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On tridimensional balancing numbers and some properties

José Chimpanzo[a], **Paula Catarino**[b], **M. Victoria Otero-Espinar**[c]

[a]Department of Exact Sciences, Higher Polytechnic Institute of Soyo, Soyo, Angola and CMAT-UTAD, Polo of CMAT, University of Minho, Campus de Gualtar, 4710-057 Braga, Portugal
jalchimpanzo@gmail.com

[b]Department of Mathematics, University of Trás-os-Montes e Alto Douro, 5000-801 Vila Real and CMAT-UTAD, Polo of CMAT, University of Minho, 4710-057 Braga, Portugal
pcatarin@utad.pt

[c]Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela and Galician Center for Mathematical Research and Technology (CITMAga), 15782 Santiago de Compostela, Spain
mvictoria.otero@usc.es

Abstract. In this paper, we introduce the tridimensional version

$$\{B_{(n,m,p)}\}_{n,m,p\geq 0}$$

of the balancing numbers extension of its unidimensional and bidimensional versions and study some of its properties and identities.

Keywords: Balancing numbers, bidimensional balancing, tridimensional balancing

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1. Introduction

Over the last few decades, various sequences of numbers have been the subject of investigation by many researchers. In this field of research, the study of numerical sequences has stood out. A clear example is the study of sequences of numbers known as balancing. The notion of balancing numbers under the name of numerical centers was introduced by the first time in the work of Finkelstein [7]. In this

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work, a very well-known problem found in many books of puzzles (see, e.g. [8]) was solved. This sequence was studied in [1], it is denoted by $\{B_n\}_{n\geq 0}$ and defined by the recurrence relation

$$B_{n+2} = 6B_{n+1} - B_n, \tag{1.1}$$

with the initial conditions $B_0 = 0$ and $B_1 = 1$.

Since then, several studies have emerged as a result of this work. For example, in [2], a new explicit formula for square-triangular numbers was presented and a link with balancing numbers was established. In [4], some formulas were explicitly given for the numerical sequences of balancing, Lucas-balancing, cobalancing and Lucas-cobalancing. Another expression was also given for the general term of each of these sequences, using the ordinary generating function. In [13], some properties of balancing, modified Lucas-balancing and Lucas-balancing sequences were studied, and their Binet formulas, generating functions and Simson formulas were presented. Summation formulas for these sequences were also presented. In [14], studies were carried out on balancing and Lucas-balancing numbers, where in particular the application to cryptography of these numbers through their recurrence relations was examined. In [9], some interesting properties and results about balancing, cobalancing and all kinds of generalized balancing number sequences were analyzed. In [10], the links between the balancing and cobalancing numbers with the Pell numbers and the associated Pell numbers were established. In [12], study was made on balancing numbers, where some connections of balancing numbers and cobalancing numbers with Pell numbers and associated Pell numbers were presented. Some important properties of balancing numbers and sequences of numbers related to them were also presented.

The bidimensional version around the balancing numbers has also been the subject of research. For example, in [5], a brief study was made of this new sequence, in particular, the authors defined it as a sequence denoted by $\{B_{(n,m)}\}_{n,m\geq 0}$ that satisfies the following recurrence relations

$$\begin{cases} B_{(n+1,m)} = 6B_{(n,m)} - B_{(n-1,m)}, \\ B_{(n,m+1)} = 6B_{(n,m)} - B_{(n,m-1)}, \end{cases}$$

with the initial conditions $B_{(0,0)} = 0$, $B_{(1,0)} = 1$, $B_{(0,1)} = i$, $B_{(1,1)} = 1 + i$ and $i^2 = -1$. Chimpanzo et. al. [5], give the following explicit form for the entry $B_{(n,m)}$

$$B_{(n,m)} = B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i$$

that will be useful in part in the proof of some results stated in the next Section of this article. Finally, in [6], a brief study of bidimensional Lucas-balancing numbers (a special case of balancing numbers) was also made. In this study, in particular, the authors defined this new number sequence as a sequence denoted by $\{C_{(n,m)}\}_{n,m\geq 0}$ that satisfies the following recurrence relations

$$\begin{cases} C_{(n+1,m)} = 6C_{(n,m)} - C_{(n-1,m)}, \\ C_{(n,m+1)} = 6C_{(n,m)} - C_{(n,m-1)}, \end{cases}$$

with the initial conditions $C_{(0,0)} = 1$, $C_{(1,0)} = 3$, $C_{(0,1)} = 1 + i$, $C_{(1,1)} = 3 + i$.

Therefore, these and other works served as motivation for the study of a new tridimensional version of the balancing numerical sequence.

This article is structured as follows: in the next section, the recurrence relations of tridimensional balancing numbers will be presented. In addition, some of their properties will be presented. Section 3 will be devoted to studying some sum identities of the tridimensional sequence of balancing numbers. Finally, we finish with a brief conclusion of this work.

2. Tridimensional balancing numbers

Based on the studies carried out in [5], where $B(n, m)$ denotes the bidimensional balancing numbers, in this section we introduce the tridimensional version of these numbers by adopting the following definition.

Motivated by the $2D$ case studied in [3], we decided to dedicate to the $3D$ case and we adopt the following definition through this text.

