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On the Diophantine equation $(p^n)^x + (4^m + p)^y = z^2$ when p, $4^m + p$ are prime integers*

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Abstract. In this paper, we give methods to solve the Diophantine equation $(p^n)^x + (p + 4^m)^y = z^2$ where $p \geq 3$ and $p + 4^m$ are prime integers. Concretely, using the congruent method, one proves that this equation has no non-negative solutions if $p > 3$. For the case $p = 3$, using the elliptic curves, we will show that this equation has no solutions if $m \geq 3$. In this case, when $m = 1$ using the elliptic curves, we will show that this equation has only solution $(x, y, z) = (2, 1, 4)$ if $n = 1$ and $(x, y, z) = (1, 1, 4)$ if $n = 2$, and when $m = 2$ using the elliptic curves, we will show that this equation has only solutions is $(x, y, z) = (4, 1, 10)$ if $n = 1$ and $(x, y, z) = (2, 1, 10)$ if $n = 2$.

Keywords: Diophantine equation, elliptic curves, factor method

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1. Introduction

Equations with an exponential Diophantine nature, such as those encountered in the Fermat–Catalan and Beal's conjectures, take the form $a^m + b^n = c^k$ while imposing restrictions on the exponents. These equations arise when additional variables are introduced as exponents in a Diophantine equation. Despite efforts dedicated to addressing specific instances like Catalan's conjecture, a comprehensive theory for solving these equations is currently unavailable. Catalan's conjecture specifically states that the equation $x^p - y^q = 1$ possesses no other integer solutions besides $3^2 - 2^3 = 1$ (see [2]). Mihailescu successfully proved this conjecture

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(see [4]). Many authors have studied a generalized form of this equation, which is the Diophantine equation expressed as

$$b^x - c^y = z^2. \tag{1.1}$$

Consider the following examples: when $a^2+b^2 = c^2$ with $\gcd(a,b,c) = 1$ and a being an even number, Terai [7] proposed the conjecture that Equation (1.1) has a unique positive integer solution $(x,y,z) = (2,2,a)$. In the case where $b = q \not\equiv 7 \pmod 8$ represents an odd prime and $x \equiv 1 \pmod 2$, the Diophantine equation

$$p^x - c^y = z^2 \tag{1.2}$$

has been solved by Arif and Muriefah [1] and Zhu [8]. In [6], Terai presented various results concerning Equation (1.2) (see [6, Theorem 1.2, 1.3, 1.4]). Additionally, Terai noted in [6] that there is currently no proof establishing the absence of solutions for the equations $12^x - 23^y = z^2$ and $24^x - 47^y = z^2$ in the scenario where both x and y are odd.

Turning our attention to a different aspect, the Diophantine equations $x^n + y^t = z^m$ have garnered significant attention from mathematicians. Generally speaking, this problem poses considerable challenges. Suppose m and n are positive integers. We define a solution $(x,y,z) \in \mathbb{Z}_3$ for the equation $x^n + y^n = z^m$ as primitive if $\gcd(x,y,z) = 1$. Conversely, a solution (x,y,z) is deemed trivial if xyz belongs to the set $\{-1,0,1\}$. In 1997, Darmon and Merel [3] succeeded in proving the following two theorems by employing the Shimura–Taniyama conjecture in conjunction with Frey curves.

Theorem 1.1. *The equation $x^n + y^n = z^2$ has no nontrivial primitive solutions for prime $n \geq 7$.*

Theorem 1.2. *Assume the Shimura–Taniyama conjecture. Then the equation $x^n + y^n = z^3$ has no nontrivial primitive solutions for prime $n \geq 7$.*

After that, Poonen [5] completed the proof of the above two theorems. Concretely, he proved Theorem 1.1 and Theorem 1.2 are true for all $n \geq 3$ and $n \geq 4$, respectively.

Now, we consider the Diophantine equations

$$p^x + q^y = z^2 \tag{1.3}$$

where p and q are distinct primes. This Diophantine equation has been widely studied for various fixed values of p and q. Note that if either $y = 0$ or $x = 0$ then Equation (1.3) becomes $z^2 - (p^n)^x = 1$ or $z^2 - q^y = 1$, a special form of the Catalan conjecture. However, there is no solution to the general equations.

