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Hosszú type functional equations on Lie algebras

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Abstract. In this paper the functional equation $f(x+y-[x,y])+f([x,y]) = f(x)+f(y)$ is investigated with the unknown continuous function $f\colon L \to Y$ which satisfies the equation for all $x, y \in L$ where L is a finite dimensional normed real vector space equipped with a bilinear, anti-commutative operation $[\cdot,\cdot]\colon L^2 \to L$, and Y is finite dimensional, normed, real vector space.

Some Pexider generalisations of the above equation are also investigated in this paper.

Keywords: functional equations, Pexider functional equations, Lie brackets, Lie algebras

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1. Introduction

The main purpose of this paper is to give the continuous solution of the functional equation

$$f(x+y-[x,y])+f([x,y]) = f(x)+f(y) \quad (x, y \in L) \tag{1.1}$$

with the unknown function $f\colon L \to Y$ where L is a finite dimensional normed real vector space with a bilinear, and anti-commutative binary operation $[\cdot,\cdot]\colon L^2 \to L$, and Y is a finite dimensional normed real vector space. The continuous solution of equation (1.1) can be found in fourth section. We also investigate some Pexider generalizations of equation (1.1).

- In the fifth section the continuous solution of equation

$$f(x+y-[x,y])+f([x,y]) = h(x)+h(y) \quad (x, y \in L) \tag{1.2}$$

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with unknown functions $f\colon L\to Y$, and $h\colon L\to Y$.

- In the sixth section the continuous solution of equation

$$f(x+y-[x,y])+g([x,y])=g(x)+g(y)\quad (x,y\in L) \tag{1.3}$$

with unknown functions $f\colon L\to Y$, and $g\colon L\to Y$.

- In the seventh section the continuous solution of equation

$$f(x+y-[x,y])+g([x,y])=f(x)+f(y)\quad (x,y\in L) \tag{1.4}$$

with unknown functions $f\colon L\to Y$, $g\colon D:=\{[x,y]\mid x\in L, y\in L\}\to Y$, and $h\colon L\to Y$.

- In the eighth section the continuous solution of equation

$$f(x+y-[x,y])+g([x,y])=h(x)+h(y)\quad (x,y\in L) \tag{1.5}$$

with unknown functions $f\colon L\to Y$, $g: D\to Y$, and $h\colon L\to Y$ can be found.

Equations (1.1)–(1.5) belong to the family of the Hosszú-type functional equations. The Hosszú equation is equation

$$f(x+y-xy)+f(xy)=f(x)+f(y)\quad (x,y\in \mathbb{R}) \tag{1.6}$$

with the unknown function $f\colon\mathbb{R}\to\mathbb{R}$. This equation was proposed to solve the first time at the International Symposium on Functional Equation Conference held in Zakopane (Poland) in 1967 by M. Hosszú.

The solution of equation (1.6) is in the form

$$f(x)=A(x)+C\quad (x\in\mathbb{R}) \tag{1.7}$$

where $A\colon\mathbb{R}\to\mathbb{R}$ is a Cauchy-additive function, that is function A satisfies the functional equation

$$A(x+y)=A(x)+A(y)\quad (x,y\in\mathbb{R}), \tag{1.8}$$

and C is a real constant.

The general solution of equation (1.6) can be found in the papers of Z. Daróczy [7, 8], D. Blanusa [3], and H. Swiatak [23, 24]. Paper [18] contains general solutions of some Pexider generalizations of equation (1.6) see also [19, 20]. In paper [21] the Hosszú functional inequality is shown. In papers [9–12] the domain of unknown function is the set of all positive integers or some abstract algebraic structures for example rings and fields.

In his paper Z. Daróczy [6] investigated functional equation

$$f(x+y-x\circ y)+g(x\circ y)=g(x)+g(y)\quad (x,y\in\mathbb{R}) \tag{1.9}$$

with unknown functions $f, g\colon \mathbb{R} \to \mathbb{R}$, and the binary operation $\circ$ defined by

$$x \circ y := \ln(e^x + e^y) \quad (x, y \in \mathbb{R}). \tag{1.10}$$

Equation (1.9) is a Hosszú type functional equation. It is also worth to mention the paper [14] which contains the equation

$$f(|x + y - x \circ y|) + g(x \circ y) = g(x) + g(y) \quad (x, y \in \mathbb{R}_+) \tag{1.11}$$

with unknown functions $f, g\colon \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\} \to \mathbb{R}$, and the binary operation $\circ$ is defined by

$$x \circ y := (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c \quad (x, y \in \mathbb{R}_+) \tag{1.12}$$

where $c \in \mathbb{R} \setminus \{0, 1\}$ is a fixed constant. In this paper the general solution of equation (1.11) is given in only cases $c = 2, 3$.