Definition 2.1. The numbers $B(n, m, p)$ represent the tridimensional balancing numbers that satisfy the following recurrence relations, where n, m and p are non-negative integers

$$B_{(n,m,p)} = \begin{cases} 6B_{(n-1,m,p)} - B_{(n-2,m,p)} & \text{for all } m,\ p \text{ and } n \geq 2, \\ 6B_{(n,m-1,p)} - B_{(n,m-2,p)} & \text{for all } n,\ p \text{ and } m \geq 2, \\ 6B_{(n,m,p-1)} - B_{(n,m,p-2)} & \text{for all } m,\ n \text{ and } p \geq 2, \end{cases}$$

with the initial conditions $B_{(0,0,0)} = 0$, $B_{(1,0,0)} = 1$, $B_{(0,1,0)} = B_{(0,0,1)} = i$, $B_{(1,1,0)} = B_{(1,0,1)} = 1 + i$, $B_{(1,1,1)} = 1 + 2i$, $B_{(0,1,1)} = 2i$.

Note that these initial conditions are chosen as a particular case and taking into account the construction adopted in $2D$ case (see, [3]). Since $B_{(0,1,0)} = B_{(0,0,1)}$ and $B_{(1,1,0)} = B_{(1,0,1)}$ we can interpret this information as there exist a reflection symmetry on the plane $y = z$ if one consider the location of the value $B_{(n,m,p)}$ as the point (n, m, p) in the $3D$ Cartesian system.

The Definition 2.1 above is correct in the sense that $B_{(n,m,p)}$ does not depend on the path we use for calculation. For example, to find $B_{(2,2,2)}$, we see that

$$B_{(2,2,2)} = \begin{cases} 6B_{(1,2,2)} - B_{(0,2,2)} & \text{(path 1);} \\ 6B_{(2,1,2)} - B_{(2,0,2)} & \text{(path 2);} \\ 6B_{(2,2,1)} - B_{(2,2,0)} & \text{(path 3).} \end{cases}$$

Using path 1 we obtain that

$$B_{(1,2,2)} = \begin{cases} 6B_{(1,1,2)} - B_{(1,0,2)} & \text{(subpath } a_1\text{),} \\ 6B_{(1,2,1)} - B_{(1,2,0)} & \text{(subpath } a_2\text{),} \end{cases}$$

and

$$\begin{cases} B_{(1,1,2)} = 6B_{(1,1,1)} - B_{(1,1,0)} = 6(1+2i) - (1+i) = 5+11i, \\ B_{(1,0,2)} = 6B_{(1,0,1)} - B_{(1,0,0)} = 6(1+i) - 1 = 5+6i. \end{cases}$$

Therefore, using subpath a_1 we get

$$B_{(1,2,2)} = 6(5+11i) - (5+6i) = 25+60i.$$

On the other hand,

$$\begin{cases} B_{(1,2,1)} = 6B_{(1,1,1)} - B_{(1,0,1)} = 6(1+2i) - (1+i) = 5+11i, \\ B_{(1,2,0)} = 6B_{(1,1,0)} - B_{(1,0,0)} = 6(1+i) - 1 = 5+6i. \end{cases}$$

Since $B_{(1,2,1)} = B_{(1,1,2)}$ and $B_{(1,2,0)} = B_{(1,0,2)}$, the value of $B_{(1,2,2)}$ coincides in both supaths a_1 or a_2 and is given as $B_{(1,2,2)} = 25+60i$.

Once more, using path 1 we have

$$B_{(0,2,2)} = \begin{cases} 6B_{(0,1,2)} - B_{(0,0,2)} & \text{(subpath } b_1\text{)}, \\ 6B_{(0,2,1)} - B_{(0,2,0)} & \text{(subpath } b_2\text{)}. \end{cases}$$

Some computation gives us that

$$\begin{cases} B_{(0,1,2)} = 6B_{(0,1,1)} - B_{(0,1,0)} = 6(2i) - i = 11i, \\ B_{(0,0,2)} = 6B_{(0,0,1)} - B_{(0,0,0)} = 6i - 0 = 6i. \end{cases}$$

Therefore, using subpath b_1 we get

$$B_{(0,2,2)} = 6(11i) - 6i = 60i.$$

On the other hand,

$$\begin{cases} B_{(0,2,1)} = 6B_{(0,1,1)} - B_{(0,0,1)} = 6(2i) - i = 11i, \\ B_{(0,2,0)} = 6B_{(0,1,0)} - B_{(0,0,0)} = 6i - 0 = 6i. \end{cases}$$

Since $B_{(0,2,1)} = B_{(0,1,2)}$ and $B_{(0,2,0)} = B_{(0,0,2)}$, the value of $B_{(0,2,2)}$ coincides in both supaths b_1 or b_2 and is given by $B_{(0,2,2)} = 60i$. Hence, using path 1,

$$B_{(2,2,2)} = 6B_{(1,2,2)} - B_{(0,2,2)} = 6(25+60i) - 60i = 150+300i. \tag{2.1}$$

Similarly, by the use of path 2, we get that

$$B_{(2,1,2)} = \begin{cases} 6B_{(1,1,2)} - B_{(0,1,2)}, \\ 6B_{(2,1,1)} - B_{(2,1,0)}. \end{cases}$$

Considering the first subpath we conclude that

$$B_{(2,1,2)} = 6B_{(1,1,2)} - B_{(0,1,2)} = 6(5+11i) - 11i = 30+55i. \tag{2.2}$$

Take into account the second subpath,

$$\begin{cases} B_{(2,1,1)} = 6B_{(1,1,1)} - B_{(0,1,1)} = 6(1+2i) - 2i = 6+10i, \\ B_{(2,1,0)} = 6B_{(1,1,0)} - B_{(0,1,0)} = 6(1+2i) - i = 6+5i. \end{cases}$$

Hence

$$\begin{aligned} B_{(2,1,2)} &= 6B_{(2,1,1)} - B_{(2,1,0)} \\ &= 6(6+10i) - (6+5i) \\ &= 30+55i \end{aligned}$$

which coincides with (2.2).

