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Quantitative strong laws of large numbers for random variables with double indices

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Abstract. Recently, Neri [6] studied the rate of convergence in the strong law of large numbers. For the proof, Neri combined the method of [1] and the ideas of proof mining. In this paper, we follow the method of [6] to find the rate of convergence in the strong law of large numbers for random variables with double indices. We show that the rate of convergence for a certain subsequence implies the rate of convergence for the whole sequence. Then, we apply this result to find the rate of convergence in the strong law of large numbers for pairwise independent random variables with double indices. Quasi uncorrelated random variables are also considered.

Keywords: strong law of large numbers, double indices, rate of convergence

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1. Introduction

Let $\xi_1, \xi_2, \dots$ be identically distributed random variables with finite expectation $\mu = \mathbb{E}\xi_i$ and with partial sum $S_n = \sum_{i=1}^n \xi_i$. One of the best-known results in probability theory is the following Kolmogorov's strong law of large numbers (SLLN). If $\xi_1, \xi_2, \dots$ are independent, then $S_n/n \to \mu$ almost surely if $n \to \infty$, see [5]. Rényi in his textbook [7] presented a proof of the following Khintchine type weak law of large numbers (WLLN): if $\xi_1, \xi_2, \dots$ are pairwise independent, then $S_n/n \to \mu$ in probability if $n \to \infty$.

Etemadi in [2] gave a joint generalization of the above Kolmogorov's SLLN and Khintchine's WLLN: if $\xi_1, \xi_2, \dots$ are pairwise independent, then $S_n/n \to \mu$ almost surely if $n \to \infty$. The proof given by Etemadi is an elementary one.

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Concerning the rate of convergence in the SLLN, famous results were proved by Hsu, Robbins, Erdős, Spitzer, Baum, and Katz. A particular case of the results is the following theorem. Let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables, $S_n = \sum_{i=1}^{n} \xi_i$, $r > 1$, and assume that $\mathbb{E}|\xi_1|^r < \infty$, $\mathbb{E}\xi_1 = 0$. Then

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{P}\left(\sup_{k \ge n} |S_k/k| > \varepsilon \right) < \infty \quad \text{for all} \quad \varepsilon > 0,$$

see, e.g. [3].

In [6], Neri also studied the rate of convergence in the SLLN. Theorem 1.4 of [6] is the following.

Proposition 1.1. *Suppose $\xi_1, \xi_2, \dots$ is a sequence of pairwise independent random variables satisfying $\mathbb{E}(\xi_k) = 0$, $\mathbb{E}(|\xi_k|) \le \tau$ and $\mathrm{Var}(\xi_k) \le \sigma^2$, for all k and some $\tau > 0$, $\sigma > 0$. Let $S_n = \sum_{i=1}^{n} \xi_i$. There exists a universal constant $\kappa \le 1536$ such that for all $0 < \varepsilon \le \tau$,*

$$\mathbb{P}\left(\sup_{m \ge n} |S_m/m| > \varepsilon \right) \le \frac{\kappa \sigma^2 \tau}{n \varepsilon^3}.$$

In the proof of the above theorem, some methods of [1] were applied.

In our paper, we shall use the ideas of [6] to obtain rate of convergence results for the double indices version of the SLLN. It is known that for the multi-index SLLN we need a stronger moment assumption than for the usual single index case. By [8], Kolmogorov's SLLN is true for independent identically distributed random variables $\xi_{\boldsymbol{n}}$, $\boldsymbol{n} \in \mathbb{N}^r$, if and only if $\mathbb{E}|\xi_{\boldsymbol{n}}|(\log^+ |\xi_{\boldsymbol{n}}|)^{r-1} < \infty$. Here and in what follows $\mathbb{N}$ denotes the set of positive integers.

For pairwise independent identically distributed random variables $\xi_{i,j}$, Etemadi in [2] obtained the following SLLN. Let $S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j}$. Then condition $\mathbb{E}|\xi_{1,1}| \log^+ |\xi_{1,1}| < \infty$ implies $\lim_{m\to\infty, n\to\infty} S_{m,n} \to \mathbb{E}\xi_{1,1}$ almost surely. The main aim of our paper is to find the rate of convergence result in this SLLN using ideas of [6].

In Section 2 of our paper, we give a possible description of the rate of convergence in the case of two-dimensional indices. In Section 3, we show how we can apply certain subsequences to obtain the rate of convergence in the law of large numbers. In Section 4, we find the rate of convergence in the strong law of large numbers for pairwise independent random variables with double indices. In Section 5, we extend the results of the previous section to quasi uncorrelated random variables.

2. Basic definitions

First, we give a description of the convergence rate for sequences of random variables with double indices. In our point of view, it can be defined by a parametrized family of curves in $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$.

