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On rational decimal numbers for lower secondary mathematics teachers

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Abstract. For the purposes of mathematics teacher education, we provide an algebraic and number theoretic explanation of how one can determine, from a and b, whether the decimal expansion of a/b, where $\gcd(a,b)=1$, contains a pre-period, and if so, how long it is. To that end, we observe that explanation cannot be based solely on group theory because the residue classes of remainders do not form a subgroup of the multiplicative group modulo b. However, the collection of residue classes does form a submonoid of the multiplicative monoid modulo b.

Keywords: group, monoid, period, pre-period, rational number

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1. Introduction

One fine day, a lower secondary mathematics teacher might face a sudden question from a pupil curious to know whether one can determine from a and b if the decimal expansion of

$$x = \frac{a}{b},$$

where $\gcd(a,b)=1$ (i.e., the greatest common divisor of a and b equals 1), contains a pre-period, and if so, how long it is. Similarly, a pupil may ask whether one can foresee the length of period from a and b. Caught off guard, the teacher might buy

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some time to reflect on these questions by first explaining that x has a terminating decimal expansion if b has no prime factors other than 2 or 5. Simultaneously, the teacher might wonder: "*Did we cover anything related to this issue in our algebra courses?*"

Indeed, why does, for example, $\frac{1}{21}$ lack a pre-period, while $\frac{1}{22}$ has one? Since both 21 and 22 are composite numbers, the presence of a pre-period does not depend on whether b is a prime or not. However, examining the expansion of x where the denominator b is a prime other than 2 or 5 quickly reveals that such x lacks a pre-period.

Poranen and Haukkanen [7] explored various aspects of the decimal expansion of a rational number x through the lens of group theory. Their survey aimed to demonstrate how a solid understanding of group theory and fundamental number theory concepts can help secondary mathematics teachers gain deeper insights into secondary-level mathematical topics.

The present paper is a kind of sequel to [7]. It will turn out that addressing the questions posed in the first paragraph leads to studying monoids. Interestingly, this shift from one algebraic system to another has a parallel in mathematics educational research. Several studies (e.g., [13, 14]) have shown that a minor change in the formulation of a task can significantly alter a learner's perception of the mathematical knowledge required to solve the task. For instance, a pupil who can successfully solve

$$ax = b$$

whenever $a, b \in \mathbb{Z}_+$, may perceive the task as belonging to a fundamentally different algebraic system with new rules of calculation when a and b are negative or decimal numbers [14]. A similar phenomenon is observed even when examining how students perceive elementary mathematical concepts, such as that of equation [12]. One possible reason for these phenomena may arise from the nature of mathematics itself: seemingly similar tasks situated in different mathematical contexts can, in fact, constitute fundamentally different challenges and require entirely distinct approaches. A concrete example of this is the difference between solving the linear equation $2x = 1$ in the context of lower secondary school and solving it in an introductory real analysis course, where real numbers are treated as elements of the complete ordered field, see [11]. This observation underscores the importance of attending not only to the mathematical content but also to the context in which a problem is posed and discussed.

The aim of this paper is to establish an appropriate mathematical context for addressing the questions posed above and to provide a comprehensive answer to them. Our goal is to make the paper a self-contained learning resource suitable for use in mathematics teacher education. While the questions themselves are not new and their answers are familiar to many mathematicians, e.g., [5, Section 9.2] and [2], a concise yet complete exposition of the topic appears to be scarce in modern literature. This observation, along with findings from mathematics education research, e.g., [15], indicating that preservice teachers experience considerable difficulty in understanding the relationship between a rational number and its decimal

expansion serves as the motivation for writing this paper.

Nontheless, some recent papers have addressed aspects of the issue. For instance, Facchini and Simonetta [3] explored the connection between decimal representations of rational numbers, the structure of finite cyclic monoids, divisibility rules between integers, and divisors of the numbers of the form $99\ldots900\ldots0$. Similarly, Ross [9] considered the question, although his study primarily focused on the m-block property of fractions. Also Hall and Bastos [4] acknowledged the condition under which a pre-period exists as they explored how to convert decimal expansions of rational numbers into visual forms via string art.

