Annales Mathematicae et Informaticae
62 (2025) pp. 143–155
DOI: 10.33039/ami.2025.10.014
URL: https://ami.uni-eszterhazy.hu

Unit group of semisimple group algebra of groups of order 36

Diksha Upadhyay, Harish Chandra

Department of Mathematics and Scientific Computing,
M. M. M. University of Technology, Gorakhpur
dikshaupadhyay111@gmail.com
hcmsc@mmmut.ac.in

Abstract. In this paper, we classify the structure of the unit group of semisimple group algebras over groups of order 36. There are 14 nonisomorphic groups of this order, 10 of which are nonabelian. The structures of the unit groups of the group algebras corresponding to all abelian groups of order 36, as well as the groups $C_3 \times A_4$ and D_{36}, have already been studied. This work focuses on the remaining 8 non-abelian groups of order 36, providing a detailed examination of the unit group structures of their corresponding semisimple group algebras.

Keywords: group algebras, unit group, conjugacy classes, Wedderburn decomposition

2020 *Mathematics Subject Classification:* 16U60, 20C05

1. Introduction

Let FG represent the group algebra of a finite group G over the finite field F of order $q = p^k$. It is crucial to keep in mind that the collection of all invertible elements of FG is the unit group $U(FG)$. Finding the structure of a unit group of group rings has always been an interesting and difficult problem due to the fact that group algebraic units can be applied to both cryptography and coding theory. Additionally, units are very helpful for exploring the Lie properties of group algebras and solving isomorphism problems. Unit groups of various group algebras have been examined and described in the past few years [6, 7, 9–11, 13, 16, 18, 19, 21, 22]. Unit group of finite group algebras of abelian groups of order at most 16 are discussed in [17]. In [5], the unit group structure of non-abelian groups of order 16 is discussed. The unit group structure of group algebras of groups

Submitted: December 14, 2024
Accepted: October 3, 2025
Published online: October 21, 2025

of order 18, 20, and 24 are characterized by Sahai and Ansari [1, 14, 17]. Some recent papers related to the characterization of groups of order 26 to 34 are listed in [10, 16]. We have total 10 non-isomorphic non-abelian groups of order 36 namely D_{36}, Dic_9, $C_3^2 \rtimes C_4$, $C_3 \rtimes Dic_3$, $C_3 \cdot A_4$, S_3^2, $S_3 \times C_6$, $C_3 \times A_4$, $C_3 \times Dic_3$ and $C_2 \times C_3 \rtimes S_3$. The structure of unit groups of group algebra of abelian groups of order 36, $U(F(C_3 \times A_4))$ and $U(FD_{36})$ has already been studied in [2, 20]. In this paper, we characterize the structure of unit groups of the remaining non-abelian groups of order 36.

2. Preliminaries

Let F be any arbitrary finite field, e represent the exponent of G, and let ζ be a primitive e^{th} root of unity. Then T be the multiplicative group consisting of those elements t, taken modulo e, for which $\zeta \mapsto \zeta^t$ defines an automorphism of $F(\zeta)$ over F, i. e., $T = \{t : \zeta \mapsto \zeta^t$ is an automorphism of $F(\zeta)$ over $F\}$. For any p-regular element $g \in G$, we can denote γ_g as the sum of all of its conjugates, and the cyclotomic F-classes of γ_g are denoted by $S(\gamma_g) = \{\gamma_{g^t} : t \in T\}$.

Theorem 2.1 ([4]). *The number of simple components of $FG/J(FG)$ and the number of cyclotomic F-classes in G is equal.*

Theorem 2.2 ([4]). *Let j be the number of cyclotomic F-classes in G. If K_i, $1 \leq i \leq j$, are the simple components of the center of $FG/J(FG)$ and S_i, $1 \leq i \leq j$, are the cyclotomic F-classes in G, then $|S_i| = [K_i : F]$ for each i , after a suitable ordering of the indices.*

Theorem 2.3 ([8]). *Let F be a finite field with prime power order q. If e is such that $gcd(e, q) = 1$, ζ is the primitive e^{th} root of unity and z is the order of q modulo e, then we have $T = \{1, q, q^2, \dots, q^{z-1}\} \mod e$.*

Theorem 2.4 ([12]). *If RG is a semisimple group algebra, then*

$$RG = R(G/G') \oplus \Delta(G, G'),$$

where G' is the commutator subgroup of G, $R(G/G')$ is the sum of all commutative simple components of RG, and $\Delta(G, G')$ is the sum of all others.

Theorem 2.5 ([3]). *If $R = \bigoplus_{t=1}^{j} M_{n_t}(F_{q_t})$ is a summand of a semisimple group ring $F_qG(q = p^k)$, then p does not divide any of the n_t.*

3. Main results

The unit group structure of the group algebras of all non-isomorphic non-abelian groups of order 36 over the finite field F of positive characteristics, $p > 0$, where $p \nmid |G|$, is discussed in this section.

Theorem 3.1. *Let $G \cong C_9 \rtimes C_4$, then for a field F of characteristic $p > 3$,*

$$U(F_q(C_9 \rtimes C_4)) \cong \begin{cases} C^4_{p^k-1} \oplus GL_2(F_q)^8 & \text{for } q \equiv 1, 17 \mod 36 \\ C^4_{p^k-1} \oplus GL_2(F_q)^2 \oplus GL_2(F_{q^3})^2 & \text{for } q \equiv 29, 13, 5, 25 \mod 36 \\ C^2_{p^k-1} \oplus C_{p^{2k}-1} \oplus GL_2(F_q)^8 & \text{for } q \equiv 19, 35 \mod 36 \\ C^2_{p^k-1} \oplus C_{p^{2k}-1} \oplus GL_2(F_q)^2 \oplus GL_2(F_{q^3})^2 & \text{for } q \equiv 11, 31, 23, 7 \mod 36. \end{cases}$$

Proof. Let $G \cong C_9 \rtimes C_4 = \langle x, y, z, w \mid x^2y^{-1}, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxw^{-1}z^{-1}, w^{-1}x^{-1}wxw^{-1}, y^2, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^2w^{-2}, w^{-1}z^{-1}wz, w^3\rangle$. The Conjugacy classes description of $C_9 \rtimes C_4$ are given in Table 1.

