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# On the group of units of the semisimple group algebras of dimension 200

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**Abstract.** In this study, we examine the non-metabelian groups of order 200 and analyze the structure of the unit group within their corresponding group algebra. Among the fifty-two non-isomorphic groups of order 200, only two are non-metabelian. The paper focuses on characterizing the unit group of the semi-simple group algebras associated with these two groups over any finite field.

*Keywords:* finite field, group algebra, unit group, non-metabelian groups

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## 1. Introduction

The linear combination of elements from $G$ with coefficients from the finite field $\mathcal{F}_q$, where $G$ is a finite group, is the group algebra, denoted by $\mathcal{F}_qG$ over the field $\mathcal{F}_q$ with $q = p^k$ elements for a prime $p$ and $k \in \mathbb{Z}^+$. Maschke's theorem [19] implies that the group algebra $\mathcal{F}_qG$ is semisimple if and only if $\mathrm{char}(\mathcal{F}_q) \nmid n$. Thus, via Wedderburn decomposition theorem [19], $\mathcal{F}_qG$ is isomorphic to the direct sum of matrix algebras over division rings, that is,

$$\mathcal{F}_qG \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_l}(D_l),\ n_i, l \in \mathbb{Z}^+.$$

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One may easily determine the structure of the unit group of $\mathcal{F}_qG$ from the aforementioned isomorphism. Recall that the unit group, $\mathbb{U}(\mathcal{F}_qG)$, is set of all of the invertible elements in $\mathcal{F}_qG$. See [1, 6, 22, 24, 26, 30–32] for a few noteworthy recent studies that have been conducted to determine the structure of the unit group of the semi-simple group algebra. The applications of units in number theory [14], coding theory [16], cryptography [20], etc. make the research in this area crucial. In addition to this, the recent counterexample to the renowned Kaplansky's unit conjecture further emphasizes the need of research in this area (see [15]). Furthermore, there have been significant developments in the exploration of the unit group of modular group algebras, in addition to integral and semisimple group algebras (see [9–11, 28] and the references therein for a comprehensive and recent literature in this direction).

In view of studying the unit group of all group algebras, the groups can be categorized into two divisions: metabelian and non-metabelian. Bakshi et al. conducted a thorough analysis of the first instance in [8]. As a result, we just need to consider the non-metabelian groups. In [27], Pazderski determined possible non-metabelian group orders. The smallest non-metabelian groups have order 24, and Khan et al. [17] and Maheshwari et al. [18] have examined the unit group of the corresponding group algebras. These findings may be readily ascertained with the use of [27]. Similarly, in [7, 21, 23, 24, 26, 29], Mittal et al. and Arvind et al. categorised the unit group of group algebras of non-metabelian groups up to order 120. Mittal et al. [25] and Abhilash et al. [2–5] recently finished the work for all groups up to order 180.

It is known that there exist non-metabelian groups of order 200 as a result of [27]. There exist 52 non-isomorphic groups of order 200, two of which are non-metabelian. The objective of this paper is to take these 2 groups into consideration and use the Wedderburn decomposition to derive the unit groups of their group algebras (see [19]).

The flow of this paper is as follows. The important definitions, results and the 2 non-metablian groups to be studied in this paper are introduced in Section 2 and Section 3, respectively. Moreover, Section 3 has the main results on the unit groups of the semisimple group algebras. The final section concludes the paper in nature.

## 2. Preliminaries

The results and prerequisites definitions needed to support the main result are provided in this section. The following notations apply to this entire paper.

| | |
|---|---|
| $\mathcal{F}_q$ | finite field of order $q = p^k$ with characteristic $p$ and $k \geq 1$ |
| $G$ | finite group of order $n$ with $p \nmid n$ |
| $e$ | exponent of the group $G$ i.e., the l.c.m of the order of elements in $G$ |
| $\omega$ | primitive $e$-th root of unity over $\mathcal{F}_q$ |
| $\mathbb{G}$ | Galois group of $\mathcal{F}_q(\omega)$ over $\mathcal{F}_q$, where $\mathcal{F}_q(\omega)$ is the splitting field of $\mathcal{F}_q$ |

| | |
|---|---|
| $\mathcal{T}_{G,\mathcal{F}_q}$ | collection of all $s$ such that $\sigma(\omega)=\omega^s$, where $\sigma\in\mathbb{G}$ |
| $C_x$ | conjugacy class of $x$ |
| $[x,y]$ | denote the commutator $x^{-1}y^{-1}xy$ of $x,y\in G$ |
| 1 | identity element of $G$ |