In this paper, we consider the Diophantine equation

$$(p^n)^x + (p+4^m)^y = z^2. \tag{1.4}$$

where n is a positive integer and $p \geq 3$ are prime integers. For the case $p > 3$, using the factor method, one proves that (1.4) has no solutions. However, there is still no answer for the case $p = 3$.

2. Preliminaries

In this section, we give some results that will be used in the sequel.

Lemma 2.1. *Any odd power of an integer of the form* $4a+3, a > 0$ *is of the form* $4B + 3$*, i.e., for every odd integer* $k \geq 1$*, we have* $(4a + 3)^k = 4B + 3$*, where* B *is a positive integer.*

Proof. We prove that by induction on k. For $k = 1$, we have $(4a + 3)^1 = 4a + 3$. Hence the assertion is true when $k = 1$.

Assume that the induction is true for all odd powers less than or equal to $k = 2r + 1$, where $r \geq 0$ is an integer. Then we have

$$(4a + 3)^{2r+3} = (4a + 3)^2(4a + 3)^{2r+1}.$$

By inductive assumption, there exists a positive integer b such that

$$(4a + 3)^{2r+1} = 4b + 3.$$

Hence we have

$$\begin{aligned}(4a + 3)^{2r+3} &= (4a + 3)^2(4b + 3)\\ &= 4[b(4a + 3)^2 + 12a^2 + 18a + 6] + 3.\end{aligned}$$

We put $B = b(4a + 3)^2 + 12a^2 + 18a + 6$. Then $(4a + 3)^{2r+3} = 4B + 3$, the claim is proved. □

For the case $y = 0$, we have the following lemma.

Lemma 2.2. *The Diophantine equation*

$$(3^n)^x + 1 = z^2$$

has only solution is $(x, z) = (1, 2)$ *if* $n = 1$ *and has no non-negative integer solutions if* $n > 1$*.*

Proof. Let x and z be non-negative integers such that $(3^n)^x + 1 = z^2$. We have

$$(3^n)^x = z^2 - 1 = (z + 1)(z - 1). \tag{2.1}$$

Since 3 is prime, we get by Equation (2.1) that there exists integers $a > b$ with $a + b = nx$ such that

$$z + 1 = 3^a \quad \text{and} \quad z - 1 = 3^b. \tag{2.2}$$

From Equation (2.2), we get that

$$3^b(3^{a-b} - 1) = 2. \tag{2.3}$$

Since 2 and 3 are primes, we get by Equation (2.3) that $b = 0$, and therefore $3^{nx} = 3$. Hence $nx = 1$. Thus, if $n > 1$ then the Diophantine equation $(3^n)^x + 1 = z^2$ has no solutions, and if $n = 1$ then the Diophantine equation $(3^n)^x + 1 = z^2$ has only solution is $(x, z) = (1, 2)$. □

For the case $x = 0$, we have the following lemma.

Lemma 2.3. *Let m be a positive integer such that $4^m + 3$ is prime. Then the Diophantine*

$$1 + (4^m + 3)^y = z^2$$

has no non-negative integer solutions.

Proof. Let y and z be non-negative integers such that $(3 + 4^m)^y + 1 = z^2$. We have

$$(3 + 4^m)^y = z^2 - 1 = (z + 1)(z - 1). \tag{2.4}$$

Since 3 is prime, we get by Equation (2.4) that there exists integers $a > b$ with $a + b = y$ such that

$$z + 1 = (3 + 4^m)^a \quad \text{and} \quad z - 1 = (3 + 4^m)^b. \tag{2.5}$$

From Equation (2.5), we get that

$$(3 + 4^m)^b[(3 + 4^m)^{a-b} - 1] = 2. \tag{2.6}$$

Since 2 and $3 + 4^m$ are primes, we get by Equation (2.6) that $b = 0$, and therefore $(3 + 4^m)^y = 3$. Since $3 + 4^m > 3$, Equation (2.6) has no solutions. □

The following lemma is the key to the proof of the main results of this paper.