Table 1

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw^2, xw, xz^2, xz^2w^2, xz^2w, xzw^2, xzw, xz\}$	4
y	$\{y\}$	2
z	$\{z, z^2w\}$	9
w	$\{w, w^2\}$	3
xy	$\{xy, xyw^2, xyw, xyz^2, xyz^2w^2, xyz^2w, xyzw^2, xyzw, xyz\}$	4
yz	$\{yz, yz^2w\}$	18
yw	$\{yw, yw^2\}$	6
z^2	$\{z^2, zw\}$	9
yz^2	$\{yz^2, yzw\}$	18
zw^2	$\{zw^2, z^2w^2\}$	9
yzw^2	$\{yzw^2, yz^2w^2\}$	18

Here, the exponent of $C_9 \rtimes C_4$ is 36, $(C_9 \rtimes C_4)' = C_9$ and $(C_9 \rtimes C_4)/(C_9 \rtimes C_4)' = C_4$. Now we discuss the proof in the following four cases.

Case 1. If $p^k \equiv 1, 17 \mod 36$. In this case $T = \{1\}$ and $|S(\gamma_g)| = 1$ for all $g \in (C_9 \rtimes C_4)$. As F is a field of characteristic $p > 3$, therefore $F_q(C_9 \rtimes C_4)$ is semisimple. So Wedderburn decomposition is provided by $F_q(C_9 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t)$, where F_t is a finite extension of F_q. Theorems 2.1 and 2.2 imply that $F_q(C_9 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t)$. As $F(C_9 \rtimes C_4)/(C_9 \rtimes C_4)' \cong FC_4$, and $FC_4 \cong F^4$, for $p^k \cong 1 \mod 4$ [15]. Thus, using Theorem 2.4, we get $F_q(C_9 \rtimes C_4) \cong F_q^4 \bigoplus_{t=1}^{8} M_{n_t}(F_t)$. Now, by the dimension formula, we have $32 = \sum_{t=1}^{8} n_t^2, n_t \geq 2, \forall\ t$. Here (2, 2, 2, 2, 2, 2, 2, 2) is the only possibility of n_t's. So we have,

$$F_q(C_9 \rtimes C_4) \cong F_q^4 \oplus M_2(F_q)^8.$$

Case 2. If $p^k \equiv 29, 13, 5, 25 \mod 36$, then $T = \{1, 5, 25, 17, 13, 29\}$ and $|S(\gamma_z)| = \{\gamma_z, \gamma_{z^2}, \gamma_{zw^2}\}$, $|S(\gamma_{yz})| = \{\gamma_{yz}, \gamma_{yz^2}, \gamma_{yzw^2}\}$ and $|S(\gamma_g)| = \{\gamma_g\}$ for the remaining representatives of the conjugacy classes of $C_9 \rtimes C_4$. Now using Theorems 2.1 and 2.2, we have $F_q(C_9 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t) \bigoplus_{t=1}^{5} M_{n_t}(F_{t^3})$. Since $F(C_9 \rtimes C_4)/(C_9 \rtimes C_4)' \cong FC_4$ and $FC_4 \cong F^4$ for $p^k \cong 1 \mod 4$, therefore by Theorem 2.4, we have $F_q(C_9 \rtimes C_4) \cong F_q^4 \bigoplus_{t=1}^{2} M_{n_t}(F_t) \bigoplus_{t=3}^{4} M_{n_t}(F_{t^3})$. This implies $32 = \sum_{t=1}^{8} n_t^2, n_t \geq 2, \forall\, t$. Now (2, 2, 2, 2, 2, 2, 2, 2) is the only choice for n_t's. Thus,

$$F_q(C_9 \rtimes C_4) \cong F_q^4 \oplus M_2(F_q)^2 \oplus M_2(F_{q^3})^2.$$

Case 3. If $p^k \equiv 19, 35 \mod 36$, then $T = \{1, 19\}$, $T = \{1, 35\}$, $|S(\gamma_x)| = \{\gamma_x, \gamma_{xy}\}$ and $|S(\gamma_g)| = \{\gamma_g\}$ for the remaining representatives of the conjugacy classes of $C_9 \rtimes C_4$. Now using Theorems 2.1 and 2.2 we have $F_q(C_9 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t) \oplus M_{n_t}(F_{t^2})$. Since $F(C_9 \rtimes C_4)/(C_9 \rtimes C_4)' \cong FC_4$, and $FC_4 \cong F^2 \oplus F_2$, for $p^k \cong -1 \mod 4$, using Theorem 2.4 we have, $F_q(C_9 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \bigoplus_{t=1}^{8} M_{n_t}(F_t)$. Now by dimension formula, $32 = \sum_{t=1}^{8} n_t^2, n_t \geq 2, \forall\, t$, which further implies that the possible choice of n_t's is (2, 2, 2, 2, 2, 2, 2, 2). Therefore, we have

$$F_q(C_9 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \oplus M_2(F_q)^8.$$

Case 4. If $p^k \equiv 7, 11, 23, 31 \mod 36$, then $T = \{1, 7, 13, 19, 25, 31\}$, and $|S(\gamma_x)| = \{\gamma_x, \gamma_{xy}\}$ $|S(\gamma_z)| = \{\gamma_z, \gamma_{z^2}, \gamma_{zw^2}\}$, $|S(\gamma_{yz})| = \{\gamma_{yz}, \gamma_{yz^2}, \gamma_{yzw^2}\}$ and $|S(\gamma_g)| = \{\gamma_g\}$ for the remaining representatives of the conjugacy classes of $C_9 \rtimes C_4$. Now using Theorems 2.1 and 2.2 we have $F_q(C_9 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{2} M_{n_t}(F_t) \oplus M_{n_t}(F_{t^2}) \bigoplus_{t=3}^{4} M_{n_t}(F_{t^3})$. Since $F(C_9 \rtimes C_4)/(C_9 \rtimes C_4)' \cong FC_4$ and $FC_4 \cong F^2 \oplus F_2$, for $p^k \cong -1 \mod 4$, using Theorem 2.4 we have, $F_q(C_9 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \bigoplus_{t=1}^{2} M_{n_t}(F_t) \bigoplus_{t=3}^{4} M_{n_t}(F_{t^3})$. Now by dimension formula, $32 = n_1^2 + n_2^2 + 3\sum_{t=2}^{3} n_t^2, n_t \geq 2, \forall\, t$, which further implies that the possible choice of n_t's is (2, 2, 2, 2, 2, 2, 2, 2). Therefore, we have

$$F_q(C_9 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \oplus M_2(F_q)^2 \oplus M_2(F_{q^3})^2. \qquad \square$$