**Definition 2.1** ([13]). *(i) For any prime $p$, an element $x\in G$ is said to be $p'$-element if order of $x$ is not divisible by $p$.*
*(ii) For any $p'$-element $x\in G$, the cyclotomic $\mathcal{F}_q$-class of $\gamma_x=\sum_{h\in C_x}h$ is the set $S_{\mathcal{F}_q}(\gamma_x)=\{\gamma_{x^s}\mid s\in\mathcal{T}_{G,\mathcal{F}_q}\}$.*

Lemma 2.3 deals with the number of elements in a specific cyclotomic class, while the proposition following concerns the total count of cyclotomic $\mathcal{F}_q$-classes. Ferraz provides these two results in [13].

**Proposition 2.2.** *The set of simple components of $\mathcal{F}_qG/J(\mathcal{F}_qG)$ and the set of cyclotomic $\mathcal{F}_q$-classes in $G$, where $J(\mathcal{F}_qG)$ is the Jacobson radical of $\mathcal{F}_qG$, are in 1-1 correspondence.*

**Lemma 2.3.** *Let $l$ be the number of cyclotomic $\mathcal{F}_q$-classes in $G$. If $\mathcal{F}_{q^{m_1}},\mathcal{F}_{q^{m_2}},\cdots,\mathcal{F}_{q^{m_l}}$ are the simple components of $Z\big(\mathcal{F}_qG/J(\mathcal{F}_qG)\big)$ and $S_1,S_2,\cdots,S_l$ are the cyclotomic $\mathcal{F}_q$-classes of $G$, then $|S_i|=[\mathcal{F}_{q^{m_i}}:\mathcal{F}_q]$ with a suitable ordering of the indices, assuming that $\mathbb{G}$ is cyclic.*

In order to uniquely characterize the Wedderburn decompositions of the semisimple group algebras, we need the following important result. See [19, Chapter 3] for its proof.

**Lemma 2.4.** *(i) Let $\mathcal{F}_qG$ be a semi-simple group algebra and let $N\trianglelefteq G$. Then*

$$\mathcal{F}_qG\cong\mathcal{F}_q(G/N)\oplus\Delta(G,N),$$

*where $\Delta(G,N)$ is an ideal of $\mathcal{F}_qG$ generated by the set $\{n-1:n\in N\}$.*
*(ii) If $N=G'$ in part (i), then $\mathcal{F}_q(G/G')$ is the sum of all commutative simple components of $\mathcal{F}_qG$ and $\Delta(G,G')$ is the sum of all others.*

Further, we discuss a necessary condition for the dimension of the matrix algebra in the Wedderburn decomposition (see [12]).

**Lemma 2.5.** *Suppose $\oplus_{i=1}^{t}M_{n_i}(\mathcal{F}_{q^{m_i}})$ is the summand of the semisimple group algebra $\mathcal{F}_qG$, where $p$ is the characteristics of $\mathcal{F}_q$. Then $p$ does not divide any of the $n_i$.*

Again, we discuss two very important results which helps us to find the unique Wedderburn decomposition. For proof of Lemma 2.6, we refer to [34].

**Lemma 2.6.** *Let $p_1$ and $p_2$ be two primes. Let $\mathcal{F}_{q_1}$ be a field with $q_1=p_1^{k_1}$ elements and let $\mathcal{F}_{q_2}$ be a field with $q_2=p_2^{k_2}$ elements, where $k_1,k_2\geq 1$. Let both the group algebras $\mathcal{F}_{q_1}G,\mathcal{F}_{q_2}G$ be semisimple. Suppose that*

$$\mathcal{F}_{q_1}G\cong\oplus_{i=1}^{t}M(n_i,\mathcal{F}_{q_1}),\ n_i\geq 1$$

*and $M(n, \mathcal{F}_{q_2^r})$ is a Wedderburn component of the group algebra $\mathcal{F}_{q_2}G$ for some $r \geq 2$ and any positive integer $n$, i.e.,*

$$\mathcal{F}_{q_2}G \cong \oplus_{i=1}^{s-1} M(m_i, \mathcal{F}_{q_{2,i}}) \oplus M(n, \mathcal{F}_{q_2^r}),\ m_i \geq 1.$$

*Here $\mathcal{F}_{q_{2,i}}$ is a field extension of $\mathcal{F}_{q_2}$. Then $M(n, \mathcal{F}_{q_1})$ must be a Wedderburn component of the group algebra $\mathcal{F}_{q_1}G$ and it appears atleast $r$ times in the Wedderburn decomposition of $\mathcal{F}_{q_1}G$.*