In Section 2, we summarize the essential results from university-level algebra and number theory required in Section 3, where we present a detailed exposition of the basic theory underlying the decimal expansion of a rational number. The article concludes with Section 4 that discusses some examples.

2. Preliminaries on algebra and number theory

In this section, we give a concise summary of the algebraic and number theoretic concepts and results necessary for discussing the main theme of this article in detail. Our overview closely follows that given in [7]. For a more comprehensive review of these topics, see, for example, [1, 6, 8, 10].

2.1. Groups, monoids, and their cyclic substructures

Let G be a nonempty set equipped with a binary operation $\star\colon G\times G\to G$, i.e., for any two elements $a,b\in G$, their product $a\star b$ is a unique element of G.

Semigroups, monoids, and groups. The pair $(G,\star)$ is called a *semigroup* if the operation $\star$ is *associative*, that is,

$$(a\star b)\star c=a\star(b\star c)\quad\text{for all } a,b,c\in G.$$

Associativity ensures that the placement of parentheses does not affect the outcome of repeated operations.

A semigroup is called a *monoid* if it has a special element $e\in G$ such that

$$a\star e=e\star a=a\quad\text{for all } a\in G.$$

This element, called the *identity*, acts as a "neutral element" under $\star$ similarly as 1 does under multiplication of real numbers.

Finally, a monoid is a *group* if every element $a\in G$ has an *inverse* $a^{-1}\in G$ satisfying

$$a\star a^{-1}=a^{-1}\star a=e.$$

Thus, in a group, any element can be "undone". If, in addition,

$$a\star b=b\star a\quad\text{for all } a,b\in G,$$

then $(G, \star)$ is called an *Abelian* or *commutative* group. In school mathematics, we encounter several examples of Abelian groups, such as addition on the integers and multiplication on the positive real numbers.

Subgroups and Lagrange's theorem. A group can contain a subset that behaves self-contained similarly as the group itself. More precisely, a pair $(H, \star)$ is called a *subgroup* of $(G, \star)$ if H is a nonempty subset of G and itself forms a group under the same operation $\star$.

For a finite group G, the *finite subgroup criterion* provides a convenient test to identify its subgroups: if H is a finite nonempty subset of G, then $(H, \star)$ is a subgroup of $(G, \star)$ if and only if

$$\forall a, b \in H : a \star b \in H,$$

that is, if and only if H is closed under the operation $\star$.

A key result concerning subgroups is *Lagrange's theorem*, which states that if $(G, \star)$ is a finite group and $(H, \star)$ is its subgroup, then the number of elements in H divides the number of elements in G, i.e.,

$$|H| \mid |G|.$$

Cyclic subgroups and generators. Let $(G, \star)$ be a group and let $a \in G$. The set

$$\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$$

is called the *cyclic subgroup generated by* a. It is the smallest subgroup of $(G, \star)$ that contains a. The number of distinct elements in $\langle a \rangle$ is called the *order* of a in G, denoted by $\operatorname{ord}(a)$.

If there exists an element $a \in G$ such that $G = \langle a \rangle$, then $(G, \star)$ is a *cyclic group*, and a is called a *generator* of G. In a cyclic group, every element can be obtained by repeatedly applying the group operation to a single generator.

When G is finite, each of its cyclic subgroups can be written as

$$\langle a \rangle = \{e, a, a^2, \ldots, a^{\ell-1}\},$$

where $\ell = \operatorname{ord}(a)$ and e denotes the identity element. Such groups are among the simplest yet most fundamental structures in group theory.

Cyclic submonoids. Assume now that $(M, \star)$ is a finite monoid. The structure of cyclic submonoids is slightly harder to deal with than that of cyclic subgroups. Since we assumed that M is finite, the set $\langle a \rangle = \{a^k \mid k = 0, 1, 2, \ldots\}$ also has to be finite. Thus at least one element in the infinite sequence $1, a, a^2, \ldots$ has to be equal to infinitely many other elements. Let u be the least exponent for which a^u is equal to a subsequent element. Let a^{u+v} denote the first subsequent element that equals a^u. Now, the cyclic submonoid of M generated by the element a is given as

$$\langle a \rangle = \{1, a, a^2, \ldots, a^{u-1}, a^u, a^{u+1}, \ldots, a^{u+v-1}\}.$$