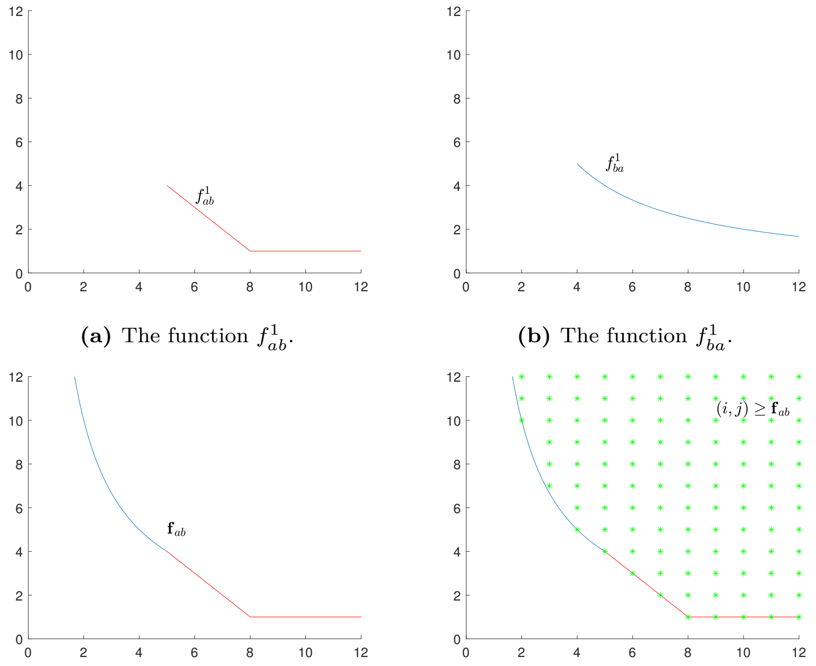
For short, we shall denote the function and its graph by the same letter. Let $a, b > 0$ be fixed. Let R^1_{ab} be the set of the graphs of all functions f^1_{ab} with the following properties: $f^1_{ab}\colon [a, \infty) \to [0, b]$ is a non-increasing continuous function with $f^1_{ab}(a) = b$. We denote by R^2_{ab} the set of those graphs f^2_{ab} which are reflections of graphs $f^1_{ab} \in R^1_{ab}$ to the line $f(x) = x$, $x \in \mathbb{R}$.

We define R_{ab} as the set of all curves $\boldsymbol{f}_{ab} = f^1_{ab} \cup f^2_{ba}$, with $f^1_{ab} \in R^1_{ab}$ and $f^2_{ba} \in R^2_{ba}$. As the point (a, b) belongs both to f^1_{ab} and f^2_{ba}, so $\boldsymbol{f}_{ab}$ is a continuous curve in $\mathbb{R}^2_+$. Now, let

$$R = \bigcup_{a,b>0} R_{ab}.$$

We see that any $\boldsymbol{f} \in R$ is a continuous curve in $\mathbb{R}^2_+$.

Any $\boldsymbol{f} \in R$ divides $\mathbb{R}^2_+$ into two disjoint parts: $A^{\boldsymbol{f}}_0$ and $A^{\boldsymbol{f}}_1$ so that $A^{\boldsymbol{f}}_0$ contains the origin $\mathbf{0} = (0, 0)$. For $\boldsymbol{p} \in \mathbb{R}^2_+$, we shall write that $\boldsymbol{f} \le \boldsymbol{p}$, if $\boldsymbol{p} \in A^{\boldsymbol{f}}_1$. For $\boldsymbol{p} \in \boldsymbol{f}$, we shall accept that $\boldsymbol{f} \le \boldsymbol{p}$. If $\boldsymbol{f} \le \boldsymbol{p}$ but $\boldsymbol{p} \notin \boldsymbol{f}$, then we shall write that $\boldsymbol{f} < \boldsymbol{p}$. For $\boldsymbol{f}, \boldsymbol{g} \in R$, we shall write that $\boldsymbol{f} \le \boldsymbol{g}$ if $\boldsymbol{f} \le \boldsymbol{p}$ for any point $\boldsymbol{p} \in \boldsymbol{g}$.



(a) The function f^1_{ab}. **(b)** The function f^1_{ba}. **(c)** The function $\boldsymbol{f}_{ab}$. **(d)** The points being greater than $\boldsymbol{f}_{ab}$.

Figure 1. An example for $\boldsymbol{f}_{ab}$.

To characterize the rate of convergence, we shall use the following type of parametrized families. Let $(\Gamma, \le)$ be a (non-empty) partially ordered set. We assume that for any $\gamma \in \Gamma$ there exists an element of R which we denote by $\boldsymbol{f}_\gamma$. We assume that $\boldsymbol{f}_\gamma \leq \boldsymbol{f}_\nu$ if $\gamma \leq \nu$, $\gamma, \nu \in \Gamma$. In the next sections, we shall see how a parametrized family of curves $\{\boldsymbol{f}_\gamma : \gamma \in \Gamma\}$ can be used to describe the rate of convergence of sequences with double indices. For short, a family $\{\boldsymbol{f}_\gamma : \gamma \in \Gamma\}$ will be called a rate.

