

Isoptic curves of cycloids

Géza Csima

Department of Algebra and Geometry, Institute of Mathematics,
Budapest University of Technology and Economics,
Műgyetem rkp. 3., H-1111 Budapest, Hungary
csgeza@math.bme.hu

Abstract. The history of the isoptic curves goes back to the 19th century, but nowadays the topic is experiencing a renaissance, providing numerous new results and new applications. First, we define the notion of isoptic curve and outline some of the well-known results for strictly convex, closed curves. Overviewing the types of centered trochoids, we will be able to give the parametric equation of the isoptic curves of hypocycloids and epicycloids. Furthermore, we will determine the corresponding class of curves. Simultaneously, we show that a generalized support function can be given to these types of curves in order to apply and extend the results for strictly convex, closed curves. The calculation methods used during the procedure provide an excellent example of the application of univariate calculus, parametric curves, and vector calculus in geometry and can therefore be processed by either advanced high school students or university students.

Keywords: isoptic curves, trochoids

AMS Subject Classification: 51M04, 53A04, 97G70

1. Introduction

In this manuscript we work in the Euclidean plane \mathbf{E}^2 . Let us introduce the following definition:

Definition 1.1 ([30]). The locus of the intersection of tangents to a curve (or curves) meeting at a constant angle α ($0 < \alpha < \pi$) is the α -isoptic of the given curve (or curves). The isoptic curve with right angle called *orthoptic curve*.

Although the name “isoptic curve” was suggested by Taylor in 1884 ([26]), reference to former results can be found in [30]. In the obscure history of isoptic curves, we can find the names of la Hire (cycloids 1704) and Chasles (conics and

epitrochoids 1837) among the contributors of the subject, however, the details of the research results are not available in English. A very interesting table of isoptic and orthoptic curves is introduced in [30], unfortunately without any exact reference of its source. *Our goal in this paper is to independently reconstruct some of the missing computations for the isoptic curves of hypocycloids and epicycloids and to extend the results presented in [2] and [11].*

However, recent works are available on the topic, which shows its timeliness. In [2] the Euclidean isoptic curves of closed strictly convex curves are studied using their support function. Papers [15, 28, 29] deal with Euclidean curves having a circle or an ellipse for an isoptic curve. Further curves appearing as isoptic curves are well studied in Euclidean plane geometry \mathbf{E}^2 , see e.g. [17, 23]. Isoptic curves of conic sections have been studied in [12] and [24]. There are results for Bezier curves by Kunkli et al. as well, see [14]. Many papers focus on the properties of isoptics, e.g. [18–20], and the references therein. There are some generalizations of the isoptics as well e.g. equioptic curves in [22] by Odehnal or secantoptics in [21, 25] by Skrzypiec.

An algorithm for convex polyhedrons has been given by the authors in [8] in order to generalize the notion of isoptic curve into the space and it has been developed by Kunkli et al. for non convex cases in [13]. The spatial case encompasses many applications in both physical and architectural aspects, see [9].

There are some results in non-Euclidean geometries as well. The isoptic curves of the hyperbolic line segment and proper conic sections are determined in [4, 6, 7]. For generalized conic sections, and for their isoptics, see [5]. The isoptics of conic sections in elliptic geometry \mathcal{E}^2 are determined in [4].

There are some results in three dimensional Thurston geometries as well. The isoptic surface of segments has been determined in [10] in \mathbf{Nil} geometry and in [3] for $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$ geometries.

2. Preliminary results

In order to conduct further investigations on isoptics we need to summarize some preliminary results on the support function. We remind the dear reader that the complex number plane and \mathbf{R}^2 are isomorphic to each other, in such a way that real and imaginary parts give the first and second coordinates respectively. Furthermore, we will use the well-known Euler formula: $e^{it} = \cos(t) + i \sin(t) \Leftrightarrow (\cos(t), \sin(t))$.

Definition 2.1. Let \mathcal{C} be a closed, strictly convex curve which surrounds the origin. Let $p(t)$ where $t \in [0, 2\pi[$ be the distance from 0 to the support line of \mathcal{C} being perpendicular to the vector e^{it} . The function p is called a support function of \mathcal{C} .