The sequence $1, a, a^2, \ldots$ is of the form

$$\underbrace{1, a, a^2, \ldots, a^{u-1}}_{\text{the pre-period}}, \underbrace{a^u, a^{u+1}, \ldots, a^{u+v-1}}_{\text{the period}}, \underbrace{a^u, a^{u+1}, \ldots, a^{u+v-1}}_{\text{the period}}, \ldots \tag{2.1}$$

with pre-period $1, a, a^2, \ldots, a^{u-1}$ of length u and period $a^u, a^{u+1}, \ldots, a^{u+v-1}$ of length v. If the monoid is a group, then $u = 0$, $v = \ell = \operatorname{ord}(a)$ and the sequence $1, a, a^2, \ldots$ is of the form

$$\underbrace{1, a, \ldots, a^{\ell-1}}_{\text{the period}}, \underbrace{1, a, \ldots, a^{\ell-1}}_{\text{the period}}, \ldots$$

without a pre-period.

2.2. Cosets

In this subsection, we show how a group can be divided into parts of equal size.

Definition. Let $(M, \star)$ be a group (resp. a monoid) and let $(H, \star)$ be its subgroup (resp. submonoid). We say that the *left coset* of $a \in M$ modulo H in M is the set

$$a \star H = \{a \star h \mid h \in H\}.$$

If e is the identity element in M, then $e \star H = H$. Also, $a \in a \star H$ for every $a \in M$.

Similarly, the *right coset* of $a \in M$ modulo H in M is the set

$$H \star a = \{h \star a \mid h \in H\}.$$

If M is commutative, then $a \star H = H \star a$ for all $a \in M$.

Partition. Let G be a group. The collection of all left cosets of H in G forms a partition of G: two left cosets are either equal or disjoint, and their union is G. Moreover,

$$a \star H = b \star H \iff a \in b \star H \iff b \in a \star H.$$

Also, $a \star H = H$ for all $a \in H$.

Each coset modulo H has the same cardinality. In particular, if H is finite, then the number of elements in each coset is the same. If G itself is finite, then the number of distinct left cosets of H in G is

$$\frac{|G|}{|H|}.$$

Analogous results hold for right cosets.

2.3. Congruences

Basics. Let m be a positive integer (≥ 2). Then $a \in \mathbb{Z}$ is said to be *congruent to* b *modulo* m if $a - b$ is divisible by m, that is, if $m \mid (a - b)$. This is denoted by

$$a \equiv b \pmod{m}.$$

Thus $a \equiv b \pmod{m}$ if and only if $a = b + mk$ for some $k \in \mathbb{Z}$.

The congruence relation $\equiv \pmod{m}$ is an equivalence relation on $\mathbb{Z}$. The equivalence classes are referred to as the *residue classes modulo* m. The residue class determined by a is denoted by $[a]$, and a is called a *representative* of $[a]$. The set of all residue classes modulo m is denoted by $\mathbb{Z}_m$, that is,

$$\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} = \{[0], [1], [2], \ldots, [m-1]\}.$$

According to the division algorithm, for each $a \in \mathbb{Z}$ there exist unique integers q and r such that $a = mq + r$ with $0 \leq r < m$. The number q is called the *quotient*, and the number r the *remainder*. It is clear that

$$[a] = [r]$$

and more generally,

$$[a] = [b] \iff a \equiv b \pmod{m}.$$

Thus the representative a of the class $[a]$ can be replaced by any integer congruent to a modulo m, for instance by the remainder r of a modulo m.

Algebraic structures. Addition on $\mathbb{Z}_m$ is defined by

$$[a] \oplus [b] = [a + b],$$

where $[a], [b] \in \mathbb{Z}_m$. This is referred to as the *addition modulo* m. Now, $(\mathbb{Z}_m, \oplus)$ is an Abelian group. In this paper, however, we are primarily concerned with multiplication modulo m, introduced below.

