Annales Mathematicae et Informaticae
Accepted manuscript
DOI: 10.33039/ami.2025.03.002
URL: https://ami.uni-eszterhazy.hu

On the Sum of Divisors function of linearly recurrent sequences

Florian Luca[a], **Makoko Campbell Manape**[b]

[a]Mathematics Division, Stellenbosch University, Stellenbosch, South Africa
fluca@sun.ac.za

[b]School of Maths, Wits University, Johannesburg, South Africa
2127079@students.wits.ac.za

Abstract. Let $k \geq 2$ be an integer and $\sigma_k(n)$ be the sum of the kth powers of the positive divisors of n. In this paper, we prove that if $(U_n)_{n\geq 1}$ is any nondegenerate linearly recurrent sequence of integers whose general term is up to sign not a polynomial in n, then the inequality $\sigma_k(|U_n|) \leq |U_{\sigma_k(n)}|$ holds for all n sufficiently large, and the inequality $\sigma(|U_{n^k}|) \leq |U_{\sigma_k(n)}|$ holds for almost all natural numbers n, where $\sigma(n) = \sigma_1(n)$. In fact, the second inequality fails on a set of $n \leq x$ of counting function $O(x/\log x)$.

Keywords: Linear recurrence sequences, Sum of Divisors function

AMS Subject Classification: 11B39

1. Introduction

Let $\mathbf{U} := (U_n)_{n\geq 1}$ be a linearly recurrent sequence of integers. Such a sequence satisfies a recurrence of the form

$$U_{n+d} = a_1 U_{n+d-1} + \cdots + a_d U_n \quad \text{for all} \quad n \geq 1$$

with integers $a_1, \ldots, a_d$, where $U_1, \ldots, U_d$ are integers. Assuming d is minimal, U_n can be represented as

$$U_n = \sum_{i=1}^{s} P_i(n) \alpha_i^n, \tag{1.1}$$

where

$$\Psi(X) := X^d - a_1 X^{d-1} - \cdots - a_d = \prod_{i=1}^{s} (X - \alpha_i)^{\nu_i}$$

Submitted: December 12, 2024
Accepted: March 13, 2025
Published online: March 27, 2025

is the characteristic polynomial of $\mathbf{U}$, $\alpha_1, \ldots, \alpha_s$ are the distinct roots of $\Psi(X)$ with multiplicities $\nu_1, \ldots, \nu_s$, respectively, and $P_i(X)$ is a polynomial of degree $\nu_i - 1$ with coefficients in $\mathbb{Q}(\alpha_i)$. The sequence is nondegenerate if α_i/α_j is not a root of 1 for any $i \neq j$ in $\{1, \ldots, s\}$.

Recently (see [4]), we proved that if $(U_n)_{n\geq 1}$ is any nondegenerate linearly recurrent sequence of integers such that $|U_n|$ is not a polynomial in n for large values of n, then the inequalities

$$\phi(|U_n|) \geq |U_{\phi(n)}| \quad \text{and} \quad \sigma(|U_n|) \leq |U_{\sigma(n)}|$$

hold for almost all natural numbers n. In fact, we showed that the set of positive integers $n \leq x$ for which the above inequalities fail has counting function $O(x/\log x)$.

Let $\sigma_k(n)$ be the sum of the kth powers of the positive divisors of n. We denote $\sigma_1(n)$ as $\sigma(n)$. In this paper, we continue where we left off in the last paper [4] by considering the inequalities of the arithmetic function $\sigma_k(n)$ for $k \geq 2$.

In this paper, we prove the following theorems. Recall that if $f(x)$ and $g(x)$ are functions defined on $\mathbb{R}_+$ with values in $\mathbb{R}_+$ we write $f(x) = O(g(x))$ and $f(x) = o(g(x))$ if the inequality $f(x) < Kg(x)$ holds with some constant $K > 0$ and all $x > x_0$, and $\lim_{x\to\infty} f(x)/g(x) = 0$, respectively. Further, the notations $f(x) \ll g(x)$ and $g(x) \gg f(x)$ are equivalent to $f(x) = O(g(x))$.

Theorem 1.1. *Let $k \geq 2$ be an integer. Let $\mathbf{U} := (U_n)_{n\geq 1}$ be a nondegenerate linearly recurrent sequence of integers such that $|U_n|$ is not a polynomial in n for all large n and let x be a large real number. Then the inequality*

$$\sigma_k(|U_n|) \leq |U_{\sigma_k(n)}|$$

holds for all n sufficiently large.