Theorem 3.2. *Let $G \cong C_3^2 \rtimes C_4$, then for a field F of characteristic $p > 3$,*

$$U(F_q(C_3^2 \rtimes C_4)) \cong \begin{cases} C_{p^k-1}^4 \oplus GL_4(F_q)^2 & \text{for } q \equiv 1, 5 \mod 12 \\ C_{p^k-1}^2 \oplus C_{p^{2k}-1} \oplus GL_4(F_q)^2 & \text{for } q \equiv 7, 11 \mod 12. \end{cases}$$

Proof. Let $G \cong C_3^2 \rtimes C_4 = \langle x, y, z, w \mid x^2y^{-1}, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxw^{-2}, w^{-1}x^{-1}wxw^{-1}z^{-2}, y^2, z^{-1}y^{-1}zyz^{-1}, w^{-1}y^{-1}wyw^{-1}, z^3, w^{-1}z^{-1}wz, w^3\rangle$. Conjugacy classes description of $C_9 \rtimes C_4$ are given in Table 2. It is clear from Table 2 that the exponent of $C_3^2 \rtimes C_4$ is 12, $(C_3^2 \rtimes C_4)' = C_3^2$ and $(C_3^2 \rtimes C_4)/(C_3^2 \rtimes C_4)' = C_4$. We discuss the proof in two cases.

Case 1. If $p^k \equiv 1, 5 \mod 12$. In this case $T = \{1, 5\}$, and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of $(C_3^2 \rtimes C_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3^2 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t)$. As $F(C_3^2 \rtimes C_4)/(C_3^2 \rtimes C_4)' \cong FC_4$, and $FC_4 \cong F^4$ for $p^k \cong 1 \mod 4$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3^2 \rtimes C_4) \cong$

Table 2

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xzw^2, xz^2w, xw, xz, xz^2w^2, xw^2, xzw, xz^2\}$	4
y	$\{y, yw^2, yw, yz^2, yz^2w^2, yz^2w, yz, yzw^2, yzw\}$	2
z	$\{z, z^2, z^2w, zw^2\}$	3
w	$\{w, zw, w^2, z^2w^2\}$	3
xy	$\{xy, xyz^2, xyz, xyz^2w^2, xyzw^2, xyw^2, xyzw, xyw, xyz^2w\}$	4

$F_q^4 \bigoplus_{t=1}^{2} M_{n_t}(F_t)$. Now, by the dimension formula, we have $32 = n_1^2 + n_2^2, n_1, n_2 \geq 2, \forall\, t$. Here (4, 4) is the only choice of n_1, n_2. Hence,

$$F_q(C_3^2 \rtimes C_4) \cong F_q^4 \oplus M_4(F_q)^2. \tag{3.1}$$

Case 2. If $p^k \equiv 7, 11 \mod 12$. Here $T = \{1, 7\}$ or $T = \{1, 11\}$, and $|S(\gamma_x)| = \{\gamma_x, \gamma_{xy}\}$ $|S(\gamma_g)| = 1$ for all remaining representatives of the conjugacy classes of $(C_3^2 \rtimes C_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3^2 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t)$. As $F(C_3^2 \rtimes C_4)/(C_3^2 \rtimes C_4)' \cong FC_4$, and $FC_4 \cong F^2 \oplus F_2$, for $p^k \cong -1 \mod 4$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3^2 \rtimes C_4) \cong F_q^4 \bigoplus_{t=1}^{2} M_{n_t}(F_t)$. Now, by the dimension formula, we have $32 = n_1^2 + n_2^2, n_1, n_2 \geq 2, \forall\, t$. Here (4, 4) is the only choice for n_1, n_2. Thus

$$F_q(C_3^2 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \oplus M_4(F_q)^2. \qquad \square$$

Theorem 3.3. *Let $G \cong C_3 \rtimes Dic_3$, then for a field F of characteristic $p > 3$,*

$$U(F_q(C_3 \rtimes Dic_3)) \cong \begin{cases} C_{p^k-1}^4 \oplus GL_2(F_q)^8 & \text{for } q \equiv 1, 5 \mod 12 \\ C_{p^k-1}^2 \oplus C_{p^{2k-1}} \oplus GL_2(F_q)^8 & \text{for } q \equiv 7, 11 \mod 12. \end{cases}$$

Proof. Let $G \cong C_3 \rtimes Dic_3 = \langle x, y, z, w \mid x^2y^{-1}, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxz^{-1}, w^{-1}x^{-1}wxw^{-1}, y^2, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^3, w^{-1}z^{-1}wz, w^3\rangle$. The Conjugacy classes description of $C_3 \rtimes Dic_3$ are given in Table 3. It is clear from Table 3 that the exponent of $C_3 \rtimes Dic_3$ is 12, $(C_3 \rtimes Dic_3)' = C_3^2$ and $(C_3 \rtimes Dic_3)/(C_3 \rtimes Dic_3)' = C_4$. We discuss the proof in two cases.

Case 1. If $p^k \equiv 1, 5 \mod 12$. In this case $T = \{1, 5\}$, and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of $(C_3^2 \rtimes C_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \rtimes Dic_3) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t)$. As $F(C_3 \rtimes Dic_3)/(C_3 \rtimes Dic_3)' \cong FC_4$, and $FC_4 \cong F^4$, for $p^k \cong 1 \mod 4$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3 \rtimes Dic_3) \cong F_q^4 \bigoplus_{t=1}^{8} M_{n_t}(F_t)$. Now using dimension formula, we have $32 = \sum_{t=1}^{8} n_t^2, n_t \geq 2, \forall\, t$, $(2, 2, 2, 2, 2, 2, 2, 2)$ is the only choice of n_t's. Hence

$$F_q(C_3 \rtimes Dic_3) \cong F_q^4 \oplus M_2(F_q)^8.$$

Table 3

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw^2, xw, xz^2, xz^2w^2, xz^2w, xz, xzw^2, xzw\}$	4
y	$\{y\}$	2
z	$\{z, z^2\}$	3
w	$\{w, w^2\}$	3
xy	$\{xy, xyw^2, xyw, xyz^2, xyz^2w^2, xyz^2w, xyz, xyzw^2, xyzw\}$	4
yz	$\{yz, yz^2\}$	6
yw	$\{yw, yw^2\}$	6
zw	$\{zw, z^2w^2\}$	3
yzw	$\{yzw, yz^2w^2\}$	6
z^2w	$\{z^2w, zw^2\}$	3
yz^2w	$\{yz^2w, yzw^2\}$	6