Lemma 2.4. *Let A be a positive integer of the form $4^m + 3$. Then $A - 1$ has always a prime divisor q such that $q \neq 2$ and $q \neq 3$ for all $m \geq 3$.*

Proof. We have $A - 1 = 4^m + 2 = 2(2 \cdot 4^{m-1} + 1)$. It clear that 2 is not divisor of $2 \cdot 4^{m-1} + 1$. Suppose that $A - 1$ has only two prime divisors, 2 and 3. Then there exists an positive integer x such that $2 \cdot 4^{m-1} + 1 = 3^x$ that means $2 \cdot 4^{m-1} = 3^x - 1$. So, we have the following equation

$$2 \cdot (2^{m-1})^2 = 3^x - 1. \tag{2.7}$$

We divided it into three cases.

1. If $x = 3t$ for some positive integer t. Then Equation (2.7) becomes

$$4^2 \cdot (2^{m-1})^2 = 2^3 3^{3t} - 2^3. \tag{2.8}$$

   We put $Y = 4 \cdot 2^{m-1}$ and $X = 2 \cdot 3^t$ then Equation (2.8) becomes

$$Y^2 = X^3 - 2^3$$

   is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) is $(2, 0)$. Hence Equation (2.8) has no non-negative integer solutions.

2. If $x = 3t + 1$ for some non-negative integer t. Then Equation (2.7) becomes

$$3^2 \cdot 4^2 \cdot (2^{m-1})^2 = 2^3 \cdot 3^{3t+3} - 2^3 \cdot 3^2. \tag{2.9}$$

   We put $Y = 12 \cdot 2^{m-1}$ and $X = 2 \cdot 3^{t+1}$ then Equation (2.9) becomes

$$Y^2 = X^3 - 2^3 \cdot 3^2$$

   is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) is $(6, 12)$. Hence $t = 0$ and $m = 1$, a contradiction since $m \geq 3$. Hence Equation (2.9) has no non-negative integer solutions.

3. If $x = 3t + 2$ for some non-negative integer t. Then Equation (2.7) becomes

$$3^4 \cdot 4^2 \cdot (2^{m-1})^2 = 2^3 \cdot 3^{3t+6} - 2^3 \cdot 3^4. \tag{2.10}$$

   We put $Y = 36 \cdot 2^{m-1}$ and $X = 2 \cdot 3^{t+2}$ then Equation (2.10) becomes

$$Y^2 = X^3 - 2^3 \cdot 3^4$$

   is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) are $(9, 9)$, $(18, 72)$, $(22, 100)$, $(54, 396)$, $(97, 955)$, $(1809, 76941)$. Hence $2.3^{t+2} = 18, 36.2^{m-1} = 72$, that means $t = 0, m = 2$, a contradiction. Therefore, Equation (2.10) has no non-negative integer solutions. □

3. Main results

The purpose of this section is to present some results on solutions to the Diophantine equation

$$(3^n)^x + (4^m + 3)^y = z^2$$

where m is an integer such that $q = 4^m + 3$ is a prime number, for example, $(m, q) \in \{(1, 7), (2, 19), (3, 67), \ldots\}$. For $x = 0$ or $y = 0$, according to Lemma 2.2 and Lemma 2.3. Therefore, from now on, we always consider $x, y \geq 1$. Firstly, we consider x an even integer, and $m \geq 3$ is an integer. Then we have the following theorem.

Theorem 3.1. *Let $m \geq 3$ be an integer such that $4^m + 3$ is prime and x is even. Then the Diophantine equation*

$$(3^n)^x + (4^m + 3)^y = z^2$$

has no non-negative integer solutions.

Proof. Since $x = 2t$ for some $t > 0$, we have

$$(4^m + 3)^y = z^2 - (3^n)^{2t} = (z + 3^{nt})(z - 3^{nt}). \tag{3.1}$$

Since $4^m + 3$ is prime and $y \geq 1$, we get by Equation (3.1) there exists integers $a > b$ such that

$$z + 3^{nt} = (4^m + 3)^a, \;\; z - p^{nt} = (4^m + 3)^b$$

where $a + b = y$. Therefore,

$$(4^m + 3)^b[(3 \cdot 4^m + 1)^{a-b} - 1] = 2 \cdot 3^{nt}. \tag{3.2}$$

Since $4^m + 3 > 3$ and 2, and 3 are distinct primes, we get by Equation (3.2) that $b = 0$. Combined with the condition $y \geq 1$, Equation (3.2) becomes

$$2 \cdot 3^{nt} = (4^m + 3)^y - 1 = (4^m + 2)[(4^m + 3)^{y-1} \ldots + 1]. \tag{3.3}$$

By Equation (3.3) that $(4^m + 2) \mid 2 \cdot 3^{nt}$. Since $m \geq 3$, we get by Lemma 2.4 that $4^m + 2$ has always a prime divisor q such that $q \neq 2$ and $q \neq 3$. Since $(4^m+2) \mid 2 \cdot 3^{nt}$, we have $q \mid 3^{nt}$, a contradiction. □

For the case $m = 1$ then the Diophantine equation $(p^n)^x + (p + 4^m)^y = z^2$ becomes $(3^n)^x + 7^y = z^2$. Then we have the following theorem.