Theorem 1.2. *Let $k \geq 1$ be an integer. Let $\mathbf{U} := (U_n)_{n\geq 1}$ be a nondegenerate linearly recurrent sequence of integers such that $|U_n|$ is not a polynomial in n for all large n and let x be a large real number. Then the inequality*

$$\sigma(|U_{n^k}|) \leq |U_{\sigma_k(n)}|$$

holds for almost all positive integers n. In fact, the set of positive integers $n \leq x$ for which it fails is of cardinality $O(x/\log x)$.

2. Preliminary results

The following lemma follows from Theorem 323 in [3].

Lemma 2.1. *Let $n \geq 3$. We then have*

$$\sigma(n) \ll n \log\log n.$$

The following lemma follows from a result on page 116 in [6].

Lemma 2.2. *Let $k \geq 2$. For all $n \geq 1$, we have that*

$$\sigma_k(n) \leq 2n^k.$$

For a positive integer n put $p(n)$ to be the smallest prime factor of n with the convention that $p(1) = 1$. For $x \geq y \geq 2$ put

$$\Phi(x, y) := \#\{n \leq x : p(n) > y\}.$$

The following inequality is a consequence of the Brun sieve and appears, for example, as an Exercise on page 11 in [2].

Lemma 2.3. *We have uniformly for $x \geq y \geq 2$,*

$$\Phi(x, y) \ll \frac{x}{\log y}.$$

Let $\Omega(n)$ be the total number of prime factors of n counting multiplicities.

Lemma 2.4. *Let $x \geq 10$. The number of positive integers $n \leq x$ such that $\Omega(n) \geq 10 \log\log x$ is $O(x/(\log x)^2)$.*

Proof. See Lemma 3 in [4]. □

Let $\tau(n)$ be the total number of divisors of n.

Lemma 2.5. *Let $x \geq 10$ and $k \geq 1$ be an integer. The number of positive integers $n \leq x$ such that $\tau(\sigma_k(n)) > \exp\left(\sqrt{\log x}\right)$ is $O(x/(\log x)^2)$.*

Proof. The remarks on page 128 in [5] show that

$$\sum_{n \leq x} \tau(\sigma_k(n)) = x \exp\left(c(x)\left(\frac{\log x}{\log\log x}\right)^{1/2}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right),$$

holds with $c(x) \in [\alpha, \beta]$, where α and β are constants. Then following the proof of Lemma 4 in [4] concludes the proof. □

Lemma 2.6. *Let $\mathbf{U} := (U_n)_{n \geq 1}$ be a nondegenerate linearly recurrent sequence of integers whose general term is given by (1.1) with $s \geq 2$ and assume that $|\alpha_1| = \max\{|\alpha_i| : i = 1, \ldots, s\} > 1$. Then for every $\epsilon \in (0, 1)$, there exist constants x_0 and $c := c(U) \in (0, \epsilon)$ such that for all $x \geq x_0$, the number of $n \leq x$ such that*

$$|U_n| \leq |\alpha_1|^{n(1-\delta)},$$

with $\delta := x^{-c}$ is of cardinality $O(x^\epsilon)$.

The proof of the above lemma is similar to the proof of Lemma 5 in [4], by considering the $n \in (x^\epsilon, x]$ because there are only $O(x^\epsilon)$ positive integers $n \leq x^\epsilon$ and by taking $c := \epsilon/7sr \in (0, \epsilon)$, where r is as defined in [4].

3. The proof of Theorem 1.1

Let $k \geq 2$ be an integer. Assume that $|\alpha_1| \geq \cdots \geq |\alpha_s|$ and $|U_n|$ is not a polynomial in n for large n. In particular, $|\alpha_1| > 1$. By an application of the subspace theorem (see page 229 in [1]), we have that

$$|U_{\sigma_k(n)}| \geq |\alpha_1|^{\sigma_k(n)/2}$$

holds for all $n \geq n_0$, for some constant n_0. By the fact that $\sigma_k(n) \geq n^k$, the above inequality becomes

$$|U_{\sigma_k(n)}| \geq |\alpha_1|^{n^k/2}$$

for all $n \geq n_0$. On the other hand, using (1.1) and the triangle inequality, we have that

$$|U_n| = |\sum_{i=1}^{s} P_i(n)\alpha_i^n| \ll n^{\beta}|\alpha_1|^n,$$

where $\beta := \max\{\nu_i - 1 : i = 1, \ldots, s\}$ and $\nu_i - 1$ is the degree of the polynomial $P_i(X)$. Using Lemma 2.2 and the above inequality, we have that