**Lemma 2.7** ([7, corollary **3.8**])**.** *Let $\mathcal{F}_qG$ be a finite semisimple group algebra. Then if there exists an irreducible representations of degree $n$ over $\mathcal{F}_q$, then one of the Wedderburn components of $\mathcal{F}_qG$ is $M_n(\mathcal{F}_q)$. Also, if there exists $k$ irreducible representations of degree $n$ over $\mathcal{F}_q$, then $M_n(\mathcal{F}_q)^k$ is a summand of the group algebra $\mathcal{F}_qG$.*

Throughout this paper, if $G$ has $j$ conjugacy classes, then the representatives of the conjugacy classes are denoted by $g_1(=1), g_2, g_3, \ldots, g_j$.

# 3. Unit groups

The structure of the unit group of the group algebras of non-metabelian groups of order 200 is covered in this section. Two of the 52 non-isomorphic groups of order 200 are not metabelian. They are as follows:

1. $G_1 := (C_5 \times C_5) \rtimes D_8$
2. $G_2 := (C_5 \times C_5) \rtimes Q_8$

The conjugacy classes and the commutator subgroups of both the groups are taken from the GAP software (see [33]).

## 3.1. $G_1 := (C_5 \times C_5) \rtimes D_8$

The group $G_1$ has the following presentation:

$G_1 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_3^{-1}, [x_3, x_1], [x_4, x_1]x_5^{-3}x_4^{-4}, [x_5, x_1]x_5^{-4}x_4^{-2}, x_2^2, [x_3, x_2], [x_4, x_2]x_5^{-1}x_4^{-4}, [x_5, x_2]x_5^{-4}x_4^{-1}, x_3^2, [x_4, x_3]x_4^{-3}, [x_5, x_3]x_5^{-3}, x_4^5, [x_5, x_4], x_5^5\rangle$.

The sizes (S), orders (O) and the representatives (R) of the 14 conjugacy classes of $G_1$ are:

| R | 1 | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_1x_2$ | $x_1x_4$ | $x_1x_5$ | $x_2x_4$ | $x_4x_5$ | $x_2x_3x_4$ | $x_4^2x_5$ | $x_4^3x_5$ | $x_4^2x_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| S | 1 | 10 | 10 | 25 | 8 | 50 | 20 | 20 | 20 | 4 | 20 | 4 | 4 | 4 |
| O | 1 | 2 | 2 | 2 | 5 | 4 | 10 | 10 | 10 | 5 | 10 | 5 | 5 | 5 |

It is clear that the exponent of $G_1$ is 20.

**Theorem 3.1.** *The unit group of the group algebra $\mathcal{F}_qG_1$ is as follows:*

1) *for $q \equiv \{1, 9, 11, 19\}$ mod 20, $\mathbb{U}(\mathcal{F}_qG_1) \cong \mathcal{F}_q^{*4} \oplus GL_2(\mathcal{F}_q) \oplus GL_4(\mathcal{F}_q)^8 \oplus GL_8(\mathcal{F}_q)$.*
2) *for $q \equiv \{3, 7, 13, 17\}$ mod 20, $\mathbb{U}(\mathcal{F}_qG_1) \cong \mathcal{F}_q^{*4} \oplus GL_2(\mathcal{F}_q) \oplus GL_8(\mathcal{F}_q) \oplus GL_4(\mathcal{F}_{q^2})^4$.*

***Proof.*** The group algebra $\mathcal{F}_qG_1$ is Artinian and semisimple. We observe that the commutator subgroup $G_1' \cong (C_5 \times C_5) \rtimes C_2$ and $\frac{G_1}{G_1'} \cong C_2 \times C_2$. Since $q = p^k$ and $p \neq 2, 5$, we split the proof in the following 2 cases.