Theorem 3.2. *Let $x = 2t$ be even for some positive integer t. Then the Diophantine equation*

$$(3^n)^x + 7^y = z^2$$

has only solution is $(x, y, z) = (2, 1, 4)$ if $n = 1$, and has no solutions if $n \geq 2$.

Proof. Since $x = 2t$ are even integers, we have

$$7^y = z^2 - (3^n)^{2t} = (z + 3^{nt})(z - 3^{nt}). \tag{3.4}$$

Since 7 is prime and $y \geq 1$, we get by Equation (3.1) there exists integers $a > b$ such that

$$z + 3^{nt} = 7^a, \;\; z - 3^{nt} = 7^b$$

where $a + b = y$. Therefore,

$$7^b[7^{a-b} - 1] = 2 \cdot 3^{nt}. \tag{3.5}$$

Since 7 and 2, and 3 are distinct primes, we get by Equation (3.5) that $b = 0$. Hence Equation (3.5) becomes

$$2 \cdot 3^{nt} = 7^y - 1. \tag{3.6}$$

We put $nt = u$ then Equation (3.6) becomes

$$2 \cdot 3^u = 7^y - 1. \tag{3.7}$$

We divided it into two cases.

1. The case $u = 2r$. Then Equation (3.7) becomes

$$2 \cdot (3^r)^2 = 7^y - 1. \tag{3.8}$$

- If $y = 3l$, then Equation (3.8) becomes

$$(4 \cdot 3^r)^2 = 2^3 \cdot 7^{3l} - 2^3. \tag{3.9}$$

We put $Y = 4 \cdot 3^r$ and $X = 2 \cdot 7^l$ then Equation (3.9) becomes

$$Y^2 = X^3 - 2^3$$

is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) is $(2, 0)$. Hence Equation (3.9) has no non-negative integer solutions.

- If $y = 3l + 1$, then Equation (3.8) becomes

$$2^3 \cdot 7^2 \cdot 2 \cdot (3^r)^2 = 2^3 \cdot 7^{3l+3} - 2^3 \cdot 7^2. \tag{3.10}$$

We put $Y = 4 \cdot 7 \cdot 3^r$ and $X = 2 \cdot 7^{l+1}$. Then Equation (3.10) becomes

$$Y^2 = X^3 - 2^3 \cdot 7^2$$

is an elliptic curve. Using the Magma Calculator this equation has no non-negative solutions (X, Y). So, in this case, Equation (3.8) has no solutions.

- If $y = 3l + 2$, then Equation (3.8) becomes

$$2^3 \cdot 7^4 \cdot 2 \cdot (3^r)^2 = 2^3 \cdot 7^{3l+6} - 2^3 \cdot 7^4. \tag{3.11}$$

We put $Y = 4 \cdot 7^2 \cdot 3^r$ and $X = 2 \cdot 7^{l+2}$. Then Equation (3.11) becomes

$$Y^2 = X^3 - 2^3 \cdot 7^4$$

is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) is $(6402, 512240)$ that means $2 \cdot 7^{l+2} = 6402$, a contradiction. So, in this case, Equation (3.11) has no solutions.

2. The case $u = 2r + 1$ is odd. Then Equation (3.7) becomes

$$(6 \cdot 3^r)^2 = 6 \cdot 7^y - 6. \tag{3.12}$$

- If $y = 3l$, then Equation (3.12) becomes

$$6^2(6 \cdot 3^r)^2 = 6^3 \cdot 7^{3l} - 6^3. \tag{3.13}$$

We put $Y = 36 \cdot 3^r$ and $X = 6 \cdot 7^l$. Then Equation (3.13) becomes

$$Y^2 = X^3 - 6^3$$

is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) are $(6, 0)$, $(10, 28)$, $(33, 189)$. So Equation (3.12) has no non-negative solutions.