Case 2. If $p^k \equiv 7, 11 \mod 12$. Here $T = \{1, 7\}$ or $T = \{1, 11\}$ and $|S(\gamma_x)| = \{\gamma_x, \gamma_{xy}\}$, $|S(\gamma_g)| = 1$ for all remaining representatives of the conjugacy classes of $(C_3 \rtimes Dic_3)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \rtimes Dic_3) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t)$. As $F(C_3 \rtimes Dic_3)/(C_3 \rtimes Dic_3)' \cong FC_4$, and $FC_4 \cong F^2 \oplus F_2$, for $p^k \cong -1 \mod 4$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3 \rtimes Dic_3) \cong F_q^4 \bigoplus_{t=1}^{2} M_{n_t}(F_t)$. Now by dimension formula, we have $32 = \sum_{t=1}^{8} n_t^2, n_t \geq 2, \forall\, t$, (2, 2, 2, 2, 2, 2, 2, 2) is the only choice of n_t's. Hence

$$F_q(C_3^2 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \oplus M_2(F_q)^8. \qquad \square$$

Theorem 3.4. *Let $G \cong C_3 \cdot A_4$, then for a field F of characteristic $p > 3$,*

$$U(F_q(C_3 \cdot A_4)) \cong \begin{cases} C_{p^k-1}^9 \oplus GL_3(F_q)^3 & \text{for } q \equiv 1 \mod 18 \\ C_{p^k-1} \oplus C_{p^{2k}-1} \oplus C_{p^{6k}-1} \oplus GL_3(F_q) \oplus GL_3(F_{q^2}) & \text{for } q \equiv 5, 11 \mod 18 \\ C_{p^k-1}^3 \oplus C_{p^{3k}-1}^2 \oplus GL_3(F_q)^3 & \text{for } q \equiv 7, 13 \mod 18 \\ C_{p^k-1} \oplus C_{p^{2k}-1}^4 \oplus GL_3(F_q) \oplus GL_3(F_{q^2}) & \text{for } q \equiv 17 \mod 18. \end{cases}$$

Proof. Let $G \cong C_3 \cdot A_4 = \langle x, y, z, w \mid x^3y^{-1}, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxw^{-1}z^{-1}, w^{-1}x^{-1}wxz^{-1}, y^3, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^2, w^{-1}z^{-1}wz, w^2\rangle$. Conjugacy classes description of $C_3 \cdot A_4$ are given in Table 4. Now from Table 4 the exponent of $C_3 \cdot A_4$ is 12, $(C_3 \cdot A_4)' = C_2^2$ and $(C_3 \cdot A_4)/(C_3 \cdot A_4)' = C_9$. We discuss the proof in four cases.

Case 1. If $p^k \equiv 1 \mod 18$. In this case $T = \{1\}$ and $|S(\gamma_g)| = 1$ for all the representative of the conjugacy classes of $(C_3 \cdot A_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \cdot A_4) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t)$. As $F(C_3 \cdot A_4)/(C_3 \cdot A_4)' \cong FC_9$, and $FC_9 \cong$

Table 4

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xz, xw, xzw\}$	9
y	$\{y\}$	3
z	$\{z, w, zw\}$	2
x^2	$\{x^2, x^2z, x^2w, x^2zw\}$	9
xy	$\{xy, xyz, xyw, xyzw\}$	9
y^2	$\{y^2\}$	3
yz	$\{yz, yw, yzw\}$	6
x^2y	$\{x^2y, x^2yz, x^2yw, x^2yzw\}$	9
xy^2	$\{xy^2, xy^2z, xy^2w, xy^2zw\}$	9
y^2z	$\{y^2z, y^2w, y^2zw\}$	6
x^2y^2	$\{x^2y^2, x^2y^2z, x^2y^2w, x^2y^2zw\}$	9

F^9, for $p^k \cong 1 \mod 9$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3 \cdot A_4) \cong F_q^9 \bigoplus_{t=1}^{3} M_{n_t}(F_t)$. Now by using dimension formula, we have $27 = \sum_{t=1}^{3} n_t^2, n_t \geq 2, \forall\, t$, $(3,3,3)$ is the only choice of n_t's. Hence

$$F_q(C_3 \cdot A_4) \cong F_q^9 \oplus M_3(F_q)^3.$$

Case 2. If $p^k \equiv 5, 11 \mod 18$. Here $T = \{1, 5, 7, 11, 13, 17\}$ and $|S(\gamma_x)| = \{\gamma_x, \gamma_{x^2}, \gamma_{xy}, \gamma_{x^2y^2}, \gamma_{x^2y}, \gamma_{xy^2}\}$, $|S(\gamma_y)| = \{\gamma_y, \gamma_{y^2}\}$, $|S(\gamma_{yz})| = \{\gamma_{yz}, \gamma_{y^2z}\}$ and $|S(\gamma_g)| = 1$ for all the remaining representative of the conjugacy classes of $(C_3 \cdot A_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \cdot A_4) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t)$. As $F(C_3 \cdot A_4)/(C_3 \cdot A_4)' \cong FC_9$, and $FC_9 \cong F \oplus F_2 \oplus F_6$, for $p^k \cong 2, -4 \mod 9$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3 \cdot A_4) \cong F_q \oplus F_{q^2} \oplus F_{q^6} \bigoplus_{t=1}^{3} M_{n_t}(F_t)$. Now by using the dimension formula, we have $27 = \sum_{t=1}^{3} n_t^2, n_t \geq 2, \forall\, t$, $(3,3,3)$ is the only choice of n_t's. Hence

$$F_q(C_3 \cdot A_4) \cong F_q \oplus F_{q^2} \oplus F_{q^6} \oplus M_3(F_q)^3.$$