In order to calculate $B_{(2,0,2)}$ we have the following ways

$$B_{(2,0,2)} = \begin{cases} 6B_{(1,0,2)} - B_{(0,0,2)}, \\ 6B_{(2,0,1)} - B_{(2,0,0)}. \end{cases}$$

In the first subpath $B_{(0,0,2)} = 6B_{(0,0,1)} - B_{(0,0,0)} = 6i$ and so

$$B_{(2,0,2)} = 6B_{(1,0,2)} - B_{(0,0,2)} = 6(5+6i) - 6i = 30+30i \tag{2.3}$$

In the second subpath $B_{(2,0,1)} = 6B_{(1,0,1)} - B_{(0,0,1)} = 6(1+i) - i = 6+5i$ and $B_{(2,0,0)} = 6B_{(1,0,0)} - B_{(0,0,0)} = 6$. Hence

$$B_{(2,0,2)} = 6B_{(2,0,1)} - B_{(2,0,0)} = 6(6+5i) - 6 = 30+30i$$

which is equal to (2.3). Therefore, using path 2,

$$\begin{aligned} B_{(2,2,2)} &= 6B_{(2,1,2)} - B_{(2,0,2)} \\ &= 6(30+55i) - (30+30i) \\ &= 180+330i-30-30i \\ &= 150+300i \end{aligned}$$

which coincides with (2.1).

Finally, using path 3, we have

$$B_{(2,2,1)} = \begin{cases} 6B_{(1,2,1)} - B_{(0,2,1)}, \\ 6B_{(2,1,1)} - B_{(2,0,1)} \end{cases}$$

and $B_{(1,2,1)} = 5+11i$, $B_{(0,2,1)} = 11i$ as already calculated in path 1. So,

$$B_{(2,2,1)} = 6B_{(1,2,1)} - B_{(0,2,1)} = 6(5+11i) - 11i = 30+55i. \tag{2.4}$$

By the use of second subpath,

$$\begin{cases} B_{(2,1,1)} = 6B_{(1,1,1)} - B_{(0,1,1)} = 6(1+2i) - 2i = 6+10i, \\ B_{(2,0,1)} = 6B_{(1,0,1)} - B_{(0,0,1)} = 6(1+i) - i = 6+5i \end{cases}$$

and so,

$$\begin{aligned} B_{(2,2,1)} &= 6B_{(2,1,1)} - B_{(2,0,1)} \\ &= 6(6+10i) - (6+5i) \\ &= 36+60i-6-5i \\ &= 30+55i \end{aligned}$$

which is equal to (2.4). On the other hand,

$$B_{(2,2,0)} = \begin{cases} 6B_{(1,2,0)} - B_{(0,2,0)}, \\ 6B_{(2,1,0)} - B_{(2,0,0)}. \end{cases}$$

Using the first subpath we have $B_{(1,2,0)} = 5+6i$, $B_{(0,2,0)} = 6i$ as already calculated in path 1. So,

$$B_{(2,2,0)} = 6(5+6i) - 6i = 30+30i. \tag{2.5}$$

Using the second subpath, $B_{(2,2,0)} = 6+5i$, $B_{(2,0,0)} = 6$ as we have calculeted in path 2 and therefore

$$\begin{aligned} B_{(2,2,0)} &= 6B_{(2,1,0)} - B_{(2,0,2)} \\ &= 6(6+5i) - 6 \\ &= 30+30i \end{aligned}$$

which is exactly the same value found in (2.5). Hence, using path 3 we conclude that

$$\begin{aligned} B_{(2,2,2)} &= 6B_{(2,2,1)} - B_{(2,2,0)} \\ &= 6(30+55i) - (30+30i) \\ &= 180+330i-30-30i \\ &= 150+300i \end{aligned}$$

which, once more, coincides with (2.1).

Next, we will present some properties related to tridimensional balancing numbers.

Lemma 2.2. *Considering one unknown the following properties are valid for tridimensional balancing numbers*

1. $B_{(n,0,0)} = B_n$;
2. $B_{(0,m,0)} = B_{(0,0,m)} = B_m i$;
3. $B_{(n,1,0)} = B_{(n,0,1)} = B_n + (B_n - B_{n-1})i$;
4. $B_{(n,1,1)} = B_n + 2(B_n - B_{n-1})i$;
5. $B_{(1,m,0)} = B_{(1,0,m)} = (B_m - B_{m-1}) + B_m i$;

6. $B_{(0,m,1)} = B_{(0,1,m)} = (2B_m - B_{m-1})i$;
7. $B_{(1,m,1)} = (B_m - B_{m-1}) + B_{(0,m,1)}$;
8. $B_{(1,1,p)} = (B_p - B_{p-1}) + B_{(0,1,p)}$.