- If $y = 3l + 1$, then Equation (3.12) becomes
$$6^2 \cdot 7^2(6 \cdot 3^r)^2 = 6^3 \cdot 7^{3l+3} - 6^3 \cdot 7^2. \tag{3.14}$$
We put $Y = 252 \cdot 3^r$ and $X = 6 \cdot 7^{l+1}$. Then Equation (3.14) becomes
$$Y^2 = X^3 - 6^3 \cdot 7^2$$
is an elliptic curve. Using the Magma Calculator this equation has non-negative solutions (X, Y) are $(22, 8)$, $(25, 71)$, $(42, 252)$, $(105, 1071)$, $(294, 5040)$, and $(394, 7820)$. Hence $6 \cdot 7^{l+1} = 42$ or $6 \cdot 7^{l+1} = 294$, that means $l = 0$ or $l = 1$. If $l = 0$ then $y = 1$ and $nt = 1$ that means $y = 1$ and $n = t = 1$, and so $x = 2, z = 4$. If $l = 1$ then $y = 4$. So, Equation (3.7) becomes $3^{nt} = 1200$, a contradiction.
- If $y = 3l + 2$, then Equation (3.12) becomes
$$6^2 \cdot 7^4(6 \cdot 3^r)^2 = 6^3 \cdot 7^{3l+6} - 6^3 \cdot 7^4. \tag{3.15}$$
We put $Y = 6^2 \cdot 7^2 \cdot 3^r$ and $X = 6 \cdot 7^{l+2}$. Then Equation (3.15) becomes
$$Y^2 = X^3 - 6^3 \cdot 7^4$$
is an elliptic curve. Using the Magma Calculator this equation has only non-negative solution (X, Y) is $(106, 820)$. Therefore, Equation (3.15) has no non-negative solutions.
Thus if x is even, then $(x, y, z) = (2, 1, 4)$ is only solution of the Diophantine equation $(3^n)^x + 7^y = z^2$ when $n = 1$. □

For the case $m = 2$ then the Diophantine equation $(p^n)^x + (p + 4^m)^y = z^2$ becomes $(3^n)^x + 19^y = z^2$. With the same of the proof of Theorem 3.2, we have the following theorem.

Theorem 3.3. *Let $x = 2t$ for some positive integer t. Then the Diophantine equation*
$$(3^n)^x + 19^y = z^2$$
has only solution is $(x, y, z) = (4, 1, 10)$ if $n = 1$ and $(x, y, z) = (2, 1, 10)$ if $n = 2$, and has no solutions if $n \geq 3$.

Proof. Similar to the proof of Theorem 3.2, we have the equation
$$2 \cdot 3^{nt} = 19^y - 1. \tag{3.16}$$
We put $nt = u$. Then Equation (3.16) becomes
$$2 \cdot 3^u = 19^y - 1. \tag{3.17}$$
We divided it into two cases.

1. The case $u = 2r$. Then Equation (3.17) becomes

$$2 \cdot (3^r)^2 = 19^y - 1. \tag{3.18}$$

   - If $y = 3l$, then Equation (3.18) becomes

$$(4 \cdot 3^r)^2 = 2^3 \cdot 19^{3l} - 2^3. \tag{3.19}$$

   We put $Y = 4 \cdot 3^r$ and $X = 2 \cdot 19^l$. Then Equation (3.19) becomes

$$Y^2 = X^3 - 2^3. \tag{3.20}$$

   is an elliptic curve. Using the Magma Calculator, Equation (3.20) has non-negative solutions (X, Y) is $(2, 0)$. Hence $4 \cdot 3^r = 0$, a contradiction. Therefore, Equation (3.19) has no non-negative integer solutions.