Case 3. If $p^k \equiv 7, 13 \mod 18$. Here $T = \{1, 7, 13\}$ and $|S(\gamma_x)| = \{\gamma_x, \gamma_{xy}, \gamma_{y^2},\}$ $|S(\gamma_{x^2})| = \{\gamma_{x^2}, \gamma_{x^2y}, \gamma_{x^2y^2}\}$ $|S(\gamma_g)| = 1$ for all the remaining representatives of the conjugacy classes of $(C_3 \cdot A_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \cdot A_4) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t)$. As $F(C_3 \cdot A_4)/(C_3 \cdot A_4)' \cong FC_9$, and $FC_9 \cong F^3 \oplus F_3^2$, for $p^k \cong -2, 4 \mod 9$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3 \cdot A_4) \cong F_q^3 \oplus F_{q^3}^2 \bigoplus_{t=1}^{3} M_{n_t}(F_t)$. Now, by using the dimension formula, we have $27 = \sum_{t=1}^{3} n_t^2, n_t \geq 2, \forall\, t$. Here (3,3,3) is the only choice of n_t's. Hence

$$F_q(C_3 \cdot A_4) \cong F_q^3 \oplus F_{q^3}^2 \oplus M_3(F_q)^3.$$

Case 4. If $p^k \equiv 17 \mod 18$. Here $T = \{1, 17\}$ and $|S(\gamma_x)| = \{\gamma_x, \gamma_{x^2y^2}\}$, $|S(\gamma_y)| = \{\gamma_y, \gamma_{y^2}\}$, $|S(\gamma_{x^2})| = \{\gamma_{x^2}, \gamma_{xy^2}\}$, $|S(\gamma_{xy})| = \{\gamma_{xy}, \gamma_{x^2y}\}$, $|S(\gamma_{yz})| = \{\gamma_{yz}, \gamma_{y^2z}\}$,

$|S(\gamma_g)| = 1$ for all the remaining representatives of the conjugacy classes of $(C_3 \cdot A_4)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \cdot A_4) \cong F_q \oplus M_{n_t}(F_t) \bigoplus_{t=1}^{5} M_{n_t}(F_{t^2})$. As $F(C_3 \cdot A_4)/(C_3 \cdot A_4)' \cong FC_9$, and $FC_9 \cong F \oplus F_2^4$, for $p^k \cong -1 \mod 9$ [15]. Thus, using Theorem 2.4, we get $F_q(C_3 \cdot A_4) \cong F_q \oplus F_{q^2}^4 \oplus M_{n_t}(F_t) \oplus M_{n_t}(F_{t^2})$. Now, by using the dimension formula, we have $27 = n_1^2 + 2n_2^2, n_1, n_2 \geq 2$. Now (3,3,3) is the only choice of n_1, n_2. Hence

$$F_q(C_3 \cdot A_4) \cong F_q \oplus F_{q^2}^4 \oplus M_3(F_q) \oplus M_3(F_{q^2}). \qquad \square$$

Theorem 3.5. *Let $G \cong S_3^2$, then for a field F of characteristic $p > 3$,*

$$U(F_q S_3^2) \cong C_{p^k-1}^4 \oplus GL_2(F_q)^4 \oplus GL_4(F_q) \quad \text{for } q \equiv 1, 5 \mod 6.$$

Proof. Let $G \cong S_3^2 = \langle x, y, z, w \mid x^2, y^{-1}x^{-1}yx, z^{-1}x^{-1}zx, w^{-1}x^{-1}wxw^{-1}, y^2, z^{-1}y^{-1}zyz^{-1}, w^{-1}y^{-1}wy, z^3, w^{-1}z^{-1}wz, w^3 \rangle$. Conjugacy classes description of S_3^2 are given in Table 5. It is clear from Table 5 that the exponent of S_3^2 is 6,

Table 5

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw, xw^2\}$	2
y	$\{y, yz, yz^2\}$	2
z	$\{z, z^2\}$	3
w	$\{w, w^2\}$	3
xy	$\{xy, xyw^2, xyw, xyz^2, xyz^2w^2, xyz^2w, xyz, xyzw^2, xyzw\}$	2
xz	$\{xz, xzw^2, xzw, xz^2, xz^2w^2, xz^2w\}$	6
yw	$\{yw, yz^2w, yzw, yw^2, yz^2w^2, yzw^2\}$	6
zw	$\{zw, z^2w, zw^2, z^2w^2\}$	3

$(S_3^2)' = C_3^2$ and $(S_3^2)/(S_3^2)' = C_2^2$. Now, if $p^k \equiv 1, 5 \mod 6$, then $T = \{1, 5\}$ and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of (S_3^2). Theorems 2.1 and 2.2 imply that $F_q(S_3^2) \cong F_q \bigoplus_{t=1}^{8} M_{n_t}(F_t)$. As $F(S_3^2)/(S_3^2)' \cong FC_2^2$, and $FC_2^2 \cong F^4$, for $p^k \cong 1 \mod 6$ [15]. Thus, using Theorem 2.4, we get $F_q(S_3^2) \cong F_q^4 \bigoplus_{t=1}^{5} M_{n_t}(F_t)$. Now, using the dimension formula, we have $32 = \sum_{t=1}^{5} n_t^2, n_t \geq 2, \forall\, t$. We have (2, 2, 2, 2, 4) as the only choice of n_t's. Hence,

$$F_q(S_3^2) \cong F_q^4 \oplus M_2(F_q)^4 \oplus M_4(F_q). \qquad \square$$

Theorem 3.6. *Let $G \cong S_3 \times C_6$, then for a field F of characteristic, $p > 3$ the Wedderburn decomposition is given by*

$$U(F_q(S_3 \times C_6)) \cong \begin{cases} C_{p^k-1}^{12} \oplus GL_2(F_q)^6 & \text{for } q \equiv 1 \mod 6 \\ C_{p^k-1}^4 \oplus C_{p^{2k-1}}^4 \oplus GL_2(F_q)^2 \oplus GL_2(F_{q^2})^2 & \text{for } q \equiv 5 \mod 6. \end{cases}$$

Proof. Let $G \cong S_3 \times C_6 = \langle x, y, z, w \mid x^2, y^{-1}x^{-1}yx, z^{-1}x^{-1}zx, w^{-1}x^{-1}wxw^{-1}, y^2, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^3, w^{-1}z^{-1}wz, w^3\rangle$. The Conjugacy classes description of $S_3 \times C_6$ are given below.