Multiplication on $\mathbb{Z}_m$ is defined by

$$[a] \odot [b] = [ab],$$

where $[a], [b] \in \mathbb{Z}_m$. The multiplication on the right-hand side is the usual integer multiplication, while the one on the left-hand side denotes multiplication *modulo* m. Thus $(\mathbb{Z}_m, \odot)$ is a commutative monoid. An element $[a] \in \mathbb{Z}_m$ possesses an inverse in $(\mathbb{Z}_m, \odot)$ if and only if $\gcd(a, m) = 1$.

We denote by

$$\mathbb{Z}_m^\times = \{[a] \in \mathbb{Z}_m \mid \gcd(a, m) = 1\}$$

the set of invertible elements in $\mathbb{Z}_m$. Then $(\mathbb{Z}_m^\times, \odot)$ is an Abelian group, called the *multiplicative group modulo* m.

Euler's totient function. *The Euler totient function* ϕ is defined by

$$\phi(m) = |\{a : 1 \leq a \leq m, \gcd(a, m) = 1\}|, \quad m \in \mathbb{Z}^+,$$

that is, $\phi(m)$ is the number of elements in $\mathbb{Z}_m^\times$, or equivalently, the number of invertible integers modulo m. An arithmetical expansion for the Euler totient function is given by

$$\phi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right),$$

where p runs through all primes dividing m. In particular, $\phi(p^k) = p^k - p^{k-1}$ for prime powers p^k with $k \geq 1$.

Let a and m (> 1) be *relatively prime*, that is, $\gcd(a, m) = 1$. Then, by Euler's theorem,

$$a^{\phi(m)} \equiv 1 \pmod{m};$$

hence there exists a least positive integer x such that $a^x \equiv 1 \pmod{m}$. This integer x is called the *order of a modulo m* and is denoted by $\mathrm{ord}_m(a)$. In group-theoretic terms, $\mathrm{ord}_m(a)$ is the order of the element $[a]$ in the multiplicative group $\mathbb{Z}_m^\times$.

3. Theory of decimal expansions

In this section, we complete the discussion initiated in [7]. Among other extentions, we include an examination of the case $\gcd(b, 10) = 1$, which constitutes the main focus of this paper. We draw also on Rosen's book [8], although his treatment of number theory is developed on a general base. Since our interest lies on decimal numbers, we adopt base 10 throughout.

For the sake of clarity, in the rest of the paper, we focus on rational numbers $x \in (0, 1)$. The *decimal expansion* of x is

$$x = \sum_{n=1}^{\infty} q_n 10^{-n} = 0.q_1 q_2 \ldots,$$

where q_n's are integers in $[0, 9]$ so that for each positive integer N there exists an integer $n > N$ such that $q_n \neq 9$. The last condition assures that, e.g., the expansion of $1/2$ is 0.5, since the expansion $0.4999\ldots$ is not appropriate.

A decimal expansion is said to be *periodic* if there exist integers $N \geq 0$ and $\lambda > 0$ such that $q_{n+\lambda} = q_n$ for all $n > N$. We then write

$$x = 0.q_1 q_2 \ldots q_N \overline{q_{N+1} q_{N+2} \cdots q_{N+\lambda}}, \tag{3.1}$$

and we say that $q_1 q_2 \ldots q_N$ is the *pre-period* of *length* N and $q_{N+1} q_{N+2} \ldots q_{N+\lambda}$ is the *period* of length λ. Note that if a decimal expansion has a period of λ symbols, it also has a period of $t\lambda$ symbols for each $t \in \mathbb{Z}^+$. In this article, we choose N and λ such that the period of an expansion always is the shortest possible and N is the smallest possible.

Before addressing the main questions of this paper, we record a theorem that specifies the condition under which x *terminates*, i.e., there exists a positive integer N such that $q_n = 0$ for all $n > N$. The theorem is derived from the following observation: $b = 2^i 5^j$ with $s = \max\{i, j\} > 0$, if and only if x can be written as

$$x = \frac{ad}{10^s}, \tag{3.2}$$

where d is an integer. Clearly, x in (3.2) terminates. For all details, see [8, Theorem 12.3].

Theorem 3.1. *The decimal expansion of $x \in (0, 1)$ terminates if and only if x can be written as*

$$x = \frac{a}{b}, \quad \gcd(a, b) = 1,$$

where b belongs to the submonoid of $(\mathbb{Z}, \cdot)$ generated by 2 and 5.