   - If $y = 3l + 1$, then Equation (3.18) becomes

$$2^3 \cdot 19^2 \cdot 2 \cdot (3^r)^2 = 2^3 \cdot 19^{3l+3} - 2^3 \cdot 19^2. \tag{3.21}$$

   We put $Y = 4 \cdot 19 \cdot 3^r$ and $X = 2 \cdot 19^{l+1}$. Then Equation (3.21) becomes

$$Y^2 = X^3 - 2^3 \cdot 19^2$$

   is an elliptic curve. Using the Magma Calculator, this equation has non-negative solutions (X, Y) are $(17, 45)$, $(38, 228)$, $(114, 1216)$ and $(209, 3021)$. Hence

$$2 \cdot 19^{l+1} = 38 \quad \text{and} \quad 4 \cdot 19 \cdot 3^r = 228$$

   that means $l = 0, r = 1$. Therefore, we have $nt = 2, y = 1$. So, Equation (3.16) has no solutions if $n \geq 3$, and the non-negative integer solutions of Equation (3.16) are $(x, y, z) = (4, 1, 10)$ if $n = 1$ and $(x, y, z) = (2, 1, 10)$ if $n = 2$.

   - If $y = 3l + 2$, then Equation (3.18) becomes

$$2^3 \cdot 19^4 \cdot 2 \cdot (3^r)^2 = 2^3 \cdot 19^{3l+6} - 2^3 \cdot 19^4. \tag{3.22}$$

   We put $Y = 4 \cdot 19^2 \cdot 3^r$ and $X = 2 \cdot 19^{l+2}$. Then Equation (3.22) becomes

$$Y^2 = X^3 - 2^3 \cdot 19^4$$

   is an elliptic curve. Using the Magma Calculator this equation has no non-negative solutions. So, in this case, Equation (3.18) has no solutions.

2. The case $u = 2r + 1$. Then Equation (3.17) becomes

$$(6 \cdot 3^r)^2 = 6 \cdot 19^y - 6. \tag{3.23}$$

- If $y = 3l$, then Equation (3.23) becomes

$$6^2(6 \cdot 3^r)^2 = 6^3 \cdot 19^{3l} - 6^3. \tag{3.24}$$

We put $Y = 36 \cdot 3^r$ and $X = 6 \cdot 19^l$. Then Equation (3.24) becomes

$$Y^2 = X^3 - 6^3$$

is an elliptic curve. Using the Magma Calculator, Equation (3.24) has non-negative solutions (X, Y) are $(6, 0), (10, 28), (33, 189)$. So, Equation (3.24) has no non-negative solutions.

- If $y = 3l + 1$, then Equation (3.23) becomes

$$6^2 \cdot 19^2(6 \cdot 3^r)^2 = 6^3 \cdot 19^{3l+3} - 6^3 \cdot 19^2. \tag{3.25}$$

We put $Y = 36 \cdot 19 \cdot 3^r$ and $X = 6 \cdot 19^{l+1}$. Then Equation (3.25) becomes

$$Y^2 = X^3 - 6^3 \cdot 19^2$$

is an elliptic curve. Using the Magma Calculator this equation has no non-negative solutions. So, in this case, Equation (3.23) has no solutions.

- If $y = 3l + 2$, then Equation (3.23) becomes

$$6^2 \cdot 19^4(6 \cdot 3^r)^2 = 6^3 \cdot 19^{3l+6} - 6^3 \cdot 19^4. \tag{3.26}$$

We put $Y = 6^2 \cdot 19^2 \cdot 3^r$ and $X = 6 \cdot 19^{l+2}$. Then Equation (3.26) becomes

$$Y^2 = X^3 - 6^3 \cdot 19^4$$

is an elliptic curve. Using the Magma Calculator this equation has no non-negative solutions. So, in this case, Equation (3.23) has no solutions. □

Next, we consider y to be an even number. Then we have the following theorem.

Theorem 3.4. *Let m be an integer such that $4^m + 3$ is prime and $y = 2s$ be even for some positive integer s. Then the Diophantine equation*

$$(3^n)^x + (4^m + 3)^y = z^2$$

has no non-negative integer solutions.

Proof. Since $y = 2s, s > 0$, the Diophantine equation

$$(3^n)^x + (4^m + 3)^y = z^2$$

becomes

$$3^{nx} = z^2 - (4^m + 3)^{2s} = [z + (4^m + 3)^s][z - (4^m + 3)^s]. \tag{3.27}$$