Proof. 1. The statement is trivial since the initial values are 0 and 1, and this provides the sequence of balancing numbers.

2. The proof is done by induction on m.

For $m = 0$ and, again, given the value of B_0 we have that $B_{(0,0,0)} = 0 = B_0 i$ and the proposition is true.

For $m = 1$ and, once again, considering the value of B_1 we have that $B_{(0,1,0)} = B_{(0,0,1)} = i = B_1 i$ and the proposition is also true.

Suppose the proposition is true for any integer less than or equal to m. Let us prove that it remains true for $m + 1$.

Then, using the reflection symmetry on the plane $y = z$, $B_{(0,m+1,0)} = B_{(0,0,m+1)}$ and hence

$$\begin{aligned} B_{(0,m+1,0)} = B_{(0,0,m+1)} &= 6B_m i - B_{m-1} i \\ &= (6B_m - B_{m-1})i \\ &= B_{m+1} i, \end{aligned}$$

which ends the proof 2.

3. The proof is done by induction on n.

For $n = 0$ and given that $B_0 = 0$ and $B_{-n} = -B_n$ in [5] and given one of the initial conditions of the sequence $\{B_{(n,m,p)}\}_{n,m,p \geq 0}$ we have that

$$B_{(0,1,0)} = B_{(0,0,1)} = i = B_0 + (B_0 - B_{-1})i,$$

and the proposition is true.

For $n = 1$ and, once again, taking into account the values of $B_0 = 0$ and $B_1 = 1$ we have that

$$B_{(1,1,0)} = B_{(1,0,1)} = 1 + i = B_1 + (B_1 - B_0)i,$$

and the proposition is valid.

Suppose the proposition is true for all values less than or equal to n. Let us show that it is still valid for $n + 1$.

Then, by the symmetry property mentioned before $B_{(n+1,1,0)} = B_{(n+1,0,1)}$ and hence

$$\begin{aligned} B_{(n+1,1,0)} = B_{(n+1,0,1)} &= 6\Big(B_n + (B_n - B_{n-1})i\Big) - \Big(B_{n-1} + (B_{n-1} - B_{n-2})i\Big) \\ &= 6B_n + 6(B_n - B_{n-1})i - B_{n-1} - (B_{n-1} - B_{n-2})i \\ &= (6B_n - B_{n-1}) + (6B_n - B_{n-1})i - (6B_{n-1} - B_{n-2})i \\ &= B_{n+1} + (B_{n+1} - B_n)i, \end{aligned}$$

and thus the property 3 is valid.

4. The proof is performed by induction on n.

For $n=0$ and, once again, given that $B_0=0$ and $B_{-n}=-B_n$ in [5] and taking into account one of the initial conditions of $\{B_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that

$$B_{(0,1,1)}=2i=B_0+2(B_0-B_{-1})i,$$

and the proposition is valid.

For $n=1$ and again, given that $B_0=0$ and $B_1=1$ and also one of the initial conditions of $\{B_{(n,m,p)}\}_{n,m,p\geq 0}$ we have that

$$B_{(1,1,1)}=1+2i=B_1+2(B_1-B_0)i,$$

and the proposition is true.

Suppose the proposition is true for any integer $k\leq n$. Let us prove that it remains true for $n+1$.

Then, by the first recurrence relation of Definition 2.1, by the induction hypothesis and by the recurrence relation expressed in (1.1), we get

$$\begin{aligned}
B_{(n+1,1,1)} &= 6B_{(n,1,1)}-B_{(n-1,1,1)}\\
&= 6\Big(B_n+2(B_n-B_{n-1})i\Big)-\Big(B_{n-1}+2(B_{n-1}-B_{n-2})i\Big)\\
&= 6B_n+12(B_n-B_{n-1})i-B_{n-1}-2(B_{n-1}-B_{n-2})i\\
&= (6B_n-B_{n-1})+2(6B_n-B_{n-1})i-2(6B_{n-1}-B_{n-2})i\\
&= B_{n+1}+2(B_{n+1}-B_n)i,
\end{aligned}$$

so the property 4 is valid.

Since the results of items 5, 6, 7 and 8 are similar to the previous ones, we have omitted their respective proofs. Note that item 5 comes directly from item 3 via the symmetry. □

Lemma 2.3. *Considering two unknowns the following properties are valid for tridimensional balancing numbers*

1. $B_{(n,m,0)}=B_{(n,0,m)}=B_n(B_m-B_{m-1})+(B_n-B_{n-1})B_mi;$
2. $B_{(0,m,p)}=B_mB_{(0,1,p)}-B_{m-1}B_pi;$
3. $B_{(n,m,1)}=B_n(B_m-B_{m-1})+(B_n-B_{n-1})B_{(0,m,1)};$
4. $B_{(n,1,p)}=B_n(B_p-B_{p-1})+(B_n-B_{n-1})B_{(0,1,p)};$
5. $B_{(1,m,p)}=(B_m-B_{m-1})(B_p-B_{p-1})+B_{(0,m,p)}.$

Proof. 1. The proof is done by induction on m.