Example 2.1. In this example, we show how to build up a curve $\boldsymbol{f}_{ab}$ (see subfigure (1c) of Figure 1) using f^1_{ab} (see subfigure (1a)) and f^1_{ba} (see subfigure (1b)). Then we visualize the points (i, j) with $(i, j) \geq \boldsymbol{f}_{ab}$ on subfigure (1d) of Figure 1.

In Section 3, we shall apply general rates. In Section 4 and Section 5, we shall use symmetric rate curves.

3. General results

We consider sequences of random variables with double indices. $\boldsymbol{n} = (n_1, n_2) \in \mathbb{N}^2$ will denote the indices. Here $\mathbb{N}$ denotes the set of positive integers. We will denote by $\mathbb{N}_0$ the set of non-negative integers. We say that the double sequence of random variables $\eta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, converges almost surely to η, if $\eta_{n_1,n_2}(\omega) \longrightarrow \eta(\omega)$ as $n_1, n_2 \longrightarrow \infty$ for $\omega \in A$, $\mathbb{P}(A) = 1$. We say that the double sequence of random variables $\eta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, converges almost surely strongly to η, if $\eta_{n_1,n_2}(\omega) \longrightarrow \eta(\omega)$ as $\max\{n_1, n_2\} \longrightarrow \infty$ for $\omega \in A$, $\mathbb{P}(A) = 1$.

Let $\alpha > 1$ and let $\boldsymbol{p} = (p_1, p_2) \in \mathbb{N}_0^2$ be fixed. Define the following set of indices

$$C_{\alpha,\boldsymbol{p}} = \{\boldsymbol{n} : \boldsymbol{n} \in \mathbb{N}^2, \alpha^{p_1} \leq n_1 < \alpha^{p_1+1}, \alpha^{p_2} \leq n_2 < \alpha^{p_2+1}\}.$$

These sets are (possibly empty) rectangles of integer lattice points.

Let $\boldsymbol{k}^-(\boldsymbol{p}) = \min C_{\alpha,\boldsymbol{p}}$ and $\boldsymbol{k}^+(\boldsymbol{p}) = \max C_{\alpha,\boldsymbol{p}}$, if $C_{\alpha,\boldsymbol{p}} \neq \emptyset$, where min and max is defined coordinate-wise. Then $\boldsymbol{k}^-(\boldsymbol{p}) \leq \boldsymbol{n} \leq \boldsymbol{k}^+(\boldsymbol{p})$ for any $\boldsymbol{n} \in C_{\alpha,\boldsymbol{p}}$ ($\leq$ is defined coordinate-wise). When $C_{\alpha,\boldsymbol{p}} = \emptyset$, let $\boldsymbol{k}^-(\boldsymbol{p}) = \boldsymbol{k}^+(\boldsymbol{p}) = \boldsymbol{0} = (0, 0)$. We shall use notation $\boldsymbol{k}^\pm(\boldsymbol{p})$ if a relation is true both for $\boldsymbol{k}^-(\boldsymbol{p})$ and $\boldsymbol{k}^+(\boldsymbol{p})$.

Let $|\boldsymbol{n}| = n_1 \cdot n_2$ if $\boldsymbol{n} = (n_1, n_2)$. We shall use the notation $\boldsymbol{1} = (1, 1) \in \mathbb{N}^2$.

Proposition 3.1. *Let $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, be non-negative random variables, and let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} \xi_{\boldsymbol{k}}$, $\mathbb{E}\xi_{\boldsymbol{n}} = \mu$ for all $\boldsymbol{n} \in \mathbb{N}^2$. If for each $\alpha > 1$,*

$$\sum_{\boldsymbol{p} \in \mathbb{N}_0^2, \boldsymbol{k}^\pm(\boldsymbol{p}) \neq \boldsymbol{0}} \mathbb{P}\left(\left| \frac{S_{\boldsymbol{k}^\pm(\boldsymbol{p})}}{|\boldsymbol{k}^\pm(\boldsymbol{p})|} - \mu \right| > \varepsilon \right) < \infty, \quad \textit{for all} \quad \varepsilon > 0,$$

then

$$\frac{S_{\boldsymbol{n}}}{|\boldsymbol{n}|} \to \mu \quad \textit{a.s. strongly as} \quad \boldsymbol{n} \to \infty. \tag{3.1}$$

Proof. By the Borel-Cantelli lemma, only finitely many of the events $\left\{ \left| \frac{S_{\boldsymbol{k}^\pm(\boldsymbol{p})}}{|\boldsymbol{k}^\pm(\boldsymbol{p})|} - \mu \right| > \varepsilon \right\}$ occur almost surely. So

$$\frac{S_{\boldsymbol{k}^\pm(\boldsymbol{p})}}{|\boldsymbol{k}^\pm(\boldsymbol{p})|} \to \mu \quad \text{almost surely as} \quad \max\{p_1, p_2\} \to \infty. \tag{3.2}$$