It is easy to see that if the operation $\circ$ is associative for example the operation $\circ$ is defined by (1.10) or (1.12) so the general solution of equations (1.9), can be given by using the Hosszú cycle [16] which is also a functional equation

$$F(x \circ y, z) + F(x, y) = F(x, y \circ z) + F(y, z)$$

with unknown function F over the required structure.

The general solution of equation (1.9), and (1.11) is also in the form of (1.7).

To solve functional equations we use a series of substitutions. Although the notation that we use for substitution is clear by the context, but for the sake of the readers now we explain it in detail by the following example.

Example 1.1. Let us assume that the function $A\colon \mathbb{R} \to \mathbb{R}$ satisfies equation (1.8). In this equation take the substitution

$$x \longleftarrow x + y, \quad y \longleftarrow xy,$$

in other words let us write $x + y$ instead of x, and xy instead of y thus we obtain the equation

$$A(x + y + xy) = A(x + y) + A(xy) \quad (x, y \in \mathbb{R}).$$

2. Some examples for finite dimensional real Lie algebras

Definition 2.1. $L = L([\cdot, \cdot])$ is a Lie algebra (S. Lie 1870 [4]) if L is a vector space over a field $F(+, \cdot)$, and the binary operation $[\cdot, \cdot]\colon L \times L \to L$ (so called Lie bracket)

- is bilinear, in the sense that it is linear in both variable;
- is alternating in the sense that $[x, x] = 0$ for all $x \in L$;

- has the Jacobi Identity, that is

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad (x,y,z \in L).$$

Remark 2.2. If L is an algebra over a field F with $\mathrm{char}(F) \neq 2$ ($\mathrm{char}(F)$ denotes the characteristic of the field $F(+,\cdot)$), and the binary operation $[\cdot,\cdot]\colon L^2 \to L$ is additive in its both variable then the following assertions are equivalent.

- The operation $[\cdot,\cdot]$ has alternating property;
- The operation $[\cdot,\cdot]$ is anti-commutative, that is $[x,y] = -[y,x]$ for all $x,y \in L$.

Example 2.3. $\mathbb{R}^3(\times)$ is a Lie algebra where $\mathbb{R}^3$ is the three dimensional vector space, and the operation $\times$ is the cross product of vectors defined by

$$v \times w := \det \begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix},$$

where (i,j,k) is the standard base of $\mathbb{R}^3$, $v = (v_1,v_2,v_3)^\top$, $w = (w_1,w_2,w_3)^\top$ with respect to the standard base.

Example 2.4. Now we collect additional examples for finite dimensional normed real algebras equipped with a bilinear, and anti-commutative binary operator $[\cdot,\cdot]$.

The simplest examples for such algebras (say denoted by X) that the multiplication is not commutative, and the binary operator $[\cdot,\cdot]$ is the well-known commutator defined by

$$[x,y] := xy - yx \quad (x,y \in X).$$

There exist many famous algebras having the above properties.

- The algebra of the square matrices, in a bit more details, $X := \mathbb{R}^{n\times n}$ where $n > 1$, and the norm on X is the well-known Frobenius norm [13] which is defined by

  $$\|M\| := \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |m_{ij}|^2\right)^{\frac{1}{2}} \quad (M \in \mathbb{R}^{n\times n}),$$

  that is, this norm is the usual Euclidean norm on $\mathbb{R}^{n\times n}$ [22].

- The Quaternion $\mathbb{H}$ W. R. Hamilton 1843 [15]

  $$\mathbb{H} := \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}\}.$$

  is a normed division algebra which is non-commutative.

- The Octonion J. T. Graves 1843, and independently A. Cayley 1845 [5]

  $$\mathbb{O} := \{a_0e_0 + a_1e_1 + \cdots + a_7e_7 \mid a_i \in \mathbb{R}\ (i = 0,1,\ldots,7)\}$$

  is a normed division algebra which is non-commutative, and non-associative.

3. On continuous additive functions

Definition 3.1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be Cauchy additive if

$$f(x+y) = f(x) + f(y) \tag{3.1}$$

for all $x, y \in \mathbb{R}$. In more general, if $X = X(+)$, $Y = Y(+)$ are groupoids, and the function $f\colon X \to Y$ satisfies equation (3.1) for all $x, y \in X$ then the function f is said to be additive [1, 2, 17].