For $m = 0$ and, once again, given that $B_0 = 0$ and $B_{-m} = -B_m$ in [5] and by item 1 of Lemma 2.3 we have that

$$B_{(n,0,0)} = B_n = B_n(B_0 - B_{-1}) + (B_n - B_{n-1})B_0 i,$$

and equality is true.

For $m = 1$ and again, given the values of $B_0 = 0$ and $B_1 = 1$ and Lemma 2.3, item 3 we have that

$$B_{(n,1,0)} = B_{(n,0,1)} = B_n + (B_n - B_{n-1})i = B_n(B_1 - B_0) + (B_n - B_{n-1})B_1 i,$$

which is true.

Suppose the proposition is valid for any integer less than or equal to m. Let us prove that it remains true for $m + 1$.

Then, using the second recurrence of Definition 2.1, the induction hypothesis and the recurrence relation expressed in (1.1), we get

$$\begin{aligned} B_{(n,m+1,0)} &= 6B_{(n,m,0)} - B_{(n,m-1,0)} \\ &= 6B_{(n,0,m)} - B_{(n,0,m-1)} \\ &= B_{(n,0,m+1)}, \end{aligned}$$

thus

$$\begin{aligned} B_{(n,m+1,0)} = B_{(n,0,m+1)} &= 6\Big(B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i\Big) \\ &\quad - \Big(B_n(B_{m-1} - B_{m-2}) + (B_n - B_{n-1})B_{m-1} i\Big) \\ &= 6B_n(B_m - B_{m-1}) + 6(B_n - B_{n-1})B_m i \\ &\quad - B_n(B_{m-1} - B_{m-2}) - (B_n - B_{n-1})B_{m-1} i \\ &= B_n(6B_m - 6B_{m-1} - B_{m-1} + B_{m-2}) \\ &\quad + (B_n - B_{n-1})(6B_m - B_{m-1})i \\ &= B_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{m+1} i, \end{aligned}$$

as we wanted to prove.

2. The proof is first done by induction on p.

For $p = 0$ and, again, given the value of B_0, the value of $B_{(0,1,0)}$ and Lemma 2.3, item 2 we have that

$$B_{(0,m,0)} = B_m i = B_m B_{(0,1,0)} - B_{m-1} B_0 i,$$

and the equality is valid.

For $p = 1$ and, once again, taking into account the value of B_1 and item 6 of Lemma 2.3 we have that

$$B_{(0,m,1)} = (2B_m - B_{m-1})i = B_m B_{(0,1,1)} - B_{m-1} B_1 i,$$

and the equality holds.

Suppose that the proposition is true for any integer $k \leq p$. Let us prove that it remains true for $p+1$. Then, by the third recurrence relation of Definition 2.1, the induction hypothesis and the recurrence relation (1.1), we get

$$\begin{aligned}
B_{(0,m,p+1)} &= 6B_{(0,m,p)} - B_{(0,m,p-1)} \\
&= 6\big(B_m B_{(0,1,p)} - B_{m-1}B_p i\big) - \big(B_m B_{(0,1,p-1)} - B_{m-1}B_{p-1}i\big) \\
&= 6B_m B_{(0,1,p)} - 6B_{m-1}B_p i - B_m B_{(0,1,p-1)} + B_{m-1}B_{p-1}i \\
&= B_m\big(6B_{(0,1,p)} - B_{(0,1,p-1)}\big) - B_{m-1}(6B_p - B_{p-1})i \\
&= B_m B_{(0,1,p+1)} - B_{m-1}B_{p+1}i,
\end{aligned}$$

what we wanted to prove.

3. The proof is first performed by induction on m.

For $m = 0$ and again, given that $B_0 = 0$ and $B_{-m} = -B_m$ in [5] and using Lemma 2.3, item 3 we have that

$$B_{(n,0,1)} = B_n + (B_n - B_{n-1})i = B_n(B_0 - B_{-1}) + (B_n - B_{n-1})B_{(0,0,1)},$$

and the equality is valid.

For $m = 1$ and again, given that $B_0 = 0$ and $B_1 = 1$ in [5], given the value of $B_{(0,1,1)}$ and item 4 of Lemma 2.3 we have that

$$B_{(n,1,1)} = B_n + 2(B_n - B_{n-1})i = B_n(B_1 - B_0) + (B_n - B_{n-1})B_{(0,1,1)},$$

which is correct.

Suppose that the proposition is valid for any integer $k \leq m$. Let us show that it is still valid for $m+1$. Then, using the second recurrence relation from Definition 2.1, the induction hypothesis and recurrence relation (1.1), we get

$$\begin{aligned}
B_{(n,m+1,1)} &= 6B_{(n,m,1)} - B_{(n,m-1,1)} \\
&= 6\Big(B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_{(0,m,1)}\Big) \\
&\quad - \Big(B_n(B_{m-1} - B_{m-2}) + (B_n - B_{n-1})B_{(0,m-1,1)}\Big) \\
&= 6B_n(B_m - B_{m-1}) + 6(B_n - B_{n-1})B_{(0,m,1)} \\
&\quad - B_n(B_{m-1} - B_{m-2}) - (B_n - B_{n-1})B_{(0,m-1,1)} \\
&= B_n\Big((6B_m - B_{m-1}) - (6B_{m-1} - B_{m-2})\Big) \\
&\quad + (B_n - B_{n-1})\big(6B_{(0,m,1)} - B_{(0,m-1,1)}\big) \\
&= B_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{(0,m+1,1)},
\end{aligned}$$

what we wanted to show.