**Case 1**: $q \equiv \{1, 9, 11, 19\}$ mod 20. In this case, the cardinality of every cyclotomic $\mathcal{F}_q$-class is 1. Therefore, the decomposition of the group algebra by using Proposition 2.2 and Lemma 2.3 is $\mathcal{F}_qG_1 \cong \mathcal{F}_q \oplus_{i=1}^{13} M_{n_i}(\mathcal{F}_q), n_i \geq 1$. By applying (ii) of Lemma 2.4, we further deduce that

$$\mathcal{F}_qG_1 \cong \mathcal{F}_q^4 \oplus_{i=1}^{10} M_{n_i}(\mathcal{F}_q),\ n_i \geq 2.$$

Since the dimensions of both the sides are the same, we end up with $196 = \sum_{i=1}^{10} n_i^2$. This equation has 34 different solutions. By incorporating Lemma 2.5, we conclude that $p$ can not be 3 and 7. This means that we are remaining with 5 choices of $n_i'$s given as follows:

$$(2^8, 8, 10), (2^5, 4^3, 8^2), (2^4, 4, 5^4, 8), (2, 4^8, 8), (4^6, 5^3).$$

We observe that the subgroup $N := C_5 \times C_5$ is normal in $G_1$ and $F = G_1/N \cong D_8$. Using [8], we note that $\mathcal{F}_qF \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q)$. Due to (i) of Lemma 2.4, we are remaining with $(2^8, 8, 10), (2^5, 4^3, 8^2), (2^4, 4, 5^4, 8), (2, 4^8, 8)$ choices of $n_i'$s. Next, we define four group homomorphism $f_1, f_2, f_3, f_4$ from $G_1 \to GL_4(\mathcal{F}_{11})$, where $\mathcal{F}_{11}$ is the finite field having 11 elements, as follows:

$$x_1 \xmapsto{f_1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\quad x_2 \xmapsto{f_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad x_3 \xmapsto{f_1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_4 \xmapsto{f_1} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},\quad x_5 \xmapsto{f_1} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix},$$

$$x_1 \xmapsto{f_2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\quad x_2 \xmapsto{f_2} \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{pmatrix},\quad x_3 \xmapsto{f_2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_4 \xmapsto{f_2} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},\quad x_5 \xmapsto{f_2} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix},$$

$$x_1 \xrightarrow{f_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_2 \xrightarrow{f_3} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad x_3 \xrightarrow{f_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_4 \xrightarrow{f_3} \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad x_5 \xrightarrow{f_3} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix},$$

$$x_1 \xrightarrow{f_4} \begin{pmatrix} 0 & 10 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}, \quad x_2 \xrightarrow{f_4} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad x_3 \xrightarrow{f_4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_4 \xrightarrow{f_4} \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad x_5 \xrightarrow{f_4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$$

Clearly, the maps $f_i$'s are irreducible representations of $G_1$ of degree 4 over $\mathcal{F}_{11}$, i.e., $f_i$'s are group homomorphisms from $G_1$ to $GL_4(\mathcal{F}_{11})$ and $f_i$'s are irreducible, that is there is no matrix $U \in GL_4(\mathcal{F}_{11})$ such that

$$U^{-1} f_i(g) U = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix} \quad \text{for all } g \in G_1 \text{ and } i = 1, 2, 3, 4$$

where $A(g)$, $B(g)$, and $C(g)$ are square matirces with entries from $\mathcal{F}_{11}$ depending on $g$. Therefore, Lemma 2.7 implies that $M_4(\mathcal{F}_{11})^4$ must be a summand of $\mathcal{F}_{11}G_1$. This confirms that $(2, 4^8, 8)$ is the final choice of the values of $n_i'$s. Therefore, we have

$$\mathcal{F}_q G_1 \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q) \oplus M_4(\mathcal{F}_q)^8 \oplus M_8(\mathcal{F}_q).$$

**Case 2**: $q \equiv \{3, 7, 13, 17\} \bmod 20$. In this case, the cyclotomic $\mathcal{F}_q$-class corresponding to $g_7$ includes $g_8$, $g_9$ includes $g_{11}$, $g_{10}$ includes $g_{14}$ and $g_{12}$ includes $g_{13}$, while rest of the $g_i'$s forms individual classes. Therefore, as per Proposition 2.2, Lemma 2.3 and (ii) of Lemma 2.4, the decomposition for this scenario is given by

$$\mathcal{F}_q G_1 \cong \mathcal{F}_q^4 \oplus_{i=1}^{2} M_{n_i}(\mathcal{F}_q) \oplus_{i=3}^{6} M_{n_i}(\mathcal{F}_{q^2}), \quad n_i > 1$$

which means $196 = n_1^2 + n_2^2 + 2\sum_{i=3}^{6} n_i^2$. There are 24 possibilities and by Lemma 2.6, $M_8(\mathcal{F}_q)$ must be a summand. Also, [8] implies that $\mathcal{F}_q F \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q)$. Thus, (i) of Lemma 2.4 derives that

$$\mathcal{F}_q G_1 \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q) \oplus M_8(\mathcal{F}_q) \oplus M_4(\mathcal{F}_{q^2})^4.$$