It is well-known [1] that the support function of a planar, closed, strictly convex curve \mathcal{C} is differentiable. For now we would like to express the isoptic of \mathcal{C} using

the support function. We claim the following lemma omitting the proof which can be found for example in [27].

Lemma 2.2 ([27]). *If $f(x, y, t) = 0$ is a family of straight lines, then the equation of the envelope of these lines can be obtained by eliminating the variable t from the two equations $f(x, y, t) = 0$ and $\frac{d}{dt}f(x, y, t) = 0$.*

This is used in [27] to prove the following theorem.

Theorem 2.3 ([27]). *Given a planar, closed, strictly convex curve \mathcal{C} in polar coordinates with the radius z a function of angle t , where $t \in [0, 2\pi)$. Then the following equation holds*

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}.$$

The corollary of this theorem is that we may use this parametrization to determine the isoptic curve of \mathcal{C} . The angle of $p(t)$ and $p(t + \pi - \alpha)$ is α , since the $p(t)$, $p(t + \pi - \alpha)$ and their support lines determine a cyclic quadrilateral (see Figure 1). Our goal is to determine the intersection of these tangent lines which is the fourth vertex opposite the origin. A proof can be found in [2].

Theorem 2.4 ([2]). *Let \mathcal{C} be a plane, closed, strictly convex curve and suppose that the origin is in the interior of \mathcal{C} . Let $p(t)$, $t \in [0, 2\pi]$ be the support function of \mathcal{C} . Then the α -isoptic curve of \mathcal{C} has the form*

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t) \cot(\pi - \alpha) + \frac{1}{\sin(\pi - \alpha)} p(t + \pi - \alpha) \right) ie^{it}. \quad (2.1)$$

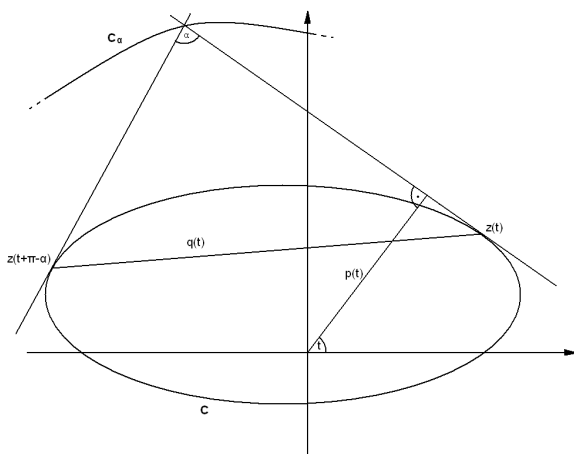


Figure 1

Definition 2.5 ([30]). *A hypocycloid is generated by a point on a circle rolling internally upon a fixed circle. An epicycloid is generated by a point on a circle*

rolling externally upon a fixed circle. A *hypotrochoid* is generated by a point rigidly attached to a circle rolling internally upon a fixed circle. An *epitrochoid* is generated by a point rigidly attached to a circle rolling externally upon a fixed circle.

We will use the following parametric equations of the hypo- and epicycloids, where we assumed that the radius of the fixed circle is 1, and the radius of the rolling circle is rational $\frac{1}{a} := \frac{p}{q} < 1$ in its lowest terms, otherwise the curve never closes, and fills the space between the circles. Then we have exactly p cusps and it is closed if and only if the length of parametric domain of t is greater than equal to $2q\pi$. In the case of hypocycloid, we also assume, that $2p \neq q$, which results in a segment. Since the parametrization of the hypo- and epicycloids are very similar, we consider them together. Hereinafter, in the notation \pm or \mp , the upper sign refers to the hypocycloid, the lower one to the epicycloid.

$$\left\{ \frac{(a \mp 1) \cos(t) \pm \cos((a \mp 1)t)}{a}, \frac{(a \mp 1) \sin(t) - \sin((a \mp 1)t)}{a} \right\}$$

Finally, we need the parametric equations of the hypo- and epitrochoids using the above sign convention:

$$\left\{ (A \pm B) \cos(t) \pm H \cos\left(\frac{A \mp B}{B}t\right), (A \mp B) \sin(t) - H \sin\left(\frac{A \mp B}{B}t\right) \right\} \quad (2.2)$$

where the radius of the fixed and the rolling circles are A and B respectively, and H is the distance of the rigid point to the center of the rolling circle (see [16]).