Because the proof of items 4 and 5 is also done in the same way as the proof of the previous results, we have omitted their corresponding demonstrations. Note that item 5 is essentially the same as item 4 again via the symmetry. □

The following result is a relation that the balancing tridimensional numerical sequence satisfies.

Theorem 2.4. *For the non-negative integers m, n and p, the tridimensional balancing numbers are described in the form*

$$\begin{aligned}B_{(n,m,p)} &= B_n(B_m - B_{m-1})(B_p - B_{p-1})\\ &\quad + (B_n - B_{n-1})\Big(B_m(B_p - B_{p-1}) + (B_m - B_{m-1})B_p\Big)i.\end{aligned}$$

Proof. We start by doing the induction on p, with n and m fixed.

For $p = 0$ and, again, given that $B_0 = 0$ and $B_{-p} = -B_p$ in [5] and given item 1 of Lemma 2.3 we have that

$$\begin{aligned}B_{(n,m,0)} &= B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i\\ &= B_n(B_m - B_{m-1})(B_0 - B_{-1})\\ &\quad + (B_n - B_{n-1})\Big(B_m(B_0 - B_{-1}) + (B_m - B_{m-1})B_0\Big)i,\end{aligned}$$

and equality is verified.

For $p = 1$ and again, given that $B_0 = 0$ and $B_1 = 1$ and according to Lemma 2.3, items 6 and 3 we have that

$$\begin{aligned}B_{(n,m,1)} &= B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_{(0,m,1)}\\ &= B_n(B_m - B_{m-1}) + (B_n - B_{n-1})(2B_m - B_{m-1})i\\ &= B_n(B_m - B_{m-1})(B_1 - B_0)\\ &\quad + (B_n - B_{n-1})\Big(B_m(B_1 - B_0) + (B_m - B_{m-1})B_1\Big)i,\end{aligned}$$

which is true.

Suppose that the proposition is valid for any integer $k \leq p$. Let us show that it is still valid for $p+1$. Then, by the third recurrence relation of Definition 2.1, the induction hypothesis and the recurrence relation described in (1.1), we get

$$\begin{aligned}B_{(n,m,p+1)} &= 6B_{(n,m,p)} - B_{(n,m,p-1)}\\ &= 6\Big(B_n(B_m - B_{m-1})(B_p - B_{p-1})\\ &\quad + (B_n - B_{n-1})\Big(B_m(B_p - B_{p-1}) + (B_m - B_{m-1})B_p\Big)i\Big)\\ &\quad - \Big(B_n(B_m - B_{m-1})(B_{p-1} - B_{p-2})\\ &\quad + (B_n - B_{n-1})\Big(B_m(B_{p-1} - B_{p-2}) + (B_m - B_{m-1})B_{p-1}\Big)i\Big)\\ &= 6B_n(B_m - B_{m-1})(B_p - B_{p-1})\\ &\quad + 6(B_n - B_{n-1})\Big(B_m(B_p - B_{p-1}) + (B_m - B_{m-1})B_p\Big)i\\ &\quad - B_n(B_m - B_{m-1})(B_{p-1} - B_{p-2})\end{aligned}$$

$$
\begin{aligned}
&- (B_n - B_{n-1})\Big(B_m(B_{p-1} - B_{p-2}) + (B_m - B_{m-1})B_{p-1}\Big)i \\
= {} & B_n(B_m - B_{m-1})\Big((6B_p - B_{p-1}) - (6B_{p-1} - B_{p-2})\Big) \\
&+ (B_n - B_{n-1})\Big(B_m\Big((6B_p - B_{p-1}) - (6B_{p-1} - B_{p-2})\Big) \\
&+ (B_m - B_{m-1})(6B_p - B_{p-1})\Big)i \\
= {} & B_n(B_m - B_{m-1})(B_{p+1} - B_p) \\
&+ (B_n - B_{n-1})\Big(B_m(B_{p+1} - B_p) + (B_m - B_{m-1})B_{p+1}\Big)i,
\end{aligned}
$$

as we wanted to prove. Therefore, the theorem is valid. □

3. Some identities of tridimensional balancing numbers

From now on, we will explore some identities of this tridimensional sequence, using certain properties inherent to it. For the proof of the following results, we need some results concerning the sequence of balancing numbers, namely, items (a) and (b) of Corollary 2.3.6 in [11] and items of Proposition 2.6 in [4]. In order to facilitate to understand the end of the proofs for the reader, we present here these results.

Lemma 3.1 (Corollary 2.3.6 in [11]). *If n is a natural number, then*

1. $B_1 + B_3 + \ldots + B_{2n-1} = B_n^2$;
2. $B_2 + B_4 + \ldots + B_{2n} = B_n B_{n+1}$.

Lemma 3.2 (Proposition 2.6 in [4]). *If B_j, C_j and ST_j are the jth terms of the balancing sequence, Lucas-balancing sequence and square triangular sequence, respectively, then*

1. $B_{2n} = 2C_n B_n$;
2. $ST_n^2 = B_n^4$;
3. $C_n^2 = 8B_n^2 + 1 = 8ST_n + 1$;
4. $C_{2n} = 16B_n^2 + 1$;
5. $B_{n+2} - B_{n-2} = B_{(1,0,m)} = 12C_n$;
6. $\sum_{j=0}^{n} B_j = \frac{-1-B_n+B_{n+1}}{4}$;
7. $\sum_{j=0}^{n} C_j = \frac{2-C_n+C_{n+1}}{4}$;
8. $\sum_{j=0}^{n} ST_j = \frac{-1-ST_n+ST_{n+1}-2n}{32}$.