Table 6

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw, xw^2\}$	2
y	$\{y\}$	2
z	$\{z\}$	3
w	$\{w, w^2\}$	3
xy	$\{xy, xyw, xyw^2\}$	2
xz	$\{xz, xzw, xzw^2\}$	6
yz	$\{yz\}$	6
yw	$\{yw, yw^2\}$	6
z^2	$\{z^2\}$	3
zw	$\{zw, zw^2\}$	3
xyz	$\{xyz, xyzw, xyzw^2\}$	6
xz^2	$\{xz^2, xz^2w, xz^2w^2\}$	6
yz^2	$\{yz^2\}$	6
yzw	$\{yzw, yzw^2\}$	6
z^2w	$\{z^2w, z^2w^2\}$	3
xyz^2	$\{xyz^2, xyz^2w, xyz^2w^2\}$	6
yz^2w	$\{yz^2w, yz^2w^2\}$	6

It is clear from Table 6 that the exponent of $S_3 \times C_6$ is 6, $(S_3 \times C_6)' = C_3$ and $(S_3 \times C_6)/(S_3 \times C_6)' = C_2 \times C_6$.

Case 1. If $p^k \equiv 1 \mod 6$, then $T = \{1\}$ and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of (S_3^2). Theorems 2.1 and 2.2 imply that $F_q(S_3^2) \cong F_q \bigoplus_{t=1}^{17} M_{n_t}(F_t)$. As $F(S_3 \times C_6)/(S_3 \times C_6)' \cong F(C_2 \times C_6)$, and $F(C_2 \times C_6) \cong F^{12}$, for $p^k \cong 1 \mod 3$ [21]. Thus, using Theorem 2.4, we get $F_q(S_3 \times C_6) \cong F_q^{12} \bigoplus_{t=1}^{6} M_{n_t}(F_t)$. Now using the dimension formula, we have $24 = \sum_{t=1}^{6} n_t^2, n_t \geq 2, \forall\, t$, (2, 2, 2, 2, 2, 2) is the only valid combination of n_t's. Thus,

$$F_q(S_3 \times C_6) \cong F_q^{12} \oplus M_2(F_q)^6.$$

Case 2. If $p^k \equiv 5 \mod 6$, then $T = \{1, 5\}$, and $|S(\gamma_z)| = \{\gamma_z, \gamma_{z^2}\}$, $|S(\gamma_{xz})| = \{\gamma_{xz}, \gamma_{xz^2}\}$, $|S(\gamma_{yz})|=\{\gamma_{yz}, \gamma_{yz^2}\}$, $|S(\gamma_{zw})|=\{\gamma_{zw}, \gamma_{z^2w}\}$, $|S(\gamma_{xyz})| = \{\gamma_{xyz}, \gamma_{xyz^2}\}$, $|S(\gamma_{yzw})| = \{\gamma_{yzw}, \gamma_{yz^2w}\}$ and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of $(S_3 \times C_6)$. Theorems 2.1 and 2.2 imply that $F_q(S_3 \times C_6) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t) \bigoplus_{t=1}^{6} M_{n_t}(F_{t^2})$. As $F(S_3 \times C_6)/(S_3 \times C_6)' \cong F(C_2 \times C_6)$, and $F(C_2 \times C_6) \cong F^4 \oplus F_2^4$, for $p^k \cong 2 \mod 3$ [21]. Thus, using Theorem 2.4, we get $F_q(S_3 \times C_6) \cong F_q^4 \oplus F_{q^2}^4 \bigoplus_{t=1}^{2} M_{n_t}(F_t) \bigoplus_{t=3}^{4} M_{n_t}(F_{t^2})$. Now using the dimension formula, we have $24 = \sum_{t=1}^{2} n_t^2 + 2\sum_{t=3}^{4} n_t^2, n_t \geq 2, \forall\, t$, $(2,2,2,2,2,2)$ is the only valid combination of n_t's. Hence

$$F_q(S_3 \times C_6) \cong F_q^4 \oplus F_{q^2}^4 \oplus M_2(F_q)^2 \oplus M_2(F_{q^2})^2. \qquad \square$$

Theorem 3.7. *Let $G \cong C_3 \times Dic_3$, then for a field F of characteristic $p > 3$,*

$$U(F_q(C_3 \times Dic_3)) \cong \begin{cases} C_{p^k-1}^{12} \oplus GL_2(F_q)^6 & \text{for } q \equiv 1 \mod 12 \\ C_{p^k-1} \oplus C_{p^{2k}-1}^4 \oplus GL_2(F_q)^2 \oplus GL_2(F_{q^2})^2 & \text{for } q \equiv 5 \mod 12 \\ C_{p^k-1}^6 \oplus C_{p^{2k}-1}^3 \oplus GL_2(F_q)^6 & \text{for } q \equiv 7 \mod 12 \\ C_{p^k-1}^2 \oplus C_{p^{2k}-1}^5 \oplus GL_2(F_q)^2 \oplus GL_2(F_{q^2})^2 & \text{for } q \equiv 11 \mod 12. \end{cases}$$

Proof. Let $G \cong C_3 \times Dic_3 = \langle x, y, z, w \mid x^2z^{-1}, y^{-1}x^{-1}yx, z^{-1}x^{-1}zx, w^{-1}x^{-1}wxw^{-1}, y^3, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^2, w^{-1}z^{-1}wz, w^3\rangle$. Conjugacy classes description of $C_3 \times Dic_3$ are given in Table 7. Now the exponent of $C_3 \times Dic_3$

Table 7

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw, xw^2\}$	4
y	$\{y\}$	3
z	$\{z\}$	2
w	$\{w, w^2\}$	3
xy	$\{xy, xyw, xyw^2\}$	12
xz	$\{xz, xzw, xzw^2\}$	4
y^2	$\{y^2\}$	3
yz	$\{yz\}$	6
yw	$\{yw, yw^2\}$	3
zw	$\{zw, zw^2\}$	6
xy^2	$\{xy^2, xy^2w, xy^2w^2\}$	12
xyz	$\{xyz, xyzw, xyzw^2\}$	12
y^2z	$\{y^2z\}$	6
y^2w	$\{y^2w, y^2w^2\}$	3
yzw	$\{yzw, yzw^2\}$	6
xy^2z	$\{xy^2z, xy^2zw, xy^2zw^2\}$	12
y^2zw	$\{y^2zw, y^2zw^2\}$	6

is 12, $(C_3 \times Dic_3)' = C_2^2$ and $(C_3 \times Dic_3)/(C_3 \times Dic_3)' = C_9$. We discuss the proof in four cases.