Let $\boldsymbol{m} \in C_{\alpha,\boldsymbol{p}}$. Then

$$\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} - \mu \geq \frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{|\boldsymbol{m}|} - \mu \geq \frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{\alpha^2|\boldsymbol{k}^-(\boldsymbol{p})|} - \mu = \left(\frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{|\boldsymbol{k}^-(\boldsymbol{p})|} - \mu\right)\frac{1}{\alpha^2} + \mu\left(\frac{1}{\alpha^2} - 1\right). \quad (3.3)$$

Here, the first inequality follows from the fact that $\{S_{\boldsymbol{n}}\}$ is monotone (since $\{\xi_{\boldsymbol{n}}\}$ is non-negative) and $\boldsymbol{k}^-(\boldsymbol{p}) \leq \boldsymbol{m}$. The second inequality is due to the fact that $|\boldsymbol{m}| \leq \alpha^2|\boldsymbol{k}^-(\boldsymbol{p})|$ (since $\boldsymbol{m} \in C_{\alpha,\boldsymbol{p}}$, so by definition $\boldsymbol{m} < \alpha^{\boldsymbol{p}+\boldsymbol{1}}$ and $\boldsymbol{k}^-(\boldsymbol{p}) \in C_{\alpha,\boldsymbol{p}}$, so, $\alpha^{\boldsymbol{p}} \leq \boldsymbol{k}^-(\boldsymbol{p})$).

Using similar arguments, we have,

$$\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} - \mu \leq \frac{S_{\boldsymbol{k}^+(\boldsymbol{p})}}{|\boldsymbol{m}|} - \mu \leq \left(\frac{S_{\boldsymbol{k}^+(\boldsymbol{p})}}{|\boldsymbol{k}^+(\boldsymbol{p})|} - \mu\right)\alpha^2 + \mu(\alpha^2 - 1). \quad (3.4)$$

Now, using (3.2) and taking $\alpha \to 1$ in inequalities (3.3)–(3.4), we obtain (3.1). □

Proposition 3.2. *Let $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, be non-negative random variables, and let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k}\leq\boldsymbol{n}} \xi_{\boldsymbol{k}}$, $\mathbb{E}\xi_{\boldsymbol{n}} = \mu$ for all $\boldsymbol{n}$. Assume that for any $\varepsilon > 0$ and any $\alpha > 1$*

$$\sum_{\boldsymbol{p}\geq\boldsymbol{l},\boldsymbol{k}^\pm(\boldsymbol{p})\neq\boldsymbol{0}} \mathbb{P}\left(\left|\frac{S_{\boldsymbol{k}^\pm(\boldsymbol{p})}}{|\boldsymbol{k}^\pm(\boldsymbol{p})|} - \mu\right| > \varepsilon\right) \leq \lambda \quad \textit{if} \quad \boldsymbol{l} \geq \boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda), \quad (3.5)$$

where $\{\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda) : \lambda > 0\}$ is a rate. Then

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}}\left|\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} - \mu\right| > \varepsilon\right) \leq \lambda \quad \textit{if} \quad \boldsymbol{n} \geq \boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda},\alpha}(\lambda),$$

where $\boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda},\alpha}(\lambda) = \alpha^{\lceil \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda)\rceil}$ and $\alpha^2 = \frac{\varepsilon}{2\mu} + 1$. Here $\lceil x \rceil$ denotes the smallest integer being not smaller than x and $\alpha^{\boldsymbol{\Gamma}} = (\alpha^{\Gamma_1}, \alpha^{\Gamma_2})$ if $\boldsymbol{\Gamma} = (\Gamma_1, \Gamma_2)$.

Proof. Let $\boldsymbol{m} \geq \boldsymbol{n}$ and assume that

$$\left|\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} - \mu\right| > \varepsilon. \quad (3.6)$$

Then $\alpha^{\boldsymbol{p}} \leq \boldsymbol{m} < \alpha^{\boldsymbol{p}+\boldsymbol{1}}$ for some $\boldsymbol{p} = (p_1, p_2) \in \mathbb{N}_0^2$. So, by (3.3) and (3.4), either

$$\frac{S_{\boldsymbol{k}^+(\boldsymbol{p})}}{|\boldsymbol{k}^+(\boldsymbol{p})|}\alpha^2 - \mu\alpha^2 + \mu(\alpha^2 - 1) > \varepsilon$$

or

$$\left(\frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{|\boldsymbol{k}^-(\boldsymbol{p})|} - \mu\right)\frac{1}{\alpha^2} + \mu\left(\frac{1}{\alpha^2} - 1\right) < -\varepsilon.$$

That is either

$$\frac{S_{\boldsymbol{k}^+(\boldsymbol{p})}}{|\boldsymbol{k}^+(\boldsymbol{p})|} - \mu > \frac{\varepsilon - \mu(\alpha^2 - 1)}{\alpha^2} \quad (3.7)$$

or

$$\frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{|\boldsymbol{k}^-(\boldsymbol{p})|} - \mu < -\left(\varepsilon - \mu\left(1 - \frac{1}{\alpha^2}\right)\right)\alpha^2. \tag{3.8}$$

