_1 \circ \varphi) + (h \circ \varphi))(y) \quad (x, y \in \mathbb{R}_+). \end{aligned} \tag{4.6}$$

Define de functions $F, G\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$F(x) : = ((A_1 \circ \varphi) + (g \circ \varphi) + C_1)(x) \qquad (x \in \mathbb{R}_+), \tag{4.7}$$

$$G(x) : = ((A_1 \circ \varphi) + (h \circ \varphi))(x) \qquad (x \in \mathbb{R}_+). \tag{4.8}$$

From equation (4.6) we have that

$$F(x + y) = G(x) + G(y) \quad (x, y \in \mathbb{R}_+). \tag{4.9}$$

From equation (4.9) by Rimán's Extension Theorem we have that there exist an additive function $A_2\colon \mathbb{R} \to \mathbb{R}$ and a constant $C_2 \in \mathbb{R}$ such that

$$F(x) = A_2(x) + 2C_2 \qquad (x \in \mathbb{R}_+), \tag{4.10}$$

$$G(x) = A_2(x) + C_2 \qquad (x \in \mathbb{R}_+). \tag{4.11}$$

From equations (4.8), and (4.11) we obtain equation (4.4). $\square$

Theorem 4.2. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly subadditive bijection. Preserving the notation of Theorem 4.1 the functions g, h is in the form*

$$g(x) = A_1(x) + A_2(\varphi^{-1}(x)) + 2C_2 - C_1 \qquad (x \in \mathbb{R}_+), \tag{4.12}$$

$$h(x) = A_1(x) + A_2(\varphi^{-1}(x)) + C_2 \qquad (x \in \mathbb{R}_+). \tag{4.13}$$

Proof. The proof is analogous to the proof of Theorem 4.1. □

Theorem 4.3. *If the functions $f, g\colon \mathbb{R}_+ \to \mathbb{R}$ satisfy the functional equation (1.5) where the operation $\circ$ is defined by (1.6) where $c \in \{2, 3\}$, then there exist an additive functions $A_1, A_2\colon \mathbb{R} \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ such that*

$$f(x) = A_1(x) + C \qquad (x \in \mathbb{R}_+), \tag{4.14}$$

$$g(x) = A_1(x) + A_2(x^{\frac{1}{c}}) + C \qquad (x \in \mathbb{R}_+). \tag{4.15}$$

Proof. The proof is evident by Theorem 2.1, Corollary 2.4, Corollary 2.5, Theorem 3.2, and Theorem 4.1. □

Theorem 4.1, Theorem 4.2, and Theorem 4.3 suggest the following open problems.

Problem A. Does Theorem 4.3 hold for all $c \in \mathbb{R}_+ \setminus \{0, 1\}$?

Our conjecture is that it remains true.

Problem B. Find the general solution of equation (1.7) with unknown functions $f, g, h\colon \mathbb{R}_+ \to \mathbb{R}$ and binary operation $\circ$ is defined by (1.6) where $c \in \mathbb{R}_+ \setminus \{0, 1\}$.

Problem C. In particular, the authors of the present article would like to know the solution of Problem B in the case of $c = -1$, in more details, find the general solution of functional equation

$$f\left(x + y - \frac{xy}{x+y}\right) + g\left(\frac{xy}{x+y}\right) = h(x) + h(y) \quad (x, y \in \mathbb{R}_+)$$

with unknown functions $f, g, h\colon \mathbb{R}_+ \to \mathbb{R}$.

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