Since 3 is prime and $x \geq 1$, we get by Equation (3.27) there exists integers $a > b$ such that

$$z + (4^m + 3)^s = 3^a, \quad z - (4^m + 3)^s = 3^b$$

where $a + b = nx$. Therefore,

$$3^b[3^{a-b} - 1] = 2 \cdot (4^m + 3)^s. \tag{3.28}$$

Since $4^m + 3 > 3$ and 2, and 3 are distinct primes, we get by Equation (3.28) that $b = 0$. Combined with the condition $x \geq 1$, Equation (3.28) becomes

$$2 \cdot (4^m + 3)^s = (3^n)^x - 1.$$

We set $nx = u$. Since $4^m + 3$ is a prime number, we have

$$3^{4^m+2} \equiv 1 \,(\mathrm{mod}\,(4^m + 3)).$$

Let i and j be integers such that $0 \leq i, j < 4^m + 2$ and $i \neq j$. Then we have

$$3^{i+(4^m+2)k} \not\equiv 3^{j+(4^m+2)k} \,(\mathrm{mod}(4^m + 3))$$

for all integers k. We divided it into two cases.

1. Let $u = i + (4^m + 2)k$ be a positive integer, where $0 < i < 4^m + 2$ and k is a non-negative integer. Then

   $$2 \cdot (4^m + 3)^s = 3^u - 1 \not\equiv 0 \,(\mathrm{mod}(4^m + 3))$$

   a contradiction.

2. Let $u = (4^m + 2)k$ be a positive integer, where k is a positive integer. Clearly, 3 is a divisor of $4^m + 2$ for all positive integers m. Hence, there exists a positive integer v such that $4^m + 2 = 3v$. Therefore,

   $$2 \cdot (4^m + 3)^s = 3^{3vk} - 1 = 27^{vk} - 1 = 2 \cdot 13 \cdot [27^{vk-1} + \ldots + 1]$$

   a contradiction, because 13 is not divisor of $2 \cdot (4^m + 3)^s$. Thus, in this case, the Diophantine equation

   $$(3^n)^x + (4^m + 3)^y = z^2$$

   has no non-negative integer solutions. □

Finally, we consider x and y to be both odd integers. Then we have the following theorem.

Theorem 3.5. *Let m be an integer such that $4^m + 3$ is prime. Suppose that $x = 2t + 1, y = 2s + 1$ for some positive integers s, t. Then the Diophantine equation*

$$(3^n)^x + (4^m + 3)^y = z^2$$

has no non-negative integer solutions for all $m \geq 3$ and has only solution is $(x, y, z) = (1, 1, 4)$ if $n = 2$ and $m = 1$, and has only solutions is $(x, y, z) = (1, 1, 10)$ if $n = 2$ and $m = 2$.

Proof. We divided it into two cases.

1. Let n is even. Then $n = 2l$ for some integer l. The equation $(3^n)^x + (4^m + 3)^y = z^2$ becomes
$$(3^l)^{2x} + (4^m + 3)^y = z^2 \qquad (3.29)$$
   - Let $m \geq 3$. By Theorem 3.1, Equation (3.29) has no non-negative integer solutions.
   - Let $m = 1$. We get by Theorem 3.2 that Equation (3.29) has only solution is $(2x, y, z) = (2, 1, 4)$ if $l = 1$. Hence $(x, y, z) = (1, 1, 4)$ if $n = 2$.
   - Let $m = 2$. Since x is odd, we get by Theorem 3.3 that Equation (3.29) has only solution is $(2x, y, z) = (2, 1, 10)$ if $l = 2$. Hence $(x, y, z) = (1, 1, 10)$ if $n = 4$.

2. Let n be odd and $x = 2t + 1$, and $y = 2s + 1$ for some positive t and s. We get by Lemma 2.1 that there exists integers A, B such that $(3^n)^{2t+1} = 4A+3$ and $(4^m + 3)^{2s+1} = 4B + 3$. Since 3 and $4^m + 3$ are odd prime numbers, we have $z^2 = (3^n)^x + (4^m + 3)^y$ is even. Hence z is even, which means $z = 2c$ for some integer c. Therefore, $z^2 = 4c^2$, which means 4 is a divisor of z^2. On the other hand, we have
$$z^2 = (3^n)^{2t+1} + (4^m + p)^{2s+1} = 4(A + B) + 2,$$
a contradiction. □

4. Conclusion

In this paper, we give methods to solve the Diophantine equation $(3^n)^x+(3+4^m)^y = z^2$. This method is perfectly applicable to similar Diophantine equations.

Conflict of interest

The authors declare that there are no potential conflicts of interest regarding the publication of this work. And there are no financial and personal relationships with other people or organizations that can inappropriately influence our work.

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