Annales Mathematicae et Informaticae
62 (2025) pp. 46–54
DOI: 10.33039/ami.2025.11.002
URL: https://ami.uni-eszterhazy.hu

Cubics and conics geodesically associated to the points of a geometric surface

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Abstract. We associate to any point P in a two-dimensional Riemannian manifold (M, g) a cubic curve inspired by the third-order differential equation of geodesics of g. This construction is more simple when the value in P of the Christoffel symbol Γ^1_{22} of g squares to $+1$; on this way some special types of points on M are considered. A large part of this note concerns with examples. If $\Gamma^1_{22}(P) = 0$ we associate a conic which is discussed directly on examples.

Keywords: two-dimensional Riemannian manifold, cubic curve, Weierstrass point

2020 *Mathematics Subject Classification:* 53A05, 53A45, 53A55, 53B20, 14H52

1. Introduction

The paper [9] provides a very good illustration of how elliptic curves naturally occur in a number of main areas of Mathematics. This makes having a variety of techniques to get or derive elliptic curves from other (geometrical) objects desirable.

In a previous paper, namely [5], we associate an elliptic curve to any point of a spacelike curve in the Lorentz plane. This idea is continued in the paper [4] were we associate cubic curves (as more general as the elliptic curves) to the points of an Euclidean curve. In both papers the fundamental tool is a differential equation of third order satisfied by the given curve.

The present work continues this line of research by starting with the most general differential two-dimensional setting i.e. a *Riemannian manifold* (M^2, g), and with its main curves namely *geodesics*. Again, a given geodesic g, with some technical conditions, is described locally by a third-order differential equation which is the model of an associated cubic curve to any point of g. We point out that several

Submitted: October 15, 2025
Accepted: November 26, 2025
Published online: December 1, 2025

times the case of Finsler metrics is also discussed.

The contents is as follows. In the next section we consider the reduction for the equation of cubic curve to a Weierstrass form which allow us to compute its discriminant. We point out that in order to do this approach we impose some algebraic conditions to a some Christoffel symbols of g. The following section concerns with the large examples given by important classes of metrics: a warped product, an isothermal one and a pseudospherical-type one. We note that the first example is, in fact, the inverse metric of the well-known Poincaré metric from the upper-half-plane model of hyperbolic geometry. Remarkable again, the Levi-Civita connection of the second example is constant i.e. its Christoffel symbols are constant. The last section deals with the case $\Gamma^1_{22}(P) = 0$ when we associate a conic to the given point $P \in M$.

2. From geodesics of a geometric surface to cubic curves

The framework of this study is a *geometric surface* i.e. ([6]) a smooth, two-dimensional Riemannian manifold (M^2, g). We fix a coordinates chart $(u, v) = (u^1, u^2)$ on M. Then the metric g is a 2×2 symmetric matrix of smooth functions $g = (g_{ij}(\cdot,\cdot))_{i,j=1,2}$ with the associated Christoffel symbols:

$$\Gamma^k_{ij} := \frac{g^{ka}}{2}\left(\frac{\partial g_{aj}}{\partial u^i} + \frac{\partial g_{ia}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^a}\right).$$

The differential system of the geodesics $(u^1 = u^1(t), u^2 = u^2(t))$ of g is:

$$\ddot{u}^k(t) + \Gamma^k_{ij}(u^1(t), u^2(t))\dot{u}^i(t)\dot{u}^j(t) = 0, \quad k = 1, 2 \tag{2.1}$$

and:

a) if the curves $u = u^1 = constant$ are not geodesics then (2.1) can be replaced by:

$$\frac{d^2v}{du^2} = \Gamma^1_{22}\left(\frac{dv}{du}\right)^3 + (2\Gamma^1_{12} - \Gamma^2_{22})\left(\frac{dv}{du}\right)^2 + (\Gamma^1_{11} - 2\Gamma^2_{12})\frac{dv}{du} - \Gamma^2_{11}, \tag{2.2}$$

b) if the curves $v = u^2 = constant$ are not geodesics then (2.1) can be replaced by:

$$\frac{d^2u}{dv^2} = \Gamma^2_{11}\left(\frac{du}{dv}\right)^3 + (2\Gamma^2_{12} - \Gamma^1_{11})\left(\frac{du}{dv}\right)^2 + (\Gamma^2_{22} - 2\Gamma^1_{12})\frac{du}{dv} - \Gamma^1_{22}.$$