Remark 3.2 ([17]). If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous additive function then there exists a constant $c \in \mathbb{R}$ such that

$$f(x) = cx \quad (x \in \mathbb{R}).$$

Now we prove a generalisation of Remark 3.2.

Theorem 3.3. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a continuous additive function, then there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that*

$$f(x) = Ax \quad (x \in \mathbb{R}^n). \tag{3.2}$$

Proof. Let $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, and (q_n) is the sequence of rationals with $\lim_{n\to\infty} q_n = \lambda$. Since the additive functions are $\mathbb{Q}$-homogeneous [17] we obtain that

$$f(\lambda x) = f\Big(\lim_{n\to\infty} q_n x\Big) = \lim_{n\to\infty} f(q_n x) = \lim_{n\to\infty} q_n f(x) = \lambda f(x),$$

whence f is homogeneous so it is linear. By the Matrix Representation Theorem of linear functions we obtain the equation (3.2). □

4. On the continuous solution of equation (1.1)

In the sequel L will be an n-dimensional normed real vector space equipped with a bilinear, and anti-commutative operation $[\cdot,\cdot]\colon L^2 \to L$, and Y will be an m-dimensional normed real vector space.

We will also frequently use the notation for functions $f\colon L \to Y$ that the function $f_0\colon L \to Y$ is defined by

$$f_0(x) := f(x) - f(0) \quad (x \in L).$$

It is easy to see that $f_0(0) = 0$.

The following proposition is evident.

Proposition 4.1. *If the function $f\colon L \to Y$ satisfies equation (1.1) for all $x, y \in L$ then the function f_0 also satisfies equation (1.1), that is*

$$f_0(x + y - [x,y]) + f_0([x,y]) = f_0(x) + f_0(y) \quad (x, y \in L). \tag{4.1}$$

Proposition 4.2. *If the function $f\colon L \to Y$ satisfies equation (1.1), then the function f_0 is 2-additive, that is*

$$f_0(2x) = 2f_0(x) \quad (x \in L). \tag{4.2}$$

Proof. Take the substitution $y \longleftarrow x$ in equation (4.1) thus we obtain equation (4.2). $\square$

Remark 4.3. It is well-known that if a function $f\colon L \to Y$ is 2-additive that it is also $2^{\mathbb{Z}}$-homogeneus, that is

$$2^z f(x) = f(2^z x) \quad (x \in L, z \in \mathbb{Z}).$$

Theorem 4.4. *If the continuous function $f\colon L \to Y$ satisfies equation (1.1) for all $x, y \in L$ then there exists a matrix $A \in \mathbb{R}^{m\times n}$, and the constant $C \in \mathbb{R}^m$ such that*

$$f(x) = Ax + C \quad (x \in L). \tag{4.3}$$

Proof. Let us assume that the function $f\colon L \to Y$ satisfies equation (1.1). By Proposition 4.1 the function f_0 satisfies equation (4.1). Let $n \in \mathbb{Z}_+$ be arbitrary fixed constant. Take the substitution

$$x \longleftarrow \frac{1}{2^n}x \quad y \longleftarrow \frac{1}{2^n}y$$

in equation (4.1) thus we obtain that

$$\begin{aligned} &f_0\left(\frac{1}{2^n}x + \frac{1}{2^n}y - \frac{1}{2^n}\frac{1}{2^n}[x,y]\right) + f_0\left(\frac{1}{2^n}\frac{1}{2^n}[x,y]\right) \\ &= f_0\left(\frac{1}{2^n}x\right) + f_0\left(\frac{1}{2^n}y\right) \quad (x, y \in L). \end{aligned} \tag{4.4}$$

Multiply by 2^n both side of equation (4.4) by Proposition 4.2 and Remark 4.3 we obtain that

$$f_0\left(x + y - \frac{1}{2^n}[x,y]\right) + f_0\left(\frac{1}{2^n}[x,y]\right) = f_0(x) + f_0(y) \quad (x, y \in L). \tag{4.5}$$

Take the limit $n \to \infty$ in equation (4.5) thus we have that function f_0 is additive, whence by Theorem 3.3 we obtain that there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$f_0(x) = Ax \quad (x \in L)$$

whence we obtain equation (4.3) with constant $C = f(0) \in \mathbb{R}^m$. $\square$

5. The continuous solution of equation (1.2)

Proposition 5.1. *If the functions $f, h\colon L \to Y$ satisfy the equation (1.2) then the functions f_0, and h_0 satisfy the equation*

$$f_0(x+y-[x,y]) + f_0([x,y]) = h_0(x) + h_0(y) \quad (x, y \in L) \tag{5.1}$$

Proof. Since

$$2f(0) = 2h(0) \tag{5.2}$$

thus from equation (1.2) we easily obtain equation (5.1). □

Proposition 5.2. *If the functions $f_0, g_0\colon L \to Y$ satisfy equation (5.1) then*

$$f_0 = h_0 \tag{5.3}$$

so the function f_0 satisfies equation (4.1).