Now, let $\mu(\alpha^2 - 1) = \frac{\varepsilon}{2}$, that is $\alpha^2 = \frac{\varepsilon}{2\mu} + 1$. Then (3.7) is equivalent to

$$\frac{S_{\boldsymbol{k}^+(\boldsymbol{p})}}{|\boldsymbol{k}^+(\boldsymbol{p})|} - \mu > \frac{\varepsilon}{2\alpha^2}.$$

Moreover, (3.8) implies that

$$\frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{|\boldsymbol{k}^-(\boldsymbol{p})|} - \mu < -\left(\varepsilon - \mu\left(1 - \frac{1}{\alpha^2}\right)\right)\alpha^2 = -\varepsilon\alpha^2 + \frac{\varepsilon}{2} \leq \frac{-\varepsilon}{2\alpha^2}$$

as $\alpha > 1$. So (3.6) implies that either $\left|\frac{S_{\boldsymbol{k}^+(\boldsymbol{p})}}{|\boldsymbol{k}^+(\boldsymbol{p})|} - \mu\right| > \frac{\varepsilon}{2\alpha^2}$ or $\left|\frac{S_{\boldsymbol{k}^-(\boldsymbol{p})}}{|\boldsymbol{k}^-(\boldsymbol{p})|} - \mu\right| > \frac{\varepsilon}{2\alpha^2}$. By (3.5), the total probability of these events is smaller than λ if $\boldsymbol{l} \geq \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda)$, that is if $\boldsymbol{n} \geq \alpha^{\lceil \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda) \rceil} = \boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda},\alpha}(\lambda)$. □

4. Pairwise independent random variables

Lemma 4.1. *Let $\xi_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{N}^2$, be pairwise independent random variables with $\mathbb{E}\xi_{\boldsymbol{k}} = \mu$ and $\operatorname{Var}\xi_{\boldsymbol{k}} \leq \sigma^2$ for all $\boldsymbol{k} \in \mathbb{N}^2$, $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} \xi_{\boldsymbol{k}}$. Let $\alpha > 1$. Then*

$$\sum_{\boldsymbol{n} \geq \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \neq \boldsymbol{0}} \mathbb{P}\left(\left|\frac{S_{\boldsymbol{k}^{\pm}(\boldsymbol{n})}}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|} - \mu\right| > \varepsilon\right) \leq \lambda,$$

if $p_1 + p_2 \geq \log_\alpha\left(\frac{\sigma^2\alpha^2}{\varepsilon^2\lambda(\alpha-1)^2}\right) = \rho_{\varepsilon,\alpha}(\lambda)$, that is the rate of convergence is given by $\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda)$, where $\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda)$ is determined by the curve $x + y = \rho_{\varepsilon,\alpha}(\lambda)$.

Proof.

$$\begin{aligned}
\sum_{\boldsymbol{n} \geq \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \neq \boldsymbol{0}} \mathbb{P}\left(\left|\frac{S_{\boldsymbol{k}^{\pm}(\boldsymbol{n})}}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|} - \mu\right| > \varepsilon\right) &\leq \frac{1}{\varepsilon^2} \sum_{\boldsymbol{n} \geq \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \neq \boldsymbol{0}} \frac{\operatorname{Var}(S_{\boldsymbol{k}^{\pm}(\boldsymbol{n})})}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|^2}) \\
&= \frac{1}{\varepsilon^2} \sum_{\boldsymbol{n} \geq \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \neq \boldsymbol{0}} \frac{\sum_{\boldsymbol{k} \leq \boldsymbol{k}^{\pm}(\boldsymbol{n})} \operatorname{Var}(\xi_{\boldsymbol{k}})}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|^2} \\
&\leq \frac{\sigma^2}{\varepsilon^2} \sum_{\boldsymbol{n} \geq \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \neq \boldsymbol{0}} \frac{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|^2} \\
&\leq \frac{\sigma^2}{\varepsilon^2} \sum_{\boldsymbol{n} \geq \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \neq \boldsymbol{0}} \alpha^{-\boldsymbol{n}} \\
&\leq \frac{\sigma^2}{\varepsilon^2} \sum_{n_1 = p_1}^{\infty} \alpha^{-n_1} \sum_{n_2 = p_2}^{\infty} \alpha^{-n_2}
\end{aligned}$$

$$= \frac{\sigma^2}{\varepsilon^2} \cdot \frac{\alpha^{-p_1}}{1-\frac{1}{\alpha}} \cdot \frac{\alpha^{-p_2}}{1-\frac{1}{\alpha}}$$
$$= \frac{\sigma^2}{\varepsilon^2} \cdot \frac{\alpha^{-p_1+1}\alpha^{-p_2+1}}{(\alpha-1)^2} \leq \lambda,$$

if $\alpha^{-(p_1+p_2)}\frac{\sigma^2}{\varepsilon^2}\frac{\alpha^2}{(\alpha-1)^2} \leq \lambda$.