We first consider how to determine the length of period from a and b. To that end, let us consider the long division of a/b, where $\gcd(a, b) = 1$. The sequence of quotients is $q_0, q_1, q_2, \ldots$ with $q_0 = 0$ and it forms the decimal expansion (3.1) of x. Let $r_0, r_1, r_2, \ldots$ denote the sequence of remainders. We examine properties of the sequence of remainders.

The remainders r_n satisfy the recurrence relation

$$\begin{aligned} 10r_n &= bq_{n+1} + r_{n+1}, \quad n = 0, 1, \ldots, \\ r_0 &= a. \end{aligned}$$

Thus,

$$10r_n \equiv r_{n+1} \pmod{b}, \quad n = 0, 1, \ldots,$$

giving

$$r_n \equiv 10^n a \pmod{b}, \quad n = 0, 1, \ldots . \tag{3.3}$$

If $\gcd(b, 10) = 1$, the remainders r_n, $n = 0, 1, \ldots$, are congruent to

$$a, 10a, 10^2 a, \ldots, 10^{\ell-1} a, a, 10a \ldots,$$

modulo b, where $\ell = \mathrm{ord}_b(10)$, i.e., ℓ is the least integer $t > 0$ such that

$$10^t \equiv 1 \pmod{b}.$$

The existence of such t follows from Euler's theorem, see Section 2.3. Observe also that, by the definition of ℓ and the property $\gcd(a, b) = 1$,

$$10^n a \not\equiv 10^m a \pmod{b} \quad \text{and} \quad 10^n \not\equiv 10^m \pmod{b}$$

for $0 \le n \ne m \le \ell - 1$.

Let us write

$$b = 2^i 5^j c, \text{ where } i, j \ge 0 \text{ and } \gcd(c, 10) = 1. \tag{3.4}$$

We may assume $c > 1$, since $c = 1$ leads to a terminating expansion.

Case 1. Assume that $\gcd(b,10)=1$, i.e., $i=j=0$. Then $b=c$. If $a=1$, then the remainders r_n, $n=0,1,\ldots$, are congruent to

$$1,10,10^2,\ldots,10^{\ell-1},1,10,\ldots \tag{3.5}$$

modulo b. Then a period of (3.5) represents the cyclic subgroup $\langle[10]\rangle$ of the group $(\mathbb{Z}_b^\times,\odot)$ generated by $[10]$, i.e., the group

$$L=\langle[10]\rangle=\{[1],[10],[10^2],\ldots,[10^{\ell-1}]\}.$$

Let $1\le a<b$. Then the remainders r_n, $n=0,1,\ldots$, are congruent to

$$a,10a,10^2a,\ldots,10^{\ell-1}a,a,10a,\ldots$$

modulo b. A period of this sequence represents the coset of a modulo L in $(\mathbb{Z}_b^\times,\odot)$ given as

$$[a]\odot L=\{[a],[10a],[10^2a],\ldots,[10^{\ell-1}a]\},$$

where $(a,b)=1$.

So, we notice that the length of the period of the sequence of remainders in the long division a/b is $\ell=\operatorname{ord}_b(10)$.

Case 2. Assume now that $(b,10)\neq 1$, i.e., $s=\max\{i,j\}>0$. Let $a=1$ and thus consider the long division of $1/b$. Write again

$$L=\langle[10]\rangle=\{[10^k]:k=0,1,2,\ldots\}.$$

In this case, it is crucial to notice that L is not a subgroup of the group $(\mathbb{Z}_b^\times,\odot)$. The set L is not even a subset of the set $\mathbb{Z}_b^\times$. However, L is a submonoid of the monoid $(\mathbb{Z}_b,\odot)$. From (2.1) and (3.3) we see that the remainders r_n, $n=0,1,\ldots$, are now congruent to

$$1,10,10^2,\ldots,10^{u-1},10^u,10^{u+1},\ldots,10^{u+v-1},10^u,10^{u+1},\ldots$$

modulo b. Observe that v is the smallest positive integer such that $10^{u+v}\equiv 10^u \pmod b$. Since $\gcd(c,10)=1$, this is equivalent to $10^v\equiv 1 \pmod c$. This implies $v=\operatorname{ord}_c 10$ as we assumed that $c>1$.