This completes the proof. □

### 3.2. $G_2 := (C_5 \times C_5) \rtimes Q_8$

The group $G_2$ has the following presentation:

$G_2 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2x_3^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1], [x_4, x_1]x_4^{-1}, [x_5, x_1]x_5^{-2}, x_2^2x_2^{-1}, [x_3, x_2], [x_4, x_2]x_5^{-1}x_4^{-4}, [x_5, x_2]x_5^{-4}x_4^{-4}, x_3^2, [x_4, x_3]x_4^{-3}, [x_5, x_3]x_5^{-3}, x_4^5, [x_5, x_4], x_5^5\rangle$.

The sizes, orders and the representatives of the 8 conjugacy classes of $G_2$ are given below:

| Representatives | 1 | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_1x_2$ | $x_4x_5$ | $x_4^2x_5$ |
|---|---|---|---|---|---|---|---|---|
| Size | 1 | 50 | 50 | 25 | 8 | 50 | 8 | 8 |
| Order | 1 | 4 | 4 | 2 | 5 | 4 | 5 | 5 |

It is clear that the exponent of $G_2$ is 20.

**Theorem 3.2.** *The unit group of the group algebra $\mathcal{F}_qG_2$ for any field with characteristic not equal to 2 and 5 is given by:*

$$\mathbb{U}(\mathcal{F}_qG_2) \cong \mathcal{F}_q^{*4} \oplus GL_2(\mathcal{F}_q) \oplus GL_8(\mathcal{F}_q)^3.$$

***Proof.*** The group algebra $\mathcal{F}_qG_2$ is Artinian and semisimple. We observe that the commutator subgroup $G_2' \cong (C_5 \times C_5) \rtimes C_2$ and $\frac{G_2}{G_2'} \cong C_2 \times C_2$. $T_{G,\mathcal{F}_q} = \{1, 3, 7, 9, 11, 13, 17, 19\}$. So, for any $g \in G_1$, we have

$$S_{\mathcal{F}_q}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G,\mathcal{F}_q}\} \Rightarrow S_{\mathcal{F}_q}(\gamma_g) = \{\gamma_g, \gamma_{g^3}, \gamma_{g^7}, \gamma_{g^9}, \gamma_{g^{11}}, \gamma_{g^{13}}, \gamma_{g^{17}}, \gamma_{g^{19}}\}.$$

Further, it can be verified that $g^t$, where $t \in T_{G,\mathcal{F}_q}$ belong to the conjugacy class of $g$. Consequently, $S_{\mathcal{F}_q}(\gamma_g) = \{\gamma_g\}$ for any $g \in G_1$. Therefore, we conclude that the cardinality of every cyclotomic $\mathcal{F}_q$-class is 1. Therefore, as per Proposition 2.2, Lemma 2.3 and (ii) of Lemma 2.4, the decomposition is

$$\mathcal{F}_qG_2 \cong \mathcal{F}_q^4 \oplus_{i=1}^4 M_{n_i}(\mathcal{F}_q),$$

and $196 = \sum_{i=1}^4 n_i^2$, where $n_i > 1$. The possible choices of $n_i'$s fulfilling above equation are $(2, 8, 8, 8), (3, 3, 3, 13), (3, 5, 9, 9), (4, 4, 8, 10), (5, 5, 5, 11), (7, 7, 7, 7)$. We observe that the subgroup $N := \langle x_4, x_5\rangle$ is normal in $G_2$ and $F = G_2/N \cong Q_8$. Using [8], we recall that $\mathcal{F}_qF \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q)$. Therefore, $M_2(\mathcal{F}_q)$ must be the Wedderburn components of $\mathcal{F}_qG_2$ as per (i) of Lemma 2.4. So, $(2, 8, 8, 8)$ is the required choice, which means that $\mathcal{F}_qG_2 \cong \mathcal{F}_q^4 \oplus M_2(\mathcal{F}_q) \oplus M_8(\mathcal{F}_q)^3$. This completes the proof. □

## 4. Conclusion

This work concludes the study of the unit groups of semisimple group algebras of all groups up to order 200, with the exception of the groups of order 192. We have previously investigated the unit group of the semisimple group algebras of non-metabelian groups of order 200. It is evident that in order to characterise the unit groups in a way that is distinct from one another, different strategies are needed as the group increases in size. Researchers are encouraged by this study to develop an algorithm that can calculate the semisimple group algebra of any finite group's Wedderburn decomposition.

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