We note that replacing $u^1 = constant$ in the differential system (2.1) gives:

$$\Gamma^1_{22}(constant, u^2(t)) = 0, \quad \ddot{u}^2(t) + \Gamma^2_{22}(constant, u^2(t))\left(\dot{u}^2(t)\right)^2 = 0$$

while the expression of the first coefficient from the right-hand-side of (2.2) is:

$$\Gamma^1_{22} := \frac{g^{1a}}{2}\left(2\frac{\partial g_{a2}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^a}\right). \tag{2.3}$$

Hence, if we ask for the non-vanishing of Γ^1_{22} then we exclude the Cartesian coordinates (x, y) of the Euclidean plane geometry $(\mathbb{R}^2, g_{can} = dx^2 + dy^2)$.

Remark 2.1. i) We point out that the Wunschmann-type condition for the equation (1.3) is discussed in [8]. Also, a version of the differential equation (1.3), based on Hopf (complex) data of g, appears in [7, p. 89]. An approach of the equation (1.3) based on projective differential geometry can be found in [15, p. 344].
ii) The geodesics of the tangent sphere bundle T_1M of M endowed with the Sasaki lift are characterized in the Theorem 1. of [12, p. 205].

Now, we fix the case a) which yields the following types of points on M:

Definition 2.2. The point $P(u_0, v_0) \in M$ is called *horizontal-cubic-monic* if $\Gamma^1_{22}(u_0, v_0) \in \{-1, +1\}$. Such a point is called *Weierstrass* if it satisfies also:

$$2\Gamma^1_{12}(u_0, v_0) = \Gamma^2_{22}(u_0, v_0).$$

Inspired by the equation (2.2) to a horizontal-cubic-monic point we associate *the cubic curve*:

$$\mathcal{C}(P) : y^2 = \Gamma^1_{22}[\Gamma^1_{22}X^3+(2\Gamma^1_{12}-\Gamma^2_{22})X^2+(\Gamma^1_{11}-2\Gamma^2_{12})X-\Gamma^2_{11}] = X^3+AX^2+BX+C$$

with all Γ's computed in (u_0, v_0). We recall that the method to associate an object to every point of a given manifold M is the starting point of the theory of *bundles over* M. The well-known transformation $X = x - \frac{A}{3}$ yields *the reduced form* of the cubic:

$$\mathcal{C}(P) : y^2 = x^3 + px + q, \quad p := B - \frac{A^2}{3}, \quad q := C - \frac{AB}{3} + \frac{2A^3}{27}$$

and having *the discriminant*:

$$\Delta = \Delta(u_0, v_0) := 4p^3 + 27q^2 = 4B^3 - A^2B^2 + 27C^2 + 4A^3C - 18ABC. \tag{2.4}$$

Using again the identity $(\Gamma^1_{22})^2 = 1$ we derive:

Theorem 2.3. *The coefficients of the reduced cubic curve are:*

$$\begin{cases} p = \Gamma^1_{22}(\Gamma^1_{11} - 2\Gamma^2_{12}) - \frac{(2\Gamma^1_{12}-\Gamma^2_{22})^2}{3}, \\ q = -\Gamma^1_{22}\Gamma^2_{11} - \frac{(2\Gamma^1_{12}-\Gamma^2_{22})(\Gamma^1_{11}-2\Gamma^1_{12})}{3} + \frac{2\Gamma^1_{22}(2\Gamma^1_{12}-\Gamma^2_{22})^3}{27}. \end{cases}$$

In particular, if the point $P(u_0, v_0) \in M$ is already Weierstrass then the discriminant is:

$$\Delta = 4\Gamma^1_{22}(\Gamma^1_{11} - 2\Gamma^2_{12})^3 + 27(\Gamma^2_{11})^2. \tag{2.5}$$

The formula (2.5) yields a third type of special points:

Definition 2.4. A Weierstrass point $P(u_0, v_0) \in M$ is called *singular* if:

$$4\Gamma^1_{22}(u_0, v_0)[\Gamma^1_{11}(u_0, v_0) - 2\Gamma^2_{12}(u_0, v_0)]^3 + 27[\Gamma^2_{11}(u_0, v_0)]^2 = 0.$$

In the following section we discuss three examples remarking that all the Weierstrass points of the last example are singular.