The next results are related to sums of tridimensional balancing numbers.

Proposition 3.3. *The sum of the first p numbers $B_{(n,m,t)}$ of odd index t is given by*

$$\sum_{q=1}^{p} B_{(n,m,2q-1)} = \Big(B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i\Big)\left(B_p^2 - B_p B_{p+1} + B_{2p}\right) + (B_n - B_{n-1})(B_m - B_{m-1})B_p^2 i.$$

Proof. By Theorem 2.4, we have

$$\sum_{q=1}^{p} B_{(n,m,2q-1)} = \sum_{q=1}^{p}\Big(B_n(B_m - B_{m-1})(B_{2q-1} - B_{2q-2}) + (B_n - B_{n-1})\Big(B_m(B_{2q-1} - B_{2q-2}) + (B_m - B_{m-1})B_{2q-1}\Big)i\Big).$$

Thus,

$$\begin{aligned}
\sum_{q=1}^{p} B_{(n,m,2q-1)} &= B_n(B_m - B_{m-1})\sum_{q=1}^{p}(B_{2q-1} - B_{2q-2}) \\
&\quad + (B_n - B_{n-1})\Big(B_m\sum_{q=1}^{p}(B_{2q-1} - B_{2q-2}) \\
&\quad + (B_m - B_{m-1})\sum_{q=1}^{p} B_{2q-1}\Big)i \\
&= B_n(B_m - B_{m-1})\left(\sum_{q=1}^{p} B_{2q-1} - \sum_{q=1}^{p} B_{2q-2}\right) \\
&\quad + (B_n - B_{n-1})\left(B_m\left(\sum_{q=1}^{p} B_{2q-1} - \sum_{q=1}^{p} B_{2q-2}\right)\right. \\
&\quad \left. + (B_m - B_{m-1})\sum_{q=1}^{p} B_{2q-1}\right)i \\
&= B_n(B_m - B_{m-1})\left(\sum_{q=1}^{p} B_{2q-1} - \left(B_0 + \sum_{q=1}^{p-2} B_{2q}\right)\right) \\
&\quad + (B_n - B_{n-1})\left(B_m\left(\sum_{q=1}^{p} B_{2q-1} - \left(B_0 + \sum_{q=1}^{p-2} B_{2q}\right)\right)\right. \\
&\quad \left. + (B_m - B_{m-1})\sum_{q=1}^{p} B_{2q-1}\right)i.
\end{aligned}$$

The result follows, using items (a) and (b) of Lemma 3.1 and $B_0 = 0$. □

Proposition 3.4. *The sum of the first p numbers $B_{(n,m,t)}$ of even index t can be described by*

$$\sum_{q=1}^{p} B_{(n,m,2q)} = \Big(B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i\Big)\left(B_pB_{p+1} - B_p^2\right) + (B_n - B_{n-1})(B_m - B_{m-1})B_pB_{p+1}i.$$

Proof. Applying Theorem 2.4, we have

$$\sum_{q=1}^{p} B_{(n,m,2q)} = \sum_{q=1}^{p}\Big(B_n(B_m - B_{m-1})(B_{2q} - B_{2q-1}) + (B_n - B_{n-1})(B_m(B_{2q} - B_{2q-1}) + (B_m - B_{m-1})B_{2q})i\Big).$$

Thus,

$$\begin{aligned}\sum_{q=1}^{p} B_{(n,m,2q)} &= B_n(B_m - B_{m-1})\sum_{q=1}^{p}(B_{2q} - B_{2q-1}) \\ &\quad + (B_n - B_{n-1})\Big(B_m\sum_{q=1}^{p}(B_{2q} - B_{2q-1}) \\ &\quad + (B_m - B_{m-1})\sum_{q=1}^{p}B_{2q}\Big)i \\ &= B_n(B_m - B_{m-1})\left(\sum_{q=1}^{p}B_{2q} - \sum_{q=1}^{p}B_{2q-1}\right) \\ &\quad + (B_n - B_{n-1})\left(B_m\left(\sum_{q=1}^{p}B_{2q} - \sum_{q=1}^{p}B_{2q-1}\right)\right. \\ &\quad \left.+ (B_m - B_{m-1})\sum_{q=1}^{p}B_{2q}\right)i.\end{aligned}$$

By items (b) and (a) of Lemma 3.1, the result follows. □

The third identity is the last one for the sum of the first m terms and is related to the third component of the index, this index being any generic number. It can also be understood as the identity relating to the sum of the first components of the even and odd indices, i.e. the identity resulting from the sum of the first components of the even and odd indices.