3. Isoptic curves

Since the calculations of the isoptic curves of hypo- and epicycloids are also very similar, we consider them together. Our first step, to determine the isoptic curves, is always the tangent calculation. We need the derivative of the parametrization:

$$\begin{aligned} v_{H/E}(t) &= \left\{ \mp \frac{2(a \mp 1) \sin\left(\frac{at}{2}\right) \cos\left(\frac{(a \mp 2)t}{2}\right)}{a}, \frac{2(a \mp 1) \sin\left(\frac{at}{2}\right) \sin\left(\frac{(a \mp 2)t}{2}\right)}{a} \right\} \\ &= \frac{2(a \mp 1) \sin\left(\frac{at}{2}\right)}{a} \left\{ \mp \cos\left(\frac{(a \mp 2)t}{2}\right), \sin\left(\frac{(a \mp 2)t}{2}\right) \right\} \\ &= \frac{2(a \mp 1) \sin\left(\frac{at}{2}\right)}{a} \bar{v}_{H/E}(t) \end{aligned} \quad (3.1)$$

where we applied trigonometric product-to-sum and sum-to-product identities.

Remark 3.1. The tangent vector can be a null vector for discrete parameter values if $\sin\left(\frac{at}{2}\right) = 0$, but its direction may be determined in limit so that continuity remains.

Now, it is easy to see, that the angle of two tangents is equal to the the angle of the corresponding tangent vectors. Considering the $t + \phi$ and $t - \phi$ parametric values:

$$\begin{aligned}
& \frac{\langle v_{H/E}(t - \phi), v_{H/E}(t + \phi) \rangle}{\|v_{H/E}(t - \phi)\| \|v_{H/E}(t + \phi)\|} \\
&= \langle \bar{v}_{H/E}(t - \phi), \bar{v}_{H/E}(t + \phi) \rangle \\
&= \cos\left(\frac{(a \mp 2)(t - \phi)}{2}\right) \cos\left(\frac{(a \mp 2)(t + \phi)}{2}\right) \\
&\quad + \sin\left(\frac{(a \mp 2)(t - \phi)}{2}\right) \sin\left(\frac{(a \mp 2)(t + \phi)}{2}\right) \\
&= \cos\left(\frac{(a \mp 2)(t + \phi)}{2} - \frac{(a \mp 2)(t - \phi)}{2}\right) = \cos((a \mp 2)\phi) \tag{3.2}
\end{aligned}$$

that is *independent form the parameter value of t* . This is not necessarily true for other classes of curves. This uniformity gives us the possibility to determine the isoptic curve. Let $\phi := \frac{\alpha}{a \mp 2}$ be true, if we are interested in the α -isoptic curve. Then the angle of the oriented tangents that are drawn to points corresponding to the parameter values $t - \phi$ and $t + \phi$ is α .

Remark 3.2. In the case of the astroid ($a = 4$), the value of ϕ is $\frac{\alpha}{2}$ so that the difference of considered two points in the parameter domain is exactly α .

From formula (3.1), we can derive the equation of the tangent respected to the parameter t :

$$x \sin\left(\frac{(a \mp 2)t}{2}\right) \pm y \cos\left(\frac{(a \mp 2)t}{2}\right) = \frac{(a \mp 2) \sin\left(\frac{at}{2}\right)}{a} \tag{3.3}$$

By replacing t with $t - \phi$ and $t + \phi$, we get an equation system. We are looking for the common point of the above tangents that will be a point of the isoptic curve related to the parameters t and α . When solving the system of equations, it is worth using Cramer's rule, since only 2×2 determinants need to be calculated. Due to the length of the solution process, we do not report it here, but after trigonometric simplification this is the result, which will be the parametrization of the isoptic curve, as well:

$$\begin{aligned}
x(t) &= \frac{(a \mp 2) \left(\sin\left(\frac{(a \mp 1)\alpha}{a \mp 2}\right) \cos(t) \pm \sin\left(\frac{\alpha}{a \mp 2}\right) \cos((a \mp 1)t) \right)}{a \sin(\alpha)} \\
y(t) &= \frac{(a \mp 2) \left(\sin\left(\frac{(a \mp 1)\alpha}{a \mp 2}\right) \sin(t) - \sin\left(\frac{\alpha}{a \mp 2}\right) \sin((a \mp 1)t) \right)}{a \sin(\alpha)}
\end{aligned}$$