$$\sigma_k(|U_n|) \leq 2|U_n|^k \ll n^{k\beta}|\alpha_1|^{kn}.$$

Let c_1 be the constant implied by the $\ll$ symbol in the above inequality. We claim that

$$c_1 n^{k\beta}|\alpha_1|^{kn} \leq |\alpha_1|^{n^k/2} \tag{3.1}$$

holds for all $n \geq n_0$, where n_0 is some constant. Taking the logarithm on both sides, the above inequality is equivalent to

$$2kn + O(\log n) \leq n^k,$$

which holds for all $n \geq n_0(k, \mathbf{U})$ sufficiently large because $k \geq 2$. Thus, (3.1) holds and hence,

$$\sigma_k(|U_n|) \leq |U_{\sigma_k(n)}|$$

holds for all $n \geq n_0$, where n_0 is some constant depending both on k and on the sequence $\mathbf{U}$.

4. The proof of Theorem 1.2

Let us assume that $|\alpha_1| \geq \cdots \geq |\alpha_s|$ and $|U_n|$ is not a polynomial in n for large n. In particular, $|\alpha_1| > 1$. We assume that k is an integer and $k \geq 2$ because for the case $k = 1$, it is already proved (see the proof of Theorem 1 in [4]). For a positive integer n put $p(n)$ be the smallest prime factor of n with the convention that $p(1) = 1$. Let

$$\mathcal{A}(x) = \{n \leq x : p(n)^k > x^{c_1}\},$$

where $c_1 \in (0, 1/4k)$ is a constant to be determined later. Let

$$\mathcal{C}(x) := \{n \le x : \sigma(|U_{n^k}|) > |U_{\sigma_k(n)}|\}.$$

We need to prove that the set $\mathcal{C}(x)$ has counting function of $O(x/\log x)$. Let us split the set $\mathcal{C}(x)$ into the following three subsets

$$\begin{aligned}
\mathcal{C}_1(x) &:= \{n \le x : n \in \mathcal{A}(x) \text{ and } n \in \mathcal{C}(x)\},\\
\mathcal{C}_2(x) &:= \{n \le x^{1/2k} : n \notin \mathcal{A}(x) \text{ and } n \in \mathcal{C}(x)\},\\
\mathcal{C}_3(x) &:= \{n \in (x^{1/2k}, x] : n \notin \mathcal{A}(x) \text{ and } n \in \mathcal{C}(x)\}.
\end{aligned}$$

The set $\mathcal{C}_1(x)$ is a subset of $\mathcal{A}(x)$. By Lemma 2.3 with $y := x^{c_1/k}$, we have that

$$\#\mathcal{A}(x) = \Phi(x, y) \ll \frac{x}{\log y} \ll \frac{x}{\log x}.$$

Thus, the subset $\mathcal{C}_1(x)$ has counting function of $O(x/\log x)$. The second subset has counting function of $O(x^{1/2k}) = o(x/\log x)$ because the set $\mathcal{C}_2(x)$ has at most $x^{1/2k}$ positive integers $n \le x^{1/2k}$. From now on, let $n \in \mathcal{C}_3(x)$.

Then $n \in (x^{1/2k}, x]$ and $p(n)^k \le x^{c_1}$. Since $p(n)$ divides n, then $(n/p(n))$ divides n as well. Thus, we have that

$$\sigma_k(n) \ge n^k + \frac{n^k}{p(n)^k} \ge n^k + \frac{n^k}{x^{c_1}} = n^k(1 + \delta),$$

where $\delta := 1/x^{c_1}$. The above inequality gives

$$n^k \le \frac{\sigma_k(n)}{1 + \delta} \le \sigma_k(n)(1 - \delta_1),$$

where $\delta_1 := \delta/2$. Let

$$U_n = \sum_{i=1}^{s} P_i(n)\alpha_i^n.$$

The case $s = 1$ needs to be treated separately as there are only finitely many n. Assume that $s = 1$. In this case $\Psi(X) = (X - \alpha_1)^d$ and α_1 is an integer with $|\alpha_1| \ge 2$. Thus,

$$U_n = P_1(n)\alpha_1^n,$$

where $P_1(X) \in \mathbb{Z}[X]$. Let n be large (say larger than the maximal real root of $P_1(X)$). By Lemma 2.1, we have that

$$\begin{aligned}
\sigma(|U_{n^k}|) &\ll |U_{n^k}| \log\log |U_{n^k}|\\
&\ll |P_1(n^k)||\alpha_1|^{n^k} \log\log |P_1(n^k)||\alpha_1|^{n^k}\\
&\ll n^{k(d-1)}|\alpha_1|^{\sigma_k(n)(1-\delta_1)} \log n. \qquad (4.1)
\end{aligned}$$