Now we can discuss the general case. In the long division of

$$\frac{a}{b},\quad \gcd(a,b)=1,$$

the remainders r_n, $n=0,1\ldots$, are congruent to

$$a,10a,10^2a,\ldots,10^{u-1}a,10^ua,10^{u+1}a,\ldots,10^{u+v-1}a,10^ua,10^{u+1}a,\ldots \tag{3.6}$$

modulo b. The pre-period and the period together again represents the coset of a modulo $L=\langle[10]\rangle$ in the monoid $\mathbb{Z}_b$.

We want to show that the length of period and that of pre-period in (3.6) determine also those in (3.1), i.e., that $v = \lambda$ and $u = N$. We first show that $v = \lambda$. Now

$$\begin{aligned}
a/b &= 0.q_1q_2\ldots q_N\overline{q_{N+1}q_{N+2}\cdots q_{N+\lambda}} \\
&= \left(\frac{q_1}{10} + \frac{q_2}{10^2} + \cdots + \frac{q_N}{10^N}\right) + \left(\frac{q_{N+1}}{10^{N+1}} + \frac{q_{N+2}}{10^{N+2}} + \cdots + \frac{q_{N+\lambda}}{10^{N+\lambda}}\right) \\
&\quad + \left(\frac{q_{N+1}}{10^{N+\lambda+1}} + \frac{q_{N+2}}{10^{N+\lambda+2}} + \cdots + \frac{q_{N+\lambda}}{10^{N+2\lambda}}\right) + \cdots \\
&= \left(\frac{q_1}{10} + \frac{q_2}{10^2} + \cdots + \frac{q_N}{10^N}\right) \\
&\quad + \frac{1}{10^N}\left(\frac{q_{N+1}}{10} + \frac{q_{N+2}}{10^2} + \cdots + \frac{q_{N+\lambda}}{10^\lambda}\right)\left(1 + \frac{1}{10^\lambda} + \frac{1}{10^{2\lambda}} + \cdots\right) \\
&= \frac{1}{10^N}\left(q_1 10^{N-1} + q_2 10^{N-2} + \cdots + q_N\right) \\
&\quad + \frac{1}{10^N}\left(\frac{q_{N+1}}{10} + \frac{q_{N+2}}{10^2} + \cdots + \frac{q_{N+\lambda}}{10^\lambda}\right)\left(\frac{10^\lambda}{10^\lambda - 1}\right) \\
&= \frac{(10^\lambda - 1)(q_1 10^{N-1} + q_2 10^{N-2} + \cdots + q_N)}{10^N(10^\lambda - 1)} \\
&\quad + \frac{q_{N+1}10^{\lambda-1} + q_{N+2}10^{\lambda-2} + \cdots + q_{N+\lambda}}{10^N(10^\lambda - 1)}.
\end{aligned}$$

This implies that b divides $10^N(10^\lambda - 1)$ or, more precisely, by (3.4), 2^i5^j divides 10^N and c divides $(10^\lambda - 1)$. The latter fact is equivalent to $10^\lambda \equiv 1 \pmod{c}$. This shows that $\lambda \geq \operatorname{ord}_c(10) = v$. Since $\lambda \leq v$ trivially, we have $\lambda = v$.

We next show that $N = u$. In long division, the remainder r_n uniquely determines the quotient q_{n+1}. However, two different remainders can lead to equal quotients. Therefore, it follows that $N + \lambda \leq u + v$. Since $\lambda = v$, we have $N \leq u$. On the other hand,

$$r_{N+\lambda} - r_N \equiv 10^{N+\lambda}a - 10^N a \equiv 10^N(10^\lambda - 1)a \equiv 0 \pmod{b}$$

because 2^i5^j divides 10^N and c divides $(10^\lambda - 1)$. This implies that $u \leq N$. Consequently, $N = u$.

The above reasoning contains also an implicit answer to the questions posed in the introduction. Let us make the answer explicit by stating two theorems. The first theorem follows directly from Cases 1 and 2, while the second requires an additional comment.