We can propose the following theorem realizing the similarities to (2.2):

Theorem 3.3. *Let us be given a hypo- or epicycloid with its parametrization*

$$\left\{ \frac{(a \mp 1) \cos(t) \pm \cos((a \mp 1)t)}{a}, \frac{(a \mp 1) \sin(t) - \sin((a \mp 1)t)}{a} \right\}$$

where $a = \frac{q}{p}$ and $t \in [0, 2q\pi]$ such that $p, q \in \mathbb{Z}^+ \wedge p < q \wedge 2p \neq q$. Then the α -isoptic curve of it is a hypo- or epitrochoid given by the parametrization

$$\left\{ (A \mp B) \cos(t) \pm H \cos\left(\frac{A \mp B}{B}t\right), (A \mp B) \sin(t) - H \sin\left(\frac{A \mp B}{B}t\right) \right\},$$

where

$$A = \frac{(a \mp 2) \sin\left(\frac{(a \mp 1)\alpha}{a \mp 2}\right)}{(a \mp 1) \sin(\alpha)}, \quad B = \frac{(a \mp 2) \sin\left(\frac{(a \mp 1)\alpha}{a \mp 2}\right)}{a(a \mp 1) \sin(\alpha)}, \quad H = \frac{(a \mp 2) \sin\left(\frac{\alpha}{a \mp 2}\right)}{a \sin(\alpha)}.$$

Remark 3.4. It is easy to see, that for the case of hypocycloid, if $\alpha = \frac{a-2}{a-1}\pi$, then $A = B = 0$ in Theorem 3.3, so that the resulted parametric curve is a circle, centered at the origin with radius

$$\frac{(a-2) \sin\left(\frac{\pi}{a-1}\right)}{a \sin\left(\frac{a-2}{a-1}\right)}.$$

For the epicycloid, in Theorem 3.3 $A = B = 0$ if $\alpha = \frac{a+2}{a+1}\pi$ but that angle is greater than π , therefore it is not a real isoptic curve.

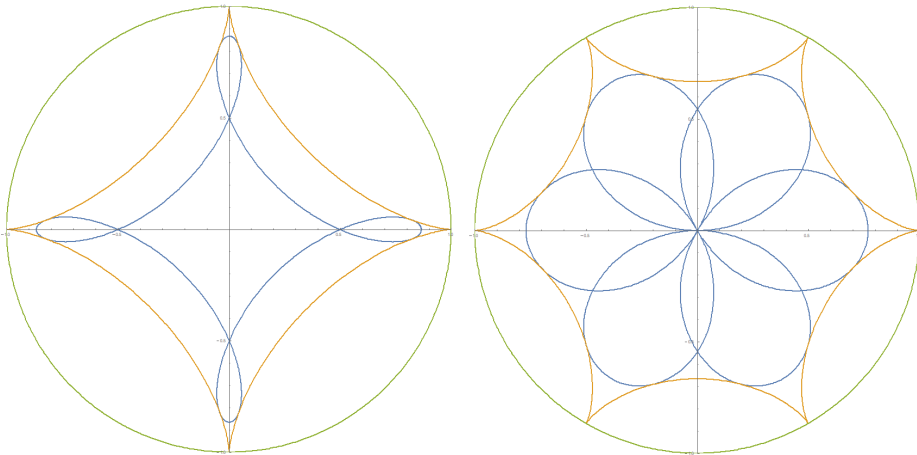


Figure 2. Isoptic curve for hypocycloid with $a = 4, \alpha = \frac{\pi}{3}$ (left) and $a = 6, \alpha = \frac{2\pi}{3}$ (right).

