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Corrigenda to the paper “Infinitary superperfect numbers”

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Abstract. We fill an gap in the author’s paper of the title and revise a table in the addendum.

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In p. 217, l.9 of the author’s paper “Infinitary superperfect numbers”, this journal vol. **47** (2017), pp. 211–218, we wrote that if $s_i = 0$ and s_j is even, then $2p_j^{s_j} \equiv \pm 2 \pmod q$ cannot be $\pm 1 \pmod q$. However, this is not true when $q = 3$. We settle the case $s_i = 0$ to complete the proof.

From (3.1) in the original paper, we see that $q^{2^l} + 1 \equiv \pm 2 \pmod q$ and therefore we must have $q = 3$. Hence, there exists no prime factor p_i such that $p_i^{2^k} + 1 = 2q$ for some integer $k > 0$. Similarly, there exists no prime factor p_j such that $p_j^2 + 1 = 2q^{2^u}$ for some integer $u > 0$.

Now (3.1) in the original paper becomes

$$3^{2^l} + 1 = 2(2 \times 3^{t_1} - 1)(2 \times 3^{t_2} - 1) \cdots (2 \times 3^{t_r} - 1)$$

for some r. We see that

$$2(2 \times 3^{t_1} - 1) \cdots (2 \times 3^{t_r} - 1) \equiv 3^{2^l} + 1 \equiv -2 \pmod 3$$

and therefore r must be odd. Moreover, we must have $l \geq 1$. Indeed, if $l = 0$, then $4 = 2(2 \times 3^{t_1} - 1) \cdots (2 \times 3^{t_r} - 1)$, which is impossible.

If $t_1 \geq 2$, then $2 \times 3^{t_i} - 1 \equiv -1 \pmod 9$ for each i and $3^{2^l} \equiv 2(-1)^r - 1 \equiv -3 \pmod 9$, which is a contradiction.

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If $t_1 = 1$ and $r \geq 3$, then $2^l > t_3 \geq 3$ and $3^{2^l} = 10(2 \times 3^{t_2} - 1) \cdots - 1$. Clearly we have $l \geq 2$. Now we see that if $t_2 > 2$, then $0 \equiv 3^{2^l} \equiv 9 \pmod{3^3}$ and if $t_2 = 2$, then $0 \equiv 3^{2^l} \equiv -1 \pm 170 \pmod{3^3}$. Thus, we have a contradiction in both cases.

Now we must have $r = 1$ and $t_1 = 1$. Hence, $3^{2^l} + 1 = 10$ and we conclude that $l = 1$. In other words, we must have $\sigma_\infty(N) = 2^f 3^2$. If $p^{2^k} \mid_\infty N$, then $p^{2^k} + 1$ divides $2^f 3^2$. Now we must have $k = 0$ since otherwise $p^{2^k} + 1 \equiv 2 \pmod 4$ and $(p^{2^k} + 1)/2$ must have a prime factor $\equiv 1 \pmod 4$.

Hence, we see that $N = \prod_i p_i$ must be squarefree with $p_i = 2^{u_i} \times 3^{t_i} - 1$ distinct primes. Since $2N = \sigma_\infty(2^f 3^2)$, $p_1 = 5$ must divide N. Since $\prod_i (p_i + 1) = \sigma_\infty(N) = 2^f 3^2$, there exists exactly one more prime p_2 such that $t_2 > 0$. Moreover, we have $t_2 = 1$ and $p_2 = 3 \times 2^{u_2} - 1$.

Since $p_2 \neq p_1 = 5$, we must have $u_2 > 1$ and therefore $p_2 \equiv 3 \pmod 4$. Since $p_2 \mid 2N = \sigma_\infty(2^f 3^2)$ and $p_2 \neq 2, 5$, we must have $p_2 \mid (2^{2^k} + 1)$ for some k. However, this is impossible. Indeed, if $p_2 \mid (2^{2^k} + 1)$, then $p_2 = 3$ with $k = 0$ or $p_2 \equiv 1 \pmod 4$ with $k > 0$, which is a contradiction. This completes the proof.

In the addendum paper (this journal vol. **49** (2019), pp. 199–201), we wrote that we found four more integers N dividing $\sigma_\infty(\sigma_\infty(N))$ up to 2^{32}. However, there exist two more such integers $N = 615517056$ and 690531840. So that, the table given in the addendum should be:

N	k
$615517056 = 2^7 \cdot 3^5 \cdot 7 \cdot 11 \cdot 257^*$	10
$690531840 = 2^9 \cdot 3^2 \cdot 5 \cdot 17 \cdot 41 \cdot 43^*$	6
$1304784000 = 2^7 \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17 \cdot 41$	7
$1680459462 = 2^9 \cdot 3^3 \cdot 11 \cdot 43 \cdot 257$	5
$4201148160 = 2^8 \cdot 3^3 \cdot 5 \cdot 11 \cdot 43 \cdot 257$	6
$4210315200 = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	8

Here * indicates integers overlooked in the addendum paper. Hence, there exist six integers N dividing $\sigma_\infty(\sigma_\infty(N))$ up to 2^{32} other than given in the original paper.

Further instances can be found in The On-Line Encyclopedia of Integer Sequences `https://oeis.org/A318182`.