On the other hand, we have that

$$|U_{\sigma_k(n)}| = |P_1(\sigma_k(n))||\alpha_1|^{\sigma_k(n)} \gg \sigma_k(n)^{d-1}|\alpha_1|^{\sigma_k(n)} \gg n^{k(d-1)}|\alpha_1|^{\sigma_k(n)}. \qquad (4.2)$$

Since $n \in \mathcal{C}_3(x)$, then

$$|U_{\sigma_k(n)}| < \sigma(|U_{n^k}|).$$

Using the above inequality together with (4.1) and (4.2), we have that

$$n^{k(d-1)}|\alpha_1|^{\sigma_k(n)} \ll |U_{\sigma_k(n)}| < \sigma(|U_{n^k}|) \ll n^{k(d-1)}|\alpha_1|^{\sigma_k(n)(1-\delta_1)} \log n.$$

The above inequality leads to the following inequality

$$|\alpha_1|^{\delta_1\sigma_k(n)} \ll \log n.$$

Let c_2 be the constant implied by the $\ll$ symbol in the above inequality. Then the above inequality is equivalent to

$$|\alpha_1|^{\delta_1\sigma_k(n)} \le c_2 \log n.$$

Taking the logarithm on both sides, this is equivalent to

$$\delta_1\sigma_k(n)\log|\alpha_1| \le \log c_2 + \log\log n.$$

Since $n \in (x^{1/2k}, x]$, then the right–hand side of the above inequality is $O(\log\log x)$. Since $\delta_1 = 0.5\delta$, then $\delta_1\sigma_k(n) \ge 0.5\delta n^k > 0.5x^{-c_1}x^{1/2} = 0.5x^{\gamma}$, where $\gamma := 0.5 - c_1 > 0.5 - 1/4k > 0$. Thus, the left–hand side is $\gg x^{\gamma}$. Then above inequality becomes

$$x^{\gamma} \ll \delta_1\sigma_k(n)\log|\alpha_1| \le \log c_2 + \log\log n \ll \log\log x,$$

which does not hold for large n because $\log\log x = o(x^{\gamma})$. Thus, x is bounded and so is n.

From now on, we assume that $s \ge 2$. Then the inequality

$$|\alpha_1|^{n^k/2} \le |U_{n^k}| \ll n^{kd}|\alpha_1|^{n^k}$$

holds for all $n \ge n_0$, where n_0 is some constant. The left-hand side is by an application of the subspace theorem (see page 229 in [1]) and the right-hand side is by triangle inequality. Thus, for $n \ge n_0$, we have that

$$\log\log|U_{n^k}| = \log n^k + O(1).$$

Using the above inequality, we have that

$$\begin{aligned}\sigma(|U_{n^k}|) &\ll |U_{n^k}|\log\log|U_{n^k}| \ll n^{kd}|\alpha_1|^{n^k}(\log n^k + O(1))\\ &\le \sigma_k(n)^d|\alpha_1|^{\sigma_k(n)(1-\delta_1)}(\log\sigma_k(n) + O(1)).\end{aligned}$$

Let c_3 and c_4 be constants implied by the $\ll$ symbol and $O(1)$ in the above inequality, respectively. We claim that with $\delta_2 := 1/x^{2c_1}$ and $m = \sigma_k(n)$, the inequality

$$c_3 m^d|\alpha_1|^{m(1-\delta_1)}(\log m + c_4) < |\alpha_1|^{m(1-\delta_2)}$$

holds for all $n \geq n_0$, where n_0 is some constant. Taking logarithm on both sides, the above inequality is equivalent to

$$\log c_3 + d \log m + \log(\log m + c_4) < m(\delta_1 - \delta_2) \log |\alpha_1|. \tag{4.3}$$