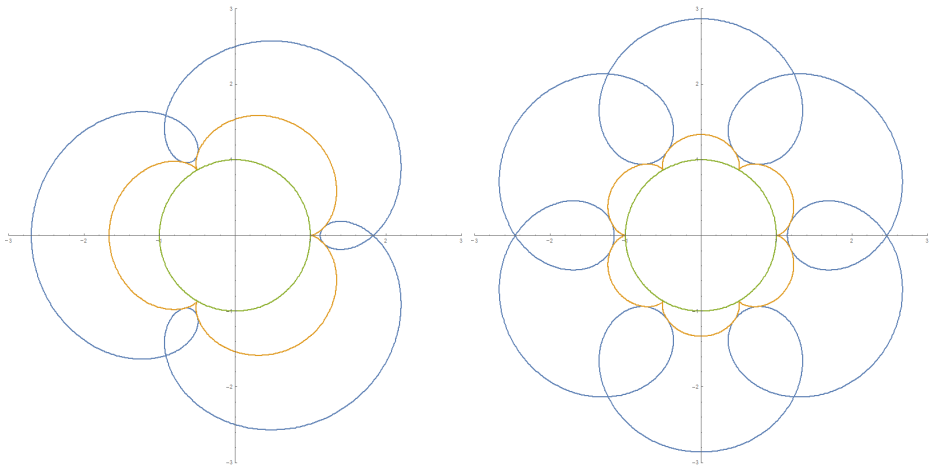


Figure 3. Isoptic curve for epicycloid with $a = 3$, $\alpha = \frac{\pi}{3}$ (left) and $a = 6$, $\alpha = \frac{\pi}{6}$ (right).

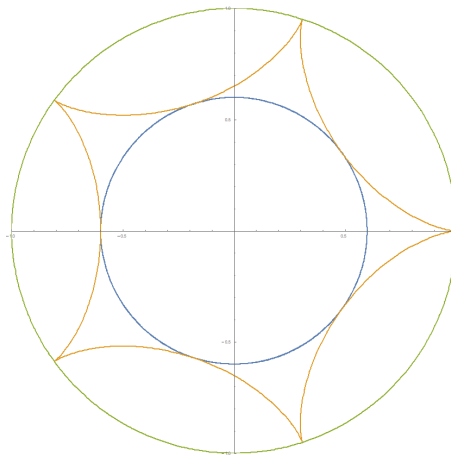


Figure 4. Isoptic curve as a circle for hypocycloid with $a = 5$, $\alpha = \frac{3\pi}{4}$.

4. Isoptic curves by support functions

One can realize that the tangents in formula (3.3) is in Hesse form, therefore it is easy to calculate the distance of the line to the origin. Despite hypocycloids and epicycloids are non-convex curves, we can define their support function nonetheless in order to give another approach of the isoptic curve by Theorem 2.4. We only

have to apply a substitution: $t = \frac{2}{a \mp 2} \left(\frac{\pi}{2} - u \right)$ in (3.1) to obtain:

$$x \cos(u) + y \sin(u) = \frac{(a \mp 2)}{a} \sin \left(\frac{a}{a \mp 2} \left(\frac{\pi}{2} - u \right) \right).$$

It is easy to see, that the transverse vector of the tangent is $e^{iu} = \{\cos(u), \sin(u)\}$ and its distance to the origin is $\frac{(a \mp 2)}{a} \sin \left(\frac{a}{a \mp 2} \left(\frac{\pi}{2} - u \right) \right)$. Then we can define the quasi-support functions:

$$p_{H/E}(u) = \frac{(a \mp 2)}{a} \sin \left(\frac{a}{a \mp 2} \left(\frac{\pi}{2} - u \right) \right). \quad (4.1)$$

From this point on, the procedure is fairly straightforward, especially with the help of some mathematical software. First, we apply (2.1) from Theorem 2.4 to (4.1), then we return to the original t variable. Finally, the resulting parametrization can be simplified with the help of the trigonometric product-to-sum and sum-to-product identities already used before, until we arrive at a form of the formulas that leads to the parametrization of the hypo- and epitrochoid by shifting the parameter range or, in the case of the epicycloid, by changing the direction as well.