Here, the first inequality follows from Chebyshev's inequality. The second inequality is derived using pairwise independence and the assumption that $\operatorname{Var}\xi_{\boldsymbol{k}} \leq \sigma^2$. The third inequality is satisfied by the condition $(\boldsymbol{k}^{\pm}(\boldsymbol{n}))_i \geq \alpha^{n_i}$ for $i = 1, 2$. The remaining steps are obtained through straightforward calculations with the application of the formula for the sum of an infinite geometric series.

The last inequality is equivalent to

$$\alpha^{p_1+p_2} \geq \frac{\sigma^2\alpha^2}{\lambda\varepsilon^2(\alpha-1)^2}$$

or

$$p_1 + p_2 \geq \log_\alpha\left(\frac{\sigma^2\alpha^2}{\lambda\varepsilon^2(\alpha-1)^2}\right). \qquad \square$$

Lemma 4.2. *Let $\xi_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{N}^2$, be pairwise independent non-negative random variables with $\mathbb{E}\xi_{\boldsymbol{k}} = \mu$ and $\operatorname{Var}\xi_{\boldsymbol{k}} \leq \sigma^2$ for all $\boldsymbol{k} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k}\leq\boldsymbol{n}}\xi_{\boldsymbol{k}}$, $\alpha^2 = \frac{\varepsilon}{2\mu}+1$. Then for all $\varepsilon > 0$, $\lambda > 0$,*

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}}\left|\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|}-\mu\right| > \varepsilon\right) \leq \lambda, \tag{4.1}$$

if $\boldsymbol{n} \geq \boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda}}(\lambda)$, where $\boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda}}(\lambda) = \alpha^{\lceil \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda)\rceil}$ and $\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda)$ is determined by the curve $x + y = \rho_{\varepsilon,\alpha}(\lambda)$ with $\rho_{\varepsilon,\alpha}(\lambda) = \log_\alpha\left(\frac{\sigma^2\alpha^2}{\lambda\varepsilon^2(\alpha-1)^2}\right)$. Inequality (4.1) is satisfied if $n_1 \cdot n_2 \geq \frac{4\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2}$.

Proof. By Lemma 4.1,

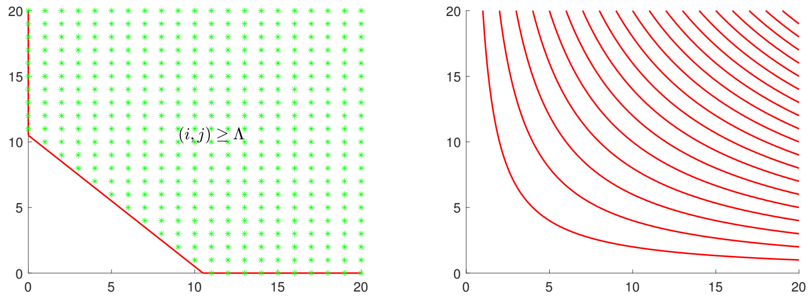
$$\sum_{\boldsymbol{n}\geq\boldsymbol{p},\boldsymbol{k}^{\pm}(\boldsymbol{n})\neq\boldsymbol{0}} \mathbb{P}\left(\left|\frac{S_{\boldsymbol{k}^{\pm}(\boldsymbol{n})}}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|}-\mu\right| > \varepsilon\right) \leq \lambda,$$

if $p_1 + p_2 \geq \log_\alpha(\frac{\sigma^2\alpha^2}{\varepsilon^2\lambda(\alpha-1)^2})$. Therefore, by Proposition 3.2,

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}}\left|\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|}-\mu\right| > \varepsilon\right) \leq \lambda \quad \text{if} \quad \boldsymbol{n} \geq \boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda},\alpha}(\lambda) = \alpha^{\lceil \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda)\rceil}.$$

We see that the above inequality is satisfied if $n_1 \cdot n_2 \geq \alpha^2 \cdot \frac{\sigma^2\alpha^2}{\lambda(\frac{\varepsilon}{2\alpha^2})^2(\alpha-1)^2} = \frac{4\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2}$. $\square$

Remark 4.3. We visualize the rates in Lemma 4.1 and Lemma 4.2 using Figure 2. On subfigure (2a), we show a rate curve in Lemma 4.1. That $\mathbf{\Lambda}_{\varepsilon,\alpha}(\lambda)$ curve is determined by a part of the x axis, a part of the y axis and the section of the line $x+y = \rho_{\varepsilon,\alpha}(\lambda)$ being in the first quadrant. On subfigure (2b), we show a sequence of rate curves in Lemma 4.2. These curves are of the shape $\mathbf{\Phi}_{\varepsilon,\mathbf{\Lambda}}(\lambda) = \alpha^{\lceil \mathbf{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda)\rceil}$. In explicit form these curves are given by the formula $x \cdot y = \frac{4\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2}$.