Proof. The statement can be easily obtained from equation (5.1) taking the substitution $y \longleftarrow 0$. □

Theorem 5.3. *If the continuous functions $f, h : L \to Y$ satisfy equation (1.2) then there exists a matrix $A \in \mathbb{R}^{m\times n}$, and a constant $C \in \mathbb{R}^m$ such that*

$$f(x) = h(x) = Ax + C \quad (x \in L). \tag{5.4}$$

Proof. By Proposition 5.2 we have that f_0 satisfies equation (4.1) whence by Theorem 4.4, by equations (5.2), and (5.3) using Theorem 3.3 we obtain equation (5.4). □

6. The continuous solution of equation (1.3)

Proposition 6.1. *If the functions $f\colon L \to Y$, and $g\colon L \to Y$ satisfy equation (1.3), then the functions f_0, and g_0 also satisfy equation (1.3), that is*

$$f_0(x+y-[x,y]) + g_0([x,y]) = g_0(x) + g_0(x) \quad (x, y \in L). \tag{6.1}$$

Proof. Since

$$f(0) = g(0) \tag{6.2}$$

thus equation (6.1) can be easily obtained from equation (1.3). □

Proposition 6.2. *If the functions f_0, and g_0 satisfy equation (6.1), then*

$$f_0 = g_0. \tag{6.3}$$

Proof. The equation (6.3) can be easily obtained from equation (6.1) taking the substitution $y \longleftarrow 0$. □

Theorem 6.3. *If the continuous functions $f\colon L \to Y$, and $g\colon L \to Y$ satisfy equation (1.3) then there exists a matrix $A \in \mathbb{R}^{m\times n}$, and constant $C \in \mathbb{R}^m$ such that*

$$f(x) = g(x) = Ax + C \quad (x \in L).$$

Proof. The proof can be easily obtained from equations (6.2), and (6.3) by Theorem 4.4, and Theorem 3.3. □

7. The continuous solution of equation (1.4)

Proposition 7.1. *If the functions $f\colon L \to Y$, and $g\colon L \to Y$ satisfy equation (1.4), then the functions f_0, and g_0 also satisfy equation (1.4), that is*

$$f_0(x + y - [x, y]) + g_0([x, y]) = f_0(x) + f_0(y) \quad (x, y \in L). \tag{7.1}$$

Proof. Since

$$f(0) = g(0) \tag{7.2}$$

thus we evidently obtain the equation (7.1) from equation (1.4). □

Proposition 7.2. *If the functions f_0 and g_0 satisfy the equation (7.1) then the function f_0 is 2-additive, that is*

$$f_0(2x) = 2f_0(x) \quad (x \in L).$$

Proof. The proof can be easily obtained from (7.1) by substitution $y \longleftarrow x$. □

In the sequence we shall use the notation for functions $f\colon L \to Y$ that the functions $f_0^-, f_0^+\colon L \to L$ are defined by

$$f_0^-(x) := f_0(x) - f_0(-x) \quad (x \in L)$$
$$f_0^+(x) := f_0(x) + f_0(-x) \quad (x \in L)$$

It is easy to see that $f_0^-(0) = 0$, and $f_0^+(0) = 0$, moreover, $f_0^-(x) = f^-(x)$, and $f_0^+(x) = f^+(x) - 2f(0)$ for all $x \in L$.