Theorem 4.1. *Let us be given a \mathcal{C} hypo- or epicycloid with its parametrization*

$$\mathcal{C}: \left\{ \frac{(a \mp 1) \cos(t) \pm \cos((a \mp 1)t)}{a}, \frac{(a \mp 1) \sin(t) - \sin((a \mp 1)t)}{a} \right\}$$

where $a = \frac{q}{p}$ and $t \in [0, 2q\pi]$ such that $p, q \in \mathbb{Z}^+ \wedge p < q \wedge 2p \neq q$. Then the α -isoptic curve of \mathcal{C} has the form

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t) \cot(\pi - \alpha) + \frac{1}{\sin(\pi - \alpha)} p(t + \pi - \alpha) \right) ie^{it},$$

where $p(t) = \frac{(a \mp 2)}{a} \sin \left(\frac{a}{a \mp 2} \left(\frac{\pi}{2} - t \right) \right)$, $t \in [0, 2\pi]$ is the support function of \mathcal{C} .

References

- [1] T. BONNESEN, W. FENCHEL: *Ergebnisse der Mathematik und Ihrer Grenzgebiete. 1. Folge*, in: *Theorie der Konvexen Körper*, Berlin, Heidelberg: Springer, 1934, DOI: [10.1007/978-3-642-47404-0](https://doi.org/10.1007/978-3-642-47404-0).
- [2] W. CIEŚLAK, A. MIERNOWSKI, W. MOZGAWA: *Isoptics of a closed strictly convex curve*, in: *Global Differential Geometry and Global Analysis*, Berlin, Heidelberg: Springer, 1991, pp. 28–35, ISBN: 978-3-540-46445-7, DOI: [10.1007/BFb0083625](https://doi.org/10.1007/BFb0083625).
- [3] G. CSIMA: *Isoptic surfaces of segments in $S^2 \times \mathbf{R}$ and $H^2 \times \mathbf{R}$ geometries*, *Journal of Geometry* 115.1 (2023), DOI: [10.1007/s00022-023-00699-x](https://doi.org/10.1007/s00022-023-00699-x).
- [4] G. CSIMA, J. SZIRMAI: *Isoptic curves of conic sections in constant curvature geometries*, *Mathematical Communications* 19.2 (2014), pp. 277–290.