Case 1. If $p^k \equiv 1 \mod 12$. In this case $T = \{1\}$ and $|S(\gamma_g)| = 1$ for all the representative of the conjugacy classes of $(C_3 \times Dic_3)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \times Dic_3) \cong F_q \bigoplus_{t=1}^{17} M_{n_t}(F_t)$. As $F(C_3 \times Dic_3)/(C_3 \times Dic_3)^{'} \cong FC_{12}$, and $FC_{12} \cong F^{12}$, for $p^k \cong 1 \mod 12$ [21]. Thus using Theorem 2.4, we get $F_q(C_3 \times Dic_3) \cong F_q^{12} \bigoplus_{t=1}^{6} M_{n_t}(F_t)$. Now by using dimension formula, we have $24 = \sum_{t=1}^{6} n_t^2, n_t \geq 2, \forall\, t$, (2, 2, 2, 2, 2, 2) is the only choice of n_t's. Hence

$$F_q(C_3 \times Dic_3) \cong F_q^{12} \oplus M_2(F_q)^6.$$

Case 2. If $p^k \equiv 5 \mod 12$. In this case $T = \{1, 5\}$, and $|S(\gamma_y)| = \{\gamma_y, \gamma_{y^2}\}$, $|S(\gamma_{xy})| = \{\gamma_{xy}, \gamma_{xy^2}\}$, $|S(\gamma_{yz})| = \{\gamma_{yz}, \gamma_{y^2z}\}$, $|S(\gamma_{yw})| = \{\gamma_{yw}, \gamma_{y^2w}\}$, $|S(\gamma_{xyz})| = \{\gamma_{xyz}, \gamma_{xy^2z}\}$, $|S(\gamma_{yzw})| = \{\gamma_{yzw}, \gamma_{y^2zw}\}$ and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of $(C_3 \times Dic_3)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \times Dic_3) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t) \bigoplus_{t=6}^{11} M_{n_t}(F_{t^2})$. As $F(C_3 \times Dic_3)/(C_3 \times Dic_3)^{'} \cong FC_{12}$, and $FC_{12} \cong F^4 \oplus F_{q^2}^4$, for $p^k \cong 5 \mod 12$ [21]. Thus, using Theorem 2.4, we get $F_q(C_3 \times Dic_3) \cong F_q^4 \oplus F_{q^2}^4 \bigoplus_{t=1}^{2} M_{n_t}(F_t) \bigoplus_{t=3}^{4} M_{n_t}(F_{t^2})$. Now by using the dimension formula, we have $24 = \sum_{t=1}^{2} n_t^2 + 2\sum_{t=3}^{4} n_t^2, n_t \geq 2, \forall\, t$, $(2, 2, 2, 2, 2, 2)$ is the only choice of n_t's. Hence

$$F_q(C_3 \times Dic_3) \cong F_q^4 \oplus F_{q^2}^4 \oplus M_2(F_q)^2 \oplus M_2(F_{q^2})^2.$$

Case 3. If $p^k \equiv 7 \mod 12$. In this case $T = \{1, 7\}$, $|S(\gamma_x)| = \{\gamma_x, \gamma_{xz}\}$ $|S(\gamma_{xy^2})| = \{\gamma_{xy^2}, \gamma_{xy^2z}\}$, and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of $(C_3 \times Dic_3)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \times Dic_3) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t) \bigoplus_{t=12}^{14} M_{n_t}(F_{t^2})$. As $F(C_3 \times Dic_3)/(C_3 \times Dic_3)^{'} \cong FC_{12}$, and $FC_{12} \cong F^6 \oplus F_{q^2}^3$, for $p^k \cong 7 \mod 12$ [21]. Thus, using Theorem 2.4, we get $F_q(C_3 \times Dic_3) \cong F_q^6 \oplus F_{q^2}^3 \bigoplus_{t=1}^{6} M_{n_t}(F_t)$. Now by using the dimension formula, we have $24 = \sum_{t=1}^{6} n_t^2, n_t \geq 2, \forall\, t$, (2, 2, 2, 2, 2, 2) is the only choice of n_t's. Hence,

$$F_q(C_3 \times Dic_3) \cong F_q^6 \oplus F_{q^2}^3 \oplus M_2(F_q)^6.$$

Case 4. If $p^k \equiv 11 \mod 12$. In this case $T = \{1, 11\}$, and $|S(\gamma_x)| = \{\gamma_x, \gamma_{xz}\}$, $|S(\gamma_y)| = \{\gamma_y, \gamma_{y^2}\}$, $|S(\gamma_{xy})| = \{\gamma_{xy}, \gamma_{xy^2z}\}$, $|S(\gamma_{yz})| = \{\gamma_{yz}, \gamma_{y^2z}\}$, $|S(\gamma_{yw})| = \{\gamma_{yw}, \gamma_{y^2w}\}$, $|S(\gamma_{xy^2})| = \{\gamma_{xy^2}, \gamma_{xyz}\}$, $|S(\gamma_{y^2w})| = \{\gamma_{y^2w}, \gamma_{y^2zw}\}$ and $|S(\gamma_g)| = 1$ for all the representative of the conjugacy classes of $(C_3 \times Dic_3)$. Theorems 2.1 and 2.2 imply that $F_q(C_3 \times Dic_3) \cong F_q \bigoplus_{t=1}^{3} M_{n_t}(F_t) \bigoplus_{t=4}^{10} M_{n_t}(F_{t^2})$. As $F(C_3 \times Dic_3)/(C_3 \times Dic_3)^{'} \cong FC_{12}$, and $FC_{12} \cong F \oplus F_{q^2}^5$, for $p^k \cong 11 \mod 12$ [21]. Thus, using Theorem 2.4, we get $F_q(C_3 \times Dic_3) \cong F_q^2 \oplus F_{q^2}^5 \bigoplus_{t=1}^{3} M_{n_t}(F_t) \bigoplus_{t=4}^{5} M_{n_t}(F_{t^2})$. Now, by using the dimension formula, we have $24 = \sum_{t=1}^{3} n_t^2 + 2\sum_{t=4}^{5} n_t^2, n_t \geq 2, \forall\, t$, (2, 2, 2, 2, 2, 2) is the only choice of n_t's. Hence,

$$F_q(C_3 \times Dic_3) \cong F_q^2 \oplus F_{q^2}^5 \oplus M_2(F_q)^2 \oplus M_2(F_{q^2})^2. \qquad \square$$

Theorem 3.8. *Let $G \cong C_2 \times C_3 \rtimes S_3$, then for a field F of characteristic $p > 3$,*

$$U(F_q(C_2 \times C_3 \rtimes S_3)) \cong C_{p^k-1}^4 \oplus GL_2(F_q)^8 \quad \textit{for } q \equiv 1, 5 \mod 6.$$

Proof. Let $G \cong C_2 \times C_3 \rtimes S_3 = \langle x, y, z, w \mid x^2, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxz^{-1}, w^{-1}x^{-1}wxw^{-1}, y^2, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^3, w^{-1}z^{-1}wz, w^3 \rangle$. Conjugacy classes description of $C_2 \times C_3 \rtimes S_3$ are given in Table 8.