Since $n \in (x^{1/2k}, x]$ and $m = \sigma_k(n) \leq 2x^k$ (by Lemma 2.2), then the left–hand side of (4.3) is $O(\log x)$. Since $\delta_1 = \delta/2$ and $\delta_2 = \delta^2$, then it follows that $\delta_1 - \delta_2 \geq 0.25\delta = 0.25x^{-c_1}$ for $x \geq x_0$, for some constant x_0. Thus, the right–hand side of (4.3) is $\gg x^\gamma$, where $\gamma := 0.5 - c_1 > 0$. Thus, since $\log x = o(x^\gamma)$, then we have that

$$\log c_3 + d \log m + \log(\log m + c_4) \ll \log x \ll x^\gamma \ll m(\delta_1 - \delta_2) \log |\alpha_1|$$

holds for all $n \geq n_0$. Thus, inequality (4.3) holds for all $n \geq n_0$ and hence,

$$|U_m| < |\alpha_1|^{m(1-\delta_2)}$$

holds for all $x \geq x_0$, for some constant x_0. Note that by Lemma 2.2 (since $k \geq 2$), $m = \sigma_k(n) \leq 2x^k$. By Lemma 2.5, we can choose $c_1 := c/2$ with $\epsilon = 1/2k$ and then the set of $m \leq 2x^k$ satisfying the above inequality is of cardinality

$$O(\sqrt{x}).$$

But this is only an upper bound on the number of distinct values of $m = \sigma_k(n)$ and we have to get the upper bound on the number of $n's$ themselves. By Lemmas 2.3 and 2.4, we may assume that $\Omega(n) \leq 10 \log\log x$ and $\tau(\sigma_k(n)) \leq \exp(\sqrt{\log x})$ since the number of $n \leq x$ for which one of the above inequalities fails is $O(x/(\log x)^2) = o(x/\log x)$. Writing

$$n = p_1^{a_1} \cdots p_\ell^{a_\ell}$$

with distinct primes $p_1, \ldots, p_\ell$ and positive exponents $a_1, \ldots, a_\ell$, so

$$\sigma_k(n) = \prod_{i=1}^{\ell} \left(\frac{p_i^{k(a_i+1)} - 1}{p_i^k - 1} \right).$$

Given $m = \sigma_k(n)$, each of $d_i := (p_i^{k(a_i+1)} - 1)/(p_i^k - 1)$ is a divisor of $\sigma_k(n)$. Additionally, given d_i and also a_i, p_i is uniquely determined. Thus, since d_i can be fixed in at most $\tau(\sigma_k(n))$ ways and $a_i \leq \Omega(n)$ can be fixed in at most $\Omega(n)$ ways, it follows that $p_i^{a_i}$ can be fixed in at most $\tau(\sigma_k(n))\Omega(n)$ ways. This is so for a fixed i, but $i \leq \ell = \omega(n) \leq \Omega(n)$. Thus, the number of such n when given $\sigma_k(n)$ and $\Omega(n)$ is at most

$$\left((10 \log\log x) \exp(\sqrt{\log x}) \right)^{10 \log\log x} < \exp\left(20 (\log\log x) \sqrt{\log x} \right)$$

for all $x \geq x_0$, for some constant x_0. Now varying $\Omega(n)$ up to $10 \log\log x$ and $\sigma_k(n)$ up to $O(\sqrt{x})$, we get that the number of possible $n \leq x$ is

$$\ll \sqrt{x} (\log\log x) \exp\left(20 (\log\log x) \sqrt{\log x} \right) = o(x/\log x)$$

as $x \to \infty$. Hence, the set $\mathcal{C}_3(x)$ has counting function $o(x/\log x)$ and thus the set $\mathcal{C}(x)$ has a counting function of $O(x/\log x)$. This concludes the proof.

Acknowledgements. We thank an anonymous referee for constructive criticism which improved the quality of our manuscript.

References

[1] J.-H. Evertse: *On sums of S-units and linear recurrences*, Compositio Math. 53 (1984), pp. 225–244.

[2] R. Hall, G. Tenenbaum: *Divisors, Cambridge Tracts in Mathematics*, 90, Cambridge: Cambridge University Press, 1988, doi: 10.1017/CBO9780511566004.

[3] G. H. Hardy, E. M. Wright: *An introduction to the theory of numbers*, Sixth edition, Oxford: Oxford University Press, 2008.

[4] F. Luca, M. C. Manape: *On the Euler function of linearly recurrent sequences*, Fibonacci Quart. 62 (2024), pp. 316–328.

[5] F. Luca, C. Pomerance: *On the average number of divisors of the Euler function*, Publ. Math. Debrecen 70 (2007), pp. 125–148.

[6] G. Tenenbaum: *Introduction to Analytic and Probabilistic Number Theory*, Third edition, 163, American Mathematical Soc.: Graduate Studies in Mathematics, 2015.