Theorem 3.2. *The length of the period in* (3.1) *is given by*

$$\lambda = \operatorname{ord}_c(10),$$

where c is as given in (3.4). *In particular, if* $\gcd(b, 10) = 1$, *the length of the period is* $\lambda = \ell = \operatorname{ord}_b(10)$.

Theorem 3.3. *The length of the pre-period of*

$$x = \frac{a}{b}$$

is $s = \max\{i, j\}$.

Proof. Let us write

$$10^s x = 2^s 5^s \frac{a}{2^i 5^j c} = \frac{2^{s-i} 5^{s-j} a}{c}.$$

Here s is the smallest integer such that the denominator of $10^s x$ is relatively prime to 10.

Let us consider the expansion

$$x = 0.q_1 q_2 \dots q_N \overline{q_{N+1} q_{N+2} \dots q_{N+\lambda}} = \frac{q_1 q_2 \dots q_N + 0.\overline{q_{N+1} q_{N+2} \dots q_{N+\lambda}}}{10^N}.$$

Here

$$0.\overline{q_{N+1} q_{N+2} \dots q_{N+\lambda}}$$

is the decimal expansion of a rational number, whose denominator is relatively prime to 10. Hence we can conclude that N is the smallest integer such that the denominator of $10^N x$ is relatively prime to 10. Thus s and N must be equal. □

4. Examples

Above we proved that the length of the pre-period of

$$x = \frac{a}{b} = \frac{a}{2^i 5^j c},$$

where $\gcd(a, b) = \gcd(10, c) = 1$, is $s = \max\{i, j\}$, and, for $c > 1$, the length of the period of x is given by $\lambda = \mathrm{ord}_c(10)$. If $c = 1$, the decimal expansion terminates. Let us consider some concrete examples.

Example 4.1 ($c = 1$)**.** For example, $\frac{13}{40}$ has got a pre-period of length $s = 3$ and the expansion terminates, because $40 = 2^3 5$. Indeed,

$$\frac{13}{40} = \frac{13}{2^3 5} = 0.325.$$

Example 4.2 ($c > 1$)**.** Then, for example, for $\frac{13}{56}$, we have $s = 3$ and $\lambda = 6$, because $56 = 2^3 7$ and $\lambda = 6$ is the smallest positive solution of

$$10^\lambda \equiv 1 \pmod 7.$$

Explicitly,

$$\frac{13}{56} = \frac{13}{2^3 7} = 0.232142857142857\ldots = 0.232\overline{142857}.$$

Example 4.3. Consider the expansion

$$\frac{1}{26} = \frac{1}{2 \cdot 13}.$$

Now, $s = 1$ and $\lambda = \mathrm{ord}_{13}(10) = 6$. All fractions of the form $a/26$, $\gcd(a, 26) = 1$, are

$$\begin{aligned}
\frac{1}{26} &= 0.0\overline{384615} & \frac{3}{26} &= 0.1\overline{153846} \\
\frac{5}{26} &= 0.1\overline{923076} & \frac{7}{26} &= 0.2\overline{692307} \\
\frac{9}{26} &= 0.3\overline{461538} & \frac{11}{26} &= 0.4\overline{230769} \\
\frac{15}{26} &= 0.5\overline{769230} & \frac{17}{26} &= 0.6\overline{538461} \\
\frac{19}{26} &= 0.7\overline{307692} & \frac{21}{26} &= 0.8\overline{076923} \\
\frac{23}{26} &= 0.8\overline{846153} & \frac{25}{26} &= 0.9\overline{615384}
\end{aligned}$$

If we look at the decimals in the above periods, we observe that the numbers

$$3, 8, 4, 6, 1, \text{and } 5 \tag{4.1}$$

occur cyclically in the periods of

$$\frac{1}{26}, \frac{3}{26}, \frac{9}{26}, \frac{17}{26}, \frac{23}{26}, \text{ and } \frac{25}{26}. \tag{4.2}$$

Similarly, the numbers 9, 2, 3, 0, 7, and 6 occur cyclically in the periods of

$$\frac{5}{26}, \frac{7}{26}, \frac{11}{26}, \frac{15}{26}, \frac{19}{26}, \text{ and } \frac{21}{26}. \tag{4.3}$$

Is this merely a coincidence? No. If we depict the decimals in the period of $\frac{1}{26}$ as shown in Figure 1, the decimals in the period of $\frac{a}{26}$, $a \in \{1, 3, 9, 17, 23, 25\}$, can be obtained by rotating the hexagon by an appropriate number of steps. The correct number of steps is determined as follows.