3. Examples

Example 3.1. Let us consider a warped product metric having a polynomial warped function: $g = du^2 + \alpha u^\beta dv^2$ on a domain $u > 0$; here α and β are real constants. This metric has only two non-zero Christoffel symbols:

$$\Gamma^1_{22} = \frac{-\alpha\beta}{2} u^{\beta-1}, \quad \Gamma^2_{12} = \frac{\beta}{2u}$$

and hence we choose $\alpha = 2$ and $\beta = 1$ for obtaining $\Gamma^1_{22} = -1$; the Gaussian curvature of the warped metric $g = du^2 + 2udv^2$ is $K(u) = \frac{1}{4u^2} > 0$. The warped function $f(u) = 2u$ is exactly the derivative of the warped function $F(u) = u^2$ which appears in the expression of the Euclidean metric $g_{can} = dx^2 + dy^2 = du^2 + F(u)dv^2$ when (u, v) are the polar coordinates on the punctured plane $\mathbb{R}^2 \setminus \{O\}$ with the usual Cartesian coordinates (x, y).

The differential system (2.1) is:

$$\ddot{u} - \dot{v}^2 = 0, \quad \ddot{v} + \frac{\dot{u}\dot{v}}{u} = 0 \tag{3.1}$$

and, apart to the solutions ($u(t) = \alpha t + \beta, v(t) = constant$) the second equation yields a first integral *of Clairaut type*:

$$u\dot{v} = \omega > 0.$$

It follows that the first differential equation (3.1) reads:

$$\ddot{u} - \frac{\omega^2}{u^2} = 0 \tag{3.2}$$

which gives a second first integral:

$$\frac{\dot{u}^2}{2} + \frac{\omega^2}{u} = constant \tag{3.3}$$

and this means that (3.2) is exactly the Euler-Lagrange equation for the autonomous Lagrangian $\mathcal{L} = \mathcal{L}(u, \dot{u}) = \frac{\dot{u}^2}{2} - \frac{\omega^2}{u}$ which is an inverse-square-law 1-dimensional Lagrangian, the multi-dimensional case is the Problem 3.3 from [17, p. 85].

The differential equation (2.2) is:

$$\frac{d^2v}{du^2} = -\left(\frac{dv}{du}\right)^3 - \frac{1}{u}\frac{dv}{du}$$

which admits the following general solution depending on an arbitrary constant $C > 0$:

$$v_C(u) = \int \frac{du}{\sqrt{Cu^2 - 2u}}, \quad u \geq \frac{2}{C} > 0.$$

If we denote with V the derivative $\frac{dv}{du}$ then (3.3) becomes the first-order differential equation:

$$\frac{dV}{du} = -V^3 - \frac{V}{u}$$

which is a Bernoulli one; as a historical issue we note that the geodesics were first investigated by Jakob Bernoulli while the term is due to Laplace.

For our study it follows that all points are Weierstrass and non-singular. The associated cubic curve is:

$$\mathcal{C}(P(u_0, v_0)) : y^2 = X^3 + \frac{1}{u_0}X = X\left(X^2 + \frac{1}{u_0}\right), \quad \Delta(u_0) = \frac{4}{u_0^3} > 0$$

since $u_0 > 0$ while the limit $u_0 \to +\infty$ gives the semicubical parabola $\mathcal{C}(+\infty)$: $y^2 = X^3$. We end this example with the remark that if the prime number $p \in \mathbb{N}^*$ satisfies $p \equiv 1 \ (mod\ 4)$ then (-1) is a quadratic residue, which implies that the polynomial $P(u_0 = 1) = x^3 + x = x(x^2 + 1)$ has three distinct zeros in the finite field $\mathbb{F}_p$.