Table 8

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw^2, xw, xz^2, xz^2w^2, xz^2w, xz, xzw^2, xzw\}$	2
y	$\{y\}$	2
z	$\{z, z^2\}$	3
w	$\{w, w^2\}$	3
xy	$\{xy, xyw^2, xyw, xyz^2, xyz^2w^2, xyz^2w, xyz, xyzw^2, xyzw\}$	2
yz	$\{yz, yz^2\}$	6
yw	$\{yw, yw^2\}$	6
zw	$\{zw, z^2w^2\}$	3
yzw	$\{yzw, yz^2w^2\}$	6
z^2w	$\{z^2w, zw^2\}$	3
yz^2w	$\{yz^2w, yzw^2\}$	6

It can be observed that the exponent of S_3^2 is 6, $(C_2 \times C_3 \rtimes S_3)' = C_3^2$ and $(C_2 \times C_3 \rtimes S_3)/(C_2 \times C_3 \rtimes S_3)' = C_2^2$. Now, if $p^k \equiv 1, 5 \mod 6$, then $T = \{1, 5\}$ and $|S(\gamma_g)| = 1$ for all the representatives of the conjugacy classes of $C_2 \times C_3 \rtimes S_3$. Theorems 2.1 and 2.2 imply that $F_q(C_2 \times C_3 \rtimes S_3) \cong F_q \bigoplus_{t=1}^{11} M_{n_t}(F_t)$. As $F(C_2 \times C_3 \rtimes S_3)/(C_2 \times C_3 \rtimes S_3)' \cong FC_2^2$, and $FC_2^2 \cong F^4$, for $p^k \cong 1 \mod 6$ [15]. Thus, using Theorem 2.4, we get $F_q(C_2 \times C_3 \rtimes S_3) \cong F_q^4 \bigoplus_{t=1}^{8} M_{n_t}(F_t)$. Now, using the dimension formula, we have $32 = \sum_{t=1}^{8} n_t^2, n_t \geq 2, \forall\, t$, (2, 2, 2, 2, 2, 2, 2, 2) is the only choice of n_t's. Hence,

$$F_q(C_2 \times C_3 \rtimes S_3) \cong F_q^4 \oplus M_2(F_q)^8. \qquad \square$$

References

[1] S. F. Ansari, M. Sahai: *Unit groups of group algebras of groups of order 20*, Quaestiones Mathematicae 44.4 (2021), pp. 503–511.

[2] H. Chandra, S. Mishra: *The group of units of group algebras of abelian groups of order 36 and C_{3}×A_{4} over any finite field*, International Electronic Journal of Algebra 32.32 (2022), pp. 176–191.

[3] C. Dietzel, G. Mittal: *Summands of finite group algebras*, Czechoslovak Mathematical Journal (2021), pp. 1–4.

[4] R. A. Ferraz: *Simple components of the center of FG/J (FG)*, Communications in Algebra® 36.9 (2008), pp. 3191–3199.

[5] J. Gildea: *Units of group algebras of non-abelian groups of order 16 and exponent 4 over F2k*, Results Math 61.3-4 (2012), pp. 245–254.

[6] J. Gildea, F. Monaghan: *Units of some group algebras of groups of order 12 over any finite field of characteristic 3*, Algebra and Discrete Mathematics (2011).

[7] M. Khan: *Structure of the unit group of FD10*, Serdica Math. J 35.1 (2009), pp. 15–24.

[8] R. Lidl: *Introduction to finite fields and their applications*, Cambridge University Press, 1994.

[9] N. Makhijani, R. Sharma, J. Srivastava: *The unit group of finite group algebra of a generalized dihedral group*, Asian-European Journal of Mathematics 7.02 (2014), p. 1450034.

[10] N. Makhijani, R. Sharma, J. Srivastava: *The unit group of Fq [D30]*, Serdica Math. J 41.2-3 (2015), pp. 185–198.

[11] N. Makhijani, R. Sharma, J. Srivastava: *Units in finite dihedral and quaternion group algebras*, Journal of the Egyptian Mathematical Society 24.1 (2016), pp. 5–7.

[12] C. P. Milies: *An Introduction to Group Rings*, Kluwer Academic Publishers, 2002.

[13] G. Mittal, R. K. Sharma: *On unit group of finite semisimple group algebras of non-metabelian groups up to order 72*, Mathematica Bohemica 146.4 (2021), pp. 429–455.

[14] M. Sahai, S. F. Ansari: *Group of units of finite group algebras for groups of order 24*, Ukrainian Mathematical Journal 75.2 (2023), pp. 244–261.

[15] M. Sahai, S. F. Ansari: *Unit groups of finite group algebras of Abelian groups of order at most 16*, Asian-European Journal of Mathematics 14.03 (2021), p. 2150030.

[16] M. Sahai, S. F. Ansari: *Unit groups of group algebras of certain dihedral groups-II*, Asian-European Journal of Mathematics 12.04 (2019), p. 1950066.

[17] M. Sahai, S. F. Ansari: *Unit groups of group algebras of groups of order 18*, Communications in Algebra 49.8 (2021), pp. 3273–3282.

[18] M. Sahai, S. F. Ansari: *Unit groups of the finite group algebras of generalized quaternion groups*, Journal of Algebra and Its Applications 19.06 (2020), p. 2050112.

[19] M. Sahai, S. Farhat Ansari: *The structure of the unit group of the group algebra* $F(C_3 \times D_{10})$, in: Annales Mathematicae et Informaticae, vol. 54, 2021, pp. 73–82.

[20] R. Sharma, Y. Kumar: *The unit group of the group algebra {F}_qD_{*36*}*, Journal of the Iranian Mathematical Society 5.1 (2024), pp. 45–53.

[21] G. Tang, Y. Gao: *The unit group of FG of group with order 12*, Int. J. Pure Appl. Math 73.2 (2011), pp. 143–158.

[22] G. Tang, Y. Wei, Y. Li: *Unit groups of group algebras of some small groups*, Czechoslovak Mathematical Journal 64.1 (2014), pp. 149–157.