For $1/26$, we see from above that $q_2 = 3$, $q_3 = 8$, $q_4 = 4$, $q_5 = 6$, $q_6 = 1$, and $q_7 = 5$. Let us compute the remainders $r_1, \ldots, r_6$ from

$$r_n \equiv 10^n \pmod{26}.$$

We get $r_1 = 10$, $r_2 = 22$, $r_3 = 12$, $r_4 = 16$, $r_5 = 4$, and, $r_6 = 14$ modulo 26.

Since the nth remainder determines the $(n+1)$th quotient, we find q_2^*, the first decimal of the period of $a/26$, $a \in \{1, 3, 9, 17, 23, 25\}$, by computing the remainder r_1^* of $a/26$ modulo 26 and determining which of the remainders $r_1 - r_6$ it is congruent to modulo 26.

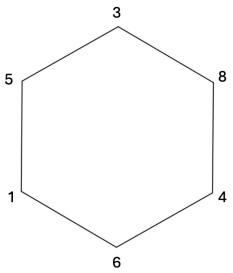

Figure 1. The decimal hexagon of 1/26.

For all a,

$$r_1^* \equiv 10a \pmod{26}.$$

So, for instance, if $a = 17$, then $r_1^* \equiv 170 \equiv 14 \equiv r_6 \pmod{26}$ implying that q_2^* of $\frac{17}{26}$ is the same as the seventh decimal q_7 in the decimal expansion of $\frac{1}{26}$.

But why do the numbers (4.1) occur in the periods of fractions (4.2) always in the same order as in the hexagon in Figure 1? Now, $r_1 - r_6$ are congruent (mod 13) to the numbers that occur in

$$S = \{1, 4, 3, 12, 9, 10\} = \{[4^k] : \ k = 0, 1, 2, \ldots\},$$

which forms a cyclic subgroup of the multiplicative group $(\mathbb{Z}_{13}^{\times}, \odot)$. So, the remainders $r_1 - r_6$ follow a cycle determined by a cyclic subgroup of the cyclic group determined by $c = 13$ in (3.4).

In a similar manner, we can observe that the cyclic occurrence of numbers 9, 2, 3, 0, 7, and 6 in the expansions of (4.3) is related to the coset of $5 \in \mathbb{Z}_{13}^{\times}$ modulo S, i.e.,

$$T = 5 \odot S = \{5, 7, 2, 8, 6, 11\}.$$

Lastly, we notice that $\mathbb{Z}_{13}^{\times}$ can be partitioned as $\mathbb{Z}_{13}^{\times} = S \cup T$. For any $a/26$, where $\gcd(a, 26) = 1$ and $a < 26$, the remainder r_1^* is congruent modulo 13 to a number belonging to S or T. Thereafter, each remainder r_n^*, $n > 1$, is congruent modulo 13 to a number belonging to the same set. This is the reason why only two different sets of numbers can occur in the expansions of (4.2) and (4.3). Namely, $r_1^* \equiv 0$ (mod 13) is not possible, otherwise a/b would terminate.

Example 4.4. Consider the following expansions.

$$\frac{1}{24} = 0.041\overline{6} \qquad \frac{5}{24} = 0.208\overline{3}$$
$$\frac{7}{24} = 0.291\overline{6} \qquad \frac{11}{24} = 0.458\overline{3}$$
$$\frac{13}{24} = 0.541\overline{6} \qquad \frac{17}{24} = 0.708\overline{3}$$
$$\frac{19}{24} = 0.791\overline{6} \qquad \frac{23}{24} = 0.958\overline{3}$$

They illustrate the impossibility of predicting the pre-period of $a/24$ from that of $1/24$. More concretely, the pre-period of $a/24$ cannot be computed solely from 0.041, as a multiple of $0.000\overline{6}$ contributes to the pre-period, too.

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