Example 3.2. Recall that on any geometrical surface there exist *isothermal coordinates*. The previous warped metric can be expressed as:

$$g = U^2[dU^2 + dv^2], \quad U = \sqrt{2u} > 0.$$

More generally, let $g = e^{2\varphi}(du^2 + dv^2)$ be a metric given in isothermal coordinates. Its Christoffel symbols are:

$$\begin{cases} \Gamma^1_{11} = \frac{\partial\varphi}{\partial u}, \quad \Gamma^1_{12} = \frac{\partial\varphi}{\partial v}, \quad \Gamma^1_{22} = -\frac{\partial\varphi}{\partial u}, \\ \Gamma^2_{11} = -\frac{\partial\varphi}{\partial v}, \quad \Gamma^2_{12} = \frac{\partial\varphi}{\partial u}, \quad \Gamma^2_{22} = \frac{\partial\varphi}{\partial v}. \end{cases}$$

and so, the equality $\Gamma^1_{22} = \varepsilon = \pm 1$ gives $\varphi(u, v) = -\varepsilon u + \phi(v)$. Since $2\Gamma^1_{12} - \Gamma^2_{22} = \frac{\partial\varphi}{\partial v}$ we get:

Proposition 3.3. *All points of* $(\mathbb{R}^2, g = e^{-2\varepsilon u}(du^2 + dv^2))$ *are Weierstrass and the only non-zero Christoffel symbols are:*

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = -\varepsilon.$$

This metric is flat, $K \equiv 0$ *and indeed, the transformation* $U = -\varepsilon e^{-\varepsilon u}$ *gives* $g = dU^2 + U^2 dv^2$ *which is the Euclidean metric expressed in the polar coordinates.*

Remark 3.4. i) A two-dimensional Riemannian metric expressed in isothermal coordinates as function of only one coordinate is called *generalized Poincaré* in [2] having as model the usual upper half-plane Poincaré metric $\frac{1}{y^2}(dx^2 + dy^2)$; from a matrix point of view the Poincaré metric is the inverse of the metric (2.4). Since the subject of the cited paper are harmonic functions of generalized Poincaré metrics we compute the Laplace operator for the metric of Example 3.1:

$$\Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{1}{2u}\left(\frac{\partial^2 f}{\partial v^2} + \frac{\partial f}{\partial u}\right)$$

and hence, the functions which are linear in the variable v are harmonic.
ii) A discussion of the geodesics in spaces with locally constant connection appears in [16]. Since the Finslerian geometry is the most natural generalization of the Riemannian geometry it is worth to point out that the two-dimensional Finsler spaces with constant Berwald connection are studied in [1].
iii) From an algebraic point of view the case $\varepsilon = -1$ is more naturally, exactly as in the first example. Indeed, given a constant linear connection $\nabla = (\Gamma^k_{ij})$ on a manifold M^n a structure of *real algebra* on $\mathbb{R}^n$ (with a fixed basis $B = \{e_1, ..., e_n\}$) is defined through the relations:

$$e_i \cdot e_j := \Gamma^k_{ij}.$$

Hence, our metric g has the associated real algebra $(\mathbb{R}^2, B, \varepsilon)$ given by:

$$e_1 \cdot e_1 := \Gamma^1_{11} e_1 = -\varepsilon e_1, \quad e_1 \cdot e_2 := \Gamma^2_{12} e_2 = -\varepsilon e_2, \quad e_2 \cdot e_2 := \Gamma^1_{22} = \varepsilon e_1$$

and therefore we obtain the complex algebra $(\mathbb{R}^2 = \mathbb{C}, B = \{e_1 = 1, e_2 = i\}, -1)$.