Proposition 7.3. *If the functions $f_0\colon L \to Y$, and $g_0\colon L \to Y$ satisfy equation (7.1) then functions f_0, f_0^- satisfy the equation*

$$f_0(x + y - [x, y]) - f_0(-x - y - [x, y]) = f_0^-(x) + f_0^-(y) \quad (x, y \in L). \tag{7.3}$$

Proof. Take the substitution

$$x \longleftarrow -x, \quad y \longleftarrow -y$$

in equation (7.1) thus we obtain equation

$$f_0(-x - y - [x, y]) + g_0([x, y]) = f_0(-x) + f_0(-y) \quad (x, y \in L). \tag{7.4}$$

Subtract the equation (7.4) from equation (7.1) thus we obtain equation (7.3). □

Proposition 7.4. *If the continuous functions f_0, f_0^- satisfy equation (7.3) then the function f_0^- is additive.*

Proof. By 2-additivity of the functions f_0, and f_0^- (Proposition 7.2), analogously to the way that we got equation (4.5), we easily obtain that the equation

$$\begin{aligned} &f_0\left(x+y-\frac{1}{2^n}[x,y]\right)-f_0\left(-x-y-\frac{1}{2^n}[x,y]\right) \\ &= f_0^-(x)+f_0^-(y) \quad (x,y\in L). \end{aligned} \tag{7.5}$$

Taking the limit $n\to\infty$ in equation (7.5) we obtain that the function f_0^- is additive. □

Proposition 7.5. *If the continuous function f_0, and g_0 satisfy equation (7.1) then*

$$f_0^+(x)=0 \quad (x\in L), \tag{7.6}$$

in other words, the function f_0 is even.

Proof. Assume that the functions f_0, and g_0 satisfy equation (7.1). Add the equation (7.4) to equation (7.1) thus we have that

$$\begin{aligned} &f_0(x+y-[x,y])+f_0(-x-y-[x,y])+2g_0([x,y]) \\ &= f_0^+(x)+f_0^+(y) \quad (x,y\in L). \end{aligned} \tag{7.7}$$

Take the substitution $y\longleftarrow -x$ in equation (7.7) whence we easily obtain equation (7.6). □

The following Corollary is the immediate consequence of Proposition 7.4, and Proposition 7.5.

Corollary 7.6. *If the functions f_0, and g_0 satisfy equation (7.1), then the function f_0 is additive.*

Theorem 7.7. *If the continuous functions $f\colon L\to Y$, $g\colon D\to L$ satisfy equation (1.4) then there exists a matrix $A\in\mathbb{R}^{m\times n}$ and a $C\in\mathbb{R}^m$ such that*

$$f(x)=g(x)=Ax+C \quad (x\in\mathbb{R}^n). \tag{7.8}$$

Proof. By Corollary 7.6 we have that the functions f_0 is additive, thus by equation (7.1) we obtain that

$$f_0(x)+f_0(y)-f_0([x,y])+g_0([x,y])=f_0(x)+f_0(y) \quad (x,y\in L),$$

that is

$$f_0(z)=g_0(z) \quad (z\in D). \tag{7.9}$$

From equation (7.2), and equation (7.9) by Theorem 3.3 we obtain that functions f and g is in the form of (7.8) with constant $C=f(0)=g(0)\in\mathbb{R}^n$. □

8. The continuous solution of equation (1.5)

Proposition 8.1. *If the functions $f\colon L \to Y$, $g\colon D \to Y$, and $h\colon L \to Y$ satisfy equation (1.5), then the functions f_0, g_0, and h_0 satisfy equation*

$$f_0(x + y - [x, y]) + g_0([x, y]) = h_0(x) + h_0(y) \quad (x, y \in L). \tag{8.1}$$

Proof. Since

$$f(0) + g(0) = 2h(0)$$

thus from equation (1.5) one can easily obtains equation (8.1). □

Proposition 8.2. *If the functions f_0, g_0, and h_0 satisfy equation (8.1) then*

$$f_0 = h_0$$

thus the functions f_0, and g_0 satisfy equation (7.1).

Proof. The proof can be easily obtained from equation (8.1) by the substitution $y \longleftarrow 0$. □

The following theorem is a simple consequence of Proposition 8.2, and Theorem 7.7.

Theorem 8.3. *If the continuous functions $f\colon L \to Y$, $g\colon D \to Y$, and $h\colon L \to Y$ satisfy equation (1.5), then there exists a matrix $A \in \mathbb{R}^{m\times n}$ and constants C_1, C_2, $C_3 \in \mathbb{R}^m$ such that*

$$\begin{aligned} f(x) &= Ax + C_1 \quad (x \in L) \\ g(x) &= Ax + C_2 \quad (x \in L) \\ h(x) &= Ax + C_3 \quad (x \in L) \end{aligned}$$

with $C_1 + C_2 = 2C_3$.

Open problem. *The general solutions of equations (1.1)–(1.5) with appropriate unknown functions are open problems, in the cases where L is an arbitrary Lie-algebra, and Y is an arbitrary vector space.*

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