(a) A curve Λ in Lemma 4.1. **(b)** Curves in Lemma 4.2.

Figure 2. Rates for pairwise independent random variables.

Proposition 4.4. *Let $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, be pairwise independent random variables with $\mathbb{E}\xi_{\boldsymbol{n}} = 0$, $\mathbb{E}|\xi_{\boldsymbol{n}}| = \tau > 0$ and $\operatorname{Var}\xi_{\boldsymbol{n}} \leq \sigma^2 > 0$ for all $\boldsymbol{n} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k}\leq\boldsymbol{n}} \xi_{\boldsymbol{k}}$. Then for all $\varepsilon > 0$, $\lambda > 0$,*

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \leq \lambda \quad \text{if} \quad |\boldsymbol{n}| \geq \frac{32\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2} \quad \text{and} \quad \alpha^2 = \frac{\varepsilon}{2\tau} + 1.$$

Proof. For the positive and the negative parts, we have $\operatorname{Var}\xi_{\boldsymbol{n}}^+ + \operatorname{Var}\xi_{\boldsymbol{n}}^- \leq \mathbb{E}(\xi_{\boldsymbol{n}}^+)^2 + \mathbb{E}(\xi_{\boldsymbol{n}}^-)^2 = \mathbb{E}\xi_{n}^2 = \operatorname{Var}\xi_{\boldsymbol{n}} \leq \sigma^2$, therefore $\operatorname{Var}\xi_{\boldsymbol{n}}^+ \leq \sigma^2$, $\operatorname{Var}\xi_{\boldsymbol{n}}^- \leq \sigma^2$. Moreover, $\mathbb{E}\xi_{\boldsymbol{n}}^+ = \mathbb{E}\xi_{\boldsymbol{n}}^- = \frac{\tau}{2} = \mu$. By Lemma 4.2,

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}} \left|\frac{S_{\boldsymbol{m}}^{\pm}}{|\boldsymbol{m}|} - \mu\right| > \frac{\varepsilon}{2}\right) \leq \frac{\lambda}{2} \quad \text{if} \quad |\boldsymbol{n}| \geq \frac{4\sigma^2\alpha^8}{(\frac{\lambda}{2})(\frac{\varepsilon}{2})^2(\alpha-1)^2},$$

where $\alpha^2 = \frac{\varepsilon}{2} \cdot \frac{1}{2\mu} + 1 = \frac{\varepsilon}{2\tau} + 1$. Then

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \leq \mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}} \left|\frac{S_{\boldsymbol{m}}^{+}}{|\boldsymbol{m}|} - \mu\right| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sup_{\boldsymbol{m}\geq\boldsymbol{n}} \left|\frac{S_{\boldsymbol{m}}^{-}}{|\boldsymbol{m}|} - \mu\right| > \frac{\varepsilon}{2}\right)$$
$$\leq \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda,$$

if $|\boldsymbol{n}| \geq \frac{32\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2}$. □

Lemma 4.5. *Let $\tau > 0$ and $\alpha^2 = \frac{\varepsilon}{2\tau} + 1$ be the values from Proposition 4.4. Then for any $C > 4$, we have*

$$\frac{\alpha^8}{(\alpha-1)^2} \le \frac{4\tau^2}{\varepsilon^2} C \tag{4.2}$$

for small enough $\varepsilon > 0$. More precisely, for any $C > 4$ there exists an $a \in (0, 1/2)$ for which $\left(\frac{(1-a)^4}{a^5}\right)^2 = C$, and with $b = (1-2a)/a^2$ the inequality (4.2) is satisfied for $0 < \varepsilon < 2\tau b$.

Proof. Analysing the function $f(x) = \sqrt{x+1} - 1$, $x \ge 0$, and the straight line ax, where $a \in (0, 1/2)$ is a parameter, we get the following.

$$ax \le \sqrt{x+1} - 1 \quad \text{if} \quad 0 \le x \le b,$$

where $b = (1-2a)/a^2$ and a is a fixed value from the interval $(0, 1/2)$.

Now, let $\alpha^2 = \frac{\varepsilon}{2\tau} + 1$ be from Proposition 4.4, and apply the above inequality with $x = \frac{\varepsilon}{2\tau}$. Then

$$\frac{\alpha^8}{(\alpha-1)^2} = \frac{\alpha^8}{\left(\sqrt{\frac{\varepsilon}{2\tau}+1} - 1\right)^2} \le \frac{\alpha^8}{\left(a\frac{\varepsilon}{2\tau}\right)^2} \le \frac{4\tau^2\alpha^8}{a^2\varepsilon^2}.$$