For the same case $\varepsilon = -1$ the metric is explicitly Euclidean in the variables $X = e^u \cos v = U \cos v$, $Y = e^u \sin v = U \sin v$ i.e. $g = dX^2 + dY^2$. The Euclidean space $\mathbb{E}^2(X, Y)$ is covered infinitely times by the transformation above $(u, v) \to (X, Y)$ since every domain $\mathcal{D}(k) := \{(u, v); u \in \mathbb{R}, (2k-1)\pi < v < (2k+1)\pi\} \subset \mathbb{R}^2$ with $k \in \mathbb{Z}$ corresponds to $\mathbb{E}^2(X, Y) \setminus \{(0, 0)\}$.

The differential system (2.1) of the geodesics of g is:

$$\ddot{u} - \varepsilon(\dot{u})^2 + \varepsilon(\dot{v})^2 = 0, \quad \ddot{v} - 2\varepsilon \dot{u}\dot{v} = 0$$

and again, apart to the solutions $(u(t) = -\frac{1}{\varepsilon} \ln(\alpha + \varepsilon t) + \beta, v(t) = constant)$ defined on the real interval given by $\alpha + \varepsilon t > 0$, we find a first integral from the second equation:

$$e^{-2\varepsilon u} \dot{v} = \omega > 0.$$

The differential equation (2.2) is:

$$\frac{d^2 v}{du^2} = \varepsilon \left[\left(\frac{dv}{du} \right)^3 + \frac{dv}{du} \right].$$

Since the Levi-Civita connection is constant the associated cubic curve is *universal* i.e. is the same for all points of $\mathbb{R}^2$:

$$\mathcal{C} : y^2 = X^3 + X, \quad \Delta = 4 > 0.$$

and its main properties are available at https://www.lmfdb.org/EllipticCurve/Q/64/a/4.

Example 3.5. Suppose now that M is a regular surface in the Euclidean 3-dimensional space $\mathbb{E}^3 := (\mathbb{R}^3, g_{can} = dx^2 + dy^2 + dz^2)$ and its metric g is the first fundamental form I. If the Gaussian curvature K is a strictly negative constant, say $-\frac{1}{\rho^2}$, then the asymptotic lines of M may be taken as parametric curve and hence the fundamental forms of M are ([14, p. 21]):

$$g = I = du^2 + 2\cos\omega du dv + dv^2, \quad II = \frac{2}{\rho}\sin\omega du dv, \quad \rho^2\frac{\partial^2\omega}{\partial u\partial v} = \sin\omega \tag{3.4}$$

with $\omega = \omega(u, v)$ the angle between the parametric lines. Now, we give up to the pseudospherical condition about K in order to allow functions ω which are not solutions to the *sine-Gordon* differential equation from the end of (3.4); the only condition about ω is to belongs to $(0, \pi)$ as g is a Riemannian metric. We derive that the non-zero Christoffel symbols are:

$$\begin{pmatrix}\Gamma^1_{11}\\ \Gamma^2_{11}\end{pmatrix} = \begin{pmatrix}-\frac{\cos\omega}{\sin\omega}\frac{\partial\omega}{\partial u}\\ -\sin\omega\frac{\partial\omega}{\partial u}\end{pmatrix}, \quad \begin{pmatrix}\Gamma^1_{22}\\ \Gamma^2_{22}\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\\ -\frac{\cos\omega}{\sin\omega}\frac{\partial\omega}{\partial v}\end{pmatrix}$$

and therefore the condition $\Gamma^1_{22} = \varepsilon$ gives the existence of a smooth function $\alpha = \alpha(u)$ such that:

$$\omega(u, v) = 2\cot^{-1}\left(e^{\alpha(u)+\varepsilon v}\right), \quad \sin\omega = \frac{1}{\cosh(\alpha(u) + \varepsilon v)}.$$

Since $\Gamma^1_{12} = 0$ it follows that $P(u_0, v_0) \in M$ is Weierstrass is and only if $\omega(u_0, v_0) = \frac{\pi}{2}$ which in turn, means $\alpha(u_0) + \varepsilon v_0 = 0$.