- [5] G. CSIMA, J. SZIRMAI: *Isoptic curves of generalized conic sections in the hyperbolic plane*, Ukrainian Mathematical Journal 71 (12 2020), pp. 1929–1944, DOI: [10.1007/s11253-020-01756-3](https://doi.org/10.1007/s11253-020-01756-3).
- [6] G. CSIMA, J. SZIRMAI: *Isoptic curves of the conic sections in the hyperbolic and elliptic plane*, Stud. Univ. Žilina 24.1 (2010), pp. 15–22.
- [7] G. CSIMA, J. SZIRMAI: *Isoptic curves to parabolas in the hyperbolic plane*, Pollac Periodica 1.1 (2012), pp. 55–64.
- [8] G. CSIMA, J. SZIRMAI: *Isoptic surfaces of polyhedra*, Computer Aided Geom. Design 47 (2016), pp. 55–60, DOI: [10.1016/j.cagd.2016.03.001](https://doi.org/10.1016/j.cagd.2016.03.001).
- [9] G. CSIMA, J. SZIRMAI: *On the isoptic hypersurfaces in the n -dimensional Euclidean space*, KoG (Scientific and professional journal of Croatian Society for Geometry and Graphics) 17 (2013), pp. 53–57.
- [10] G. CSIMA, J. SZIRMAI: *Translation-Like Isoptic Surfaces and Angle Sums of Translation Triangles in Nil Geometry*, Results in Mathematics 78 (5 2023), DOI: [10.1007/s00025-023-01961-z](https://doi.org/10.1007/s00025-023-01961-z).
- [11] T. DANA-PICARD: *An automated study of isoptic curves of an astroid*, Special Issue on Dynamic Geometry and Automated Reasoning 97 (2020), pp. 56–68, DOI: [10.1016/j.jsc.2018.12.005](https://doi.org/10.1016/j.jsc.2018.12.005).
- [12] G. HOLZMÜLLER: *Einführung in die Theorie der isogonalen Verwandtschaft*, Leipzig-Berlin: B.G. Teuber, 1882.
- [13] R. KUNKLI, F. NAGY, M. HOFFMANN: *New algorithm to find isoptic surfaces of polyhedral meshes*, Computer Aided Geometric Design 64 (1 2018), pp. 90–99, DOI: [10.1016/j.cagd.2018.04.001](https://doi.org/10.1016/j.cagd.2018.04.001).
- [14] R. KUNKLI, I. PAPP, M. HOFFMANN: *Isoptics of Bézier curves*, Computer Aided Geometric Design 30 (1 2013), pp. 78–84, DOI: [10.1016/j.cagd.2012.05.002](https://doi.org/10.1016/j.cagd.2012.05.002).
- [15] Á. KURUSA: *Is a convex plane body determined by an isoptic?*, Beiträge zur Algebra und Geometrie 53 (2012), pp. 281–294, DOI: [10.1007/s13366-011-0074-2](https://doi.org/10.1007/s13366-011-0074-2).
- [16] J. D. LAWRENCE: *A Catalog of Special Plane Curves*, Mineola, New York: Dover Publications, Inc., 1972, pp. 165–168.
- [17] G. LORIA: *Spezielle algebraische und transzendente ebene Kurve*, Leipzig-Berlin: B.G. Teuber, 1911.
- [18] M. MICHALSKA: *A sufficient condition for the convexity of the area of an isoptic curve of an oval*, Rend. Semin. Mat. Univ. Padova 110 (2003), pp. 161–169.
- [19] M. MICHALSKA, W. MOZGAWA: *α -isoptics of a triangle and their connection to α -isoptic of an oval*, Rend. Semin. Mat. Univ. Padova 133 (2015), pp. 159–172.
- [20] M. MICHALSKA, W. MOZGAWA: *On some geometric condition for convexity of isoptics*, Rend. Semin. Mat. Univ. Pol. Torino 55 (2 1997), pp. 93–98.
- [21] M. MICHALSKA, M. SKRZYPIEC: *Crofton formulas and convexity condition for secantoptics*, Bull. Belg. Math. Soc. - Simon Stevin 16 (3 2009), pp. 435–445, DOI: [10.36045/bbms/1251832370](https://doi.org/10.36045/bbms/1251832370).
- [22] B. ODEHNAL: *Equioptic curves of conic section*, J. Geom. Graphics 14 (1 2010), pp. 29–43, DOI: [10.36045/bbms/1251832370](https://doi.org/10.36045/bbms/1251832370).
- [23] T. SCH: *Spezielle ebene Kurven*, Monatshefte für Mathematik und Physik 21 (1 1910), A60–A61, DOI: [10.1007/BF01693325](https://doi.org/10.1007/BF01693325).
- [24] P. SIEBECK: *Ueber eine Gattung von Curven vierten Grades, welche mit den elliptischen Functionen zusammenhängen*. ger, Journal für die reine und angewandte Mathematik 57 (1860), pp. 359–370, URL: <http://eudml.org/doc/147795>.

- [25] M. SKRZYPIEC: *A note on secantopics*, Beiträge zur Algebra und Geometrie 49 (1 2008), pp. 205–215.
- [26] C. TAYLOR: *Note on a theory of orthoptic and isoptic loci*, Proc. R. Soc. London 37 (1884), pp. 138–141, DOI: [10.1098/rsp1.1884.0024](https://doi.org/10.1098/rsp1.1884.0024).
- [27] D. WIDDER: *Advanced Calculus, 2nd edition*, Englewood Cliffs, NJ: Prentice-Hall, 1962.
- [28] W. WUNDERLICH: *Kurven mit isoptischem Kreis*, Aequationes Mathematicae 5 (2 1970), pp. 337–338, DOI: [10.1007/BF01818468](https://doi.org/10.1007/BF01818468).
- [29] W. WUNDERLICH: *Kurven mit isoptischer Ellipse*, Monatsh. Math. 75 (1971), pp. 346–362.
- [30] R. C. YATES: *A handbook on curves and their properties*, Ann Arbor: J. W. Edwards, 1947.