Proposition 3.5. *The sum of the first p numbers $B_{(n,m,t)}$, with index t a non-negative integer, can be described as follows*

$$\sum_{q=1}^{p} B_{(n,m,q)} = B_n(B_m - B_{m-1})B_p + (B_n - B_{n-1})\Big(B_mB_p$$

$$+\frac{1}{4}(B_m - B_{m-1})(B_{p+1} - B_p - 1)\Big)i.$$

Proof. Using Theorem 2.4, we have

$$\sum_{q=1}^{p} B_{(n,m,q)} = \sum_{q=1}^{p}\Big(B_n(B_m - B_{m-1})(B_q - B_{q-1}) \\ + (B_n - B_{n-1})(B_m(B_q - B_{q-1}) + (B_m - B_{m-1})B_q)i\Big).$$

Thus,

$$\begin{aligned}\sum_{q=1}^{p} B_{(n,m,q)} &= B_n(B_m - B_{m-1})\sum_{q=1}^{p}(B_q - B_{q-1}) \\ &\quad + (B_n - B_{n-1})\Big(B_m\sum_{q=1}^{p}(B_q - B_{q-1}) + (B_m - B_{m-1})\sum_{q=1}^{p}B_q\Big)i \\ &= B_n(B_m - B_{m-1})\left(\sum_{q=1}^{p}B_q - \sum_{q=1}^{p}B_{q-1}\right) \\ &\quad + (B_n - B_{n-1})\left(B_m\left(\sum_{q=1}^{p}B_q - \sum_{q=1}^{p}B_{q-1}\right) + (B_m - B_{m-1})\sum_{q=1}^{p}B_q\right)i \\ &= B_n(B_m - B_{m-1})\left(\sum_{q=1}^{p}B_q - \left(B_0 + \sum_{q=1}^{p-1}B_q\right)\right) \\ &\quad + (B_n - B_{n-1})\left(B_m\left(\sum_{q=1}^{p}B_q - \left(B_0 + \sum_{q=1}^{p-1}B_q\right)\right)\right. \\ &\quad \left. + (B_m - B_{m-1})\sum_{q=1}^{p}B_q\right)i.\end{aligned}$$

The result follows Lemma 3.2 and the fact that $B_0 = 0$. □

Since $B_{(n,m,2q-1)} = B_{(n,2q-1,m)}$, $B_{(n,m,2q)} = B_{(n,2q,m)}$ and $B_{(n,m,q)} = B_{(n,q,m)}$ by the symmetry property, then immediately the results of Propositions 3.6, 3.7 and 3.8 follow.

Proposition 3.6. *The sum of the first m numbers $B_{(n,t,p)}$ of odd index t can be described by*

$$\sum_{l=1}^{m} B_{(n,2l-1,p)} = \Big(B_n(B_p - B_{p-1}) + (B_n - B_{n-1})B_p i\Big)(B_m^2 - B_m B_{m+1} + B_{2m}) \\ + (B_n - B_{n-1})B_m^2(B_p - B_{p-1})i.$$

Proposition 3.7. *The sum of the first m numbers $B_{(n,t,p)}$ of even index t is given by*

$$\sum_{l=1}^{m} B_{(n,2l,p)} = \Big(B_n(B_p - B_{p-1}) + (B_n - B_{n-1})B_p i\Big)\big(B_m B_{m+1} - B_m^2\big) + (B_n - B_{n-1})B_m B_{m+1}(B_p - B_{p-1})i.$$

Proposition 3.8. *The sum of the first m numbers $B_{(n,t,p)}$, with index t a non-negative integer, is given by*

$$\sum_{l=1}^{m} B_{(n,l,p)} = B_n B_m (B_p - B_{p-1}) + (B_n - B_{n-1})\Big(\frac{1}{4}(B_{m+1} - B_m - 1)(B_p - B_{p-1}) + B_m B_p\Big)i.$$

We have omitted the proofs of the following results since they are analogous to the identities concerning partial sums mentioned before.

Proposition 3.9. *The sum of the first n numbers $B_{(t,m,p)}$ of odd index t is given as follows*

$$\sum_{k=1}^{n} B_{(2k-1,m,p)} = B_n^2 (B_m - B_{m-1})(B_p - B_{p-1}) + \big(B_n^2 - B_n B_{n+1} - B_{2n}\big)\Big(B_m(B_p - B_{p-1}) + (B_m - B_{m-1})B_p\Big)i.$$

Proposition 3.10. *The sum of the first n numbers $B_{(t,m,p)}$ of even index t can be described by*

$$\sum_{k=1}^{n} B_{(2k,m,p)} = B_n B_{n+1} (B_m - B_{m-1})(B_p - B_{p-1}) + \big(B_n B_{n+1} - B_n^2\big)\Big(B_m(B_p - B_{p-1}) + (B_m - B_{m-1})B_p\Big)i.$$

Proposition 3.11. *The sum of the first n numbers $B_{(t,m,p)}$, with t a non-negative integer, can be described by*

$$\sum_{k=1}^{n} B_{(k,m,p)} = \frac{1}{4}(B_{n+1} - B_n - 1)(B_m - B_{m-1})(B_p - B_{p-1}) + B_n\Big(B_m(B_p - B_{p-1}) + (B_m - B_{m-1})B_p\Big)i.$$

4. Conclusion

This article continues the work related to the n-dimensional versions of the balancing number sequence. We introduce the tridimensional recurrence relations of the balancing number sequence and study some of its properties, as well as some of its sum identities. The results presented in this article are considered a contribution to the field of mathematics and offer an opportunity for researchers interested in number sequences.

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