The simplest case $\alpha \equiv 0$ gives the differential equation (2.2) as being:

$$\frac{d^2v}{du^2} = \varepsilon\left(\frac{dv}{du}\right)^3 - \varepsilon\cos w\left(\frac{dv}{du}\right)^2.$$

When $\varepsilon = -1$ this differential equation is:

$$\frac{d^2v}{du^2} = -\left(\frac{dv}{du}\right)^3 - \tanh(v)\left(\frac{dv}{du}\right)^2$$

and the WolframAlpha site provides an implicit solution:

$$C + u = C\sinh v - \ln(\cosh v) + 2\sinh v\arctan\left(\tanh\frac{v}{2}\right), \quad C \in \mathbb{R}.$$

The associated cubic curve in this last case $(\alpha = 0, \varepsilon = -1)$ is:

$$\mathcal{C}(P(u_0, v_0)) : y^2 = X^3 + \tanh(v_0)X^2 \tag{3.5}$$

which implies that along the geodesic curve $v = \textit{constant} = 0$ containing all Weierstrass points we have the semicubical parabola $y^2 = X^3$; it follows that all Weierstrass points are singular. In a non-Weierstrass point the curve (3.5) is a Tschirnhausen cubic.

4. Conics geodesically associated to the points of M

Suppose now that $\Gamma^1_{22}(u_0, v_0) = 0$. For example, if the given local coordinates are *orthogonal* i.e. $g_{12} = 0$ then the vanishing of all Γ^1_{22} from the relation (2.3) implies $g_{22} = g_{22}(v)$ and therefore the metric reduces to the form $g = g_{11}(u, V)du^2 + dV^2$.

In this case the equation (2.2) associates to the given point $P(u_0, v_0) \in M$ the *conic*:

$$\mathcal{C}_2(P) : y^2 = (2\Gamma^1_{12}(u_0,v_0)-\Gamma^2_{22}(u_0,v_0))x^2+(\Gamma^1_{11}(u_0,v_0)-2\Gamma^2_{12}(u_0,v_0))x-\Gamma^2_{11}(u_0,v_0)$$

and we restrict the study of it to the last two examples of Section 3.

Example 3.2 revisited. The vanishing of Γ^1_{22} in all points means that $\varphi = \varphi(v)$ and hence the metric is a warped one $g = e^{2\varphi(v)}du^2 + dV^2$ ($V = \int e^{\varphi(v)}dv$) with the only non-zero Christoffel symbols:

$$\Gamma^1_{12} = \Gamma^2_{22} = -\Gamma^2_{11} = \varphi'(v).$$

The associated conic depends only on v_0:

$$\mathcal{C}_2(u_0, v_0) : y^2 = \varphi'(v_0)(x^2 + 1).$$

Choosing $\varphi(v) = v$ it results the unit equilateral hyperbola $\mathcal{C}_2 : x^2 - y^2 + 1 = 0$ on all points of $M = \mathbb{R}^2$ while the differential equation (2.2) is:

$$\frac{d^2v}{du^2} = \left(\frac{dv}{du}\right)^2 + 1$$

with the general solution $v(u) = C_2 - \ln(\cos(u + C_1))$ with arbitrary constants $C_1 \geq 0, C_2 \in \mathbb{R}$ for the metric $g = e^{2v}(du^2 + dv^2)$. We point out that 2-dimensional Finsler metrics having hyperbolas as geodesics are discussed in [10].

Example 3.3 revisited. The vanishing of Γ^1_{22} in all points means that $\omega = \omega(u)$ and the only non-zero Christoffel symbols are:

$$\Gamma^1_{11} = -(\cot\omega(u))\omega'(u), \quad \Gamma^2_{11} = -(\sin\omega(u))\omega'(u).$$

The associated conic depends only on u_0 and is of parabolic type:

$$\mathcal{C}_2(u_0, v_0) : y^2 = \omega'(u_0)(-\cot\omega(u_0)x + \sin\omega(u_0)).$$

We remark that a relativistic metric whose geodesics are parabolas is recently discussed in [3] while a left-invariant sub-Riemannian metric on the multidimensional Heisenberg group having parabolas as geodesics is studied in [13]. The case of 2-dimensional Finsler geometry with geodesics being parabolas is treated in [11].

The author declares no conflict of interest.
This research received no external funding.

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