As now $\frac{\varepsilon}{2\tau} = x \le b$, we have

$$\frac{\alpha^8}{(\alpha-1)^2} \le \frac{4\tau^2}{\varepsilon^2}\frac{(b+1)^4}{a^2} = \frac{4\tau^2}{\varepsilon^2}\frac{\left(\frac{1-2a}{a^2}+1\right)^4}{a^2} = \frac{4\tau^2}{\varepsilon^2}C,$$

where $C = \left(\frac{(1-a)^4}{a^5}\right)^2$. Now, we consider the function $g(a) = \frac{(1-a)^4}{a^5}$ for $a \in (0, 1/2)$. Its infimum is 2. So for any given $C > 4$ we can find an appropriate a so that $C = \left(\frac{(1-a)^4}{a^5}\right)^2$. □

Theorem 4.6. *Let $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, be pairwise independent random variables with $\mathbb{E}\xi_{\boldsymbol{n}} = 0$, $\mathbb{E}|\xi_{\boldsymbol{n}}| = \tau > 0$ and $\operatorname{Var}\xi_{\boldsymbol{n}} \le \sigma^2 > 0$ for all $\boldsymbol{n} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k}\le\boldsymbol{n}} \xi_{\boldsymbol{k}}$. Then*

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\ge\boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \le \frac{K\sigma^2\tau^2}{\varepsilon^4|\boldsymbol{n}|}$$

if $K > 512$ and $\varepsilon > 0$ is small enough.

Proof. By Proposition 4.4, we have

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\ge\boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \le \frac{32\sigma^2\alpha^8}{\varepsilon^2|\boldsymbol{n}|(\alpha-1)^2}.$$

Applying Lemma 4.5, we obtain

$$\mathbb{P}\left(\sup_{\boldsymbol{m}\ge\boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \le \frac{32\sigma^2}{\varepsilon^2|\boldsymbol{n}|}\frac{4\tau^2}{\varepsilon^2}C \le \frac{K\sigma^2\tau^2}{\varepsilon^4|\boldsymbol{n}|}.$$ □

5. Quasi uncorrelated random variables

Definition 5.1. A sequence $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, of random variables is said to be quasi uncorrelated if each $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, has finite variance and there exists a positive constant c such that

$$\operatorname{Var}(S_{\boldsymbol{n}}) \leq c \sum_{\boldsymbol{k} \leq \boldsymbol{n}} \operatorname{Var}(\xi_{\boldsymbol{k}}) \tag{5.1}$$

for any $\boldsymbol{n} \in \mathbb{N}^2$, see [4].

We can see that the results for pairwise independent random variables can be adapted to the case of quasi uncorrelated random variables. So we present them without proofs. Next proposition is a version of Proposition 4.4 for quasi uncorrelated random variables.

Proposition 5.2. *Let $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, be quasi uncorrelated random variables with $\mathbb{E}\xi_{\boldsymbol{n}} = 0$, $\mathbb{E}|\xi_{\boldsymbol{n}}| = \tau > 0$ and $\operatorname{Var}\xi_{\boldsymbol{n}} \leq \sigma^2 > 0$ for all $\boldsymbol{n} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} \xi_{\boldsymbol{k}}$. Then for all $\varepsilon > 0$, $\lambda > 0$,*

$$\mathbb{P}\left(\sup_{\boldsymbol{m} \geq \boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) < \lambda \quad \text{if} \quad |\boldsymbol{n}| \geq \frac{c32\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2} \quad \text{and} \quad \alpha^2 = \frac{\varepsilon}{2\tau} + 1,$$

where c is from (5.1).

Now, we turn to the quasi uncorrelated version of Theorem 4.6.

Theorem 5.3. *Let $\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, be quasi uncorrelated random variables with $\mathbb{E}\xi_{\boldsymbol{n}} = 0$, $\mathbb{E}|\xi_{\boldsymbol{n}}| = \tau > 0$ and $\operatorname{Var}\xi_{\boldsymbol{n}} \leq \sigma^2 > 0$ for all $\boldsymbol{n} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} \xi_{\boldsymbol{k}}$. Then with c from (5.1), we have*

$$\mathbb{P}\left(\sup_{\boldsymbol{m} \geq \boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \leq \frac{Kc\sigma^2\tau^2}{\varepsilon^4|\boldsymbol{n}|}$$

if $K > 512$ and $\varepsilon > 0$ is small enough.

Now, we turn to negatively dependent random variables. Random variables ξ and η are called negatively dependent (ND) if

$$\mathbb{P}(\xi \leq x, \eta \leq y) \leq \mathbb{P}(\xi \leq x)\mathbb{P}(\eta \leq y)$$

for all real numbers x and y. A set of random variables is said to be pairwise ND if every pair of random variables in the set is ND.

Remark 5.4. For pairwise ND random variables Proposition 5.2 and Theorem 5.3 are true with $c = 1$. For the proof, we just remark that in a pairwise ND set, for two different random variables ξ and η, we have $\operatorname{Cov}(\xi, \eta) \leq 0$, see e.g. [9]. So an ND set of random variables is quasi uncorrelated with $c = 1$.

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