Units of the semisimple group algebras $\mathcal{F}_q SL(2, 8)$ and $\mathcal{F}_q SL(2, 9)$

Sivaranjani N U$^a$, E. Nandakumar$^a$, Gaurav Mittal$^b$, R. K. Sharma$^c$

$^a$Department of Mathematics, College of Engineering and Technology, SRM Institute of Science & Technology, Kattankulathur - 603 203, Tamil Nadu, INDIA
sn0926@srmist.edu.in, nanda1611@gmail.com

$^b$Defence Research and Development Organization, Near Metcalfe House, Delhi - 110054, INDIA
gaur.mittaltwins@yahoo.com

$^c$Department of Mathematics, Indian Institute of Technology, Delhi - 110016, INDIA
rksharmaiitd@gmail.com

Corresponding Author: E. Nandakumar

Abstract. In this paper, we characterize the structure of the unit group of the semisimple group algebras $\mathcal{F}_q SL(2, 8)$ and $\mathcal{F}_q SL(2, 9)$ of the special linear groups of $2 \times 2$ matrices with determinant 1 over the finite fields of order 8 and 9, respectively.

Keywords: Unit group, Group algebra, Special linear group, Wedderburn Decomposition

AMS Subject Classification: 16U60, 20C05

1. Introduction

The group algebra of the finite group $G$ over the finite field $\mathcal{F}_q$ is denoted by $\mathcal{F}_q G$. Let $q$ and $p$ be the order and characteristics of the finite field $\mathcal{F}_q$, respectively and $q = p^k$. Let $U(\mathcal{F}_q G)$ be the group of units of the group algebra $\mathcal{F}_q G$. Group theory frequently runs into the issues with unit group of the group algebra. The characterization of the unit groups is crucial for a number of applications, including the study of the isomorphism problem [14], one of the most significant research problems in the theory of group algebras, the development of convolutional codes in group algebra (see [5, 9]) and other applications. The structure of the unit
group of the semisimple group algebra $F_q G$ has been extensively studied (see [3, 4, 10, 12–18, 20, 22]). The study by Bakshi et al. [4], in which the unit groups of the semisimple group algebras of all metabelian groups are studied, is one of the most significant ones in this field. As a result, the majority of research in this field focuses on understanding the unit group of non-metabelian group algebras. The unit groups of the group algebras of non-metabelian groups up to order 72 were described by Mittal et al. in [16]. Further, Sharma et al. determined the unit group of the semisimple group algebra (SGA) of the respective groups $SL(2, 3)$ (special linear group over the finite field of 3 elements) and $SL(2, 5)$ (See [11] and [19]). In continuation, Sivaranjani et al. [21] determined the unit group of the SGA of the group $SL(2, 7)$. In addition, Arvind et al. [1] investigated the unit group of the SGA of the group $SL(3, \mathbb{Z}_2)$. The main objective of this paper is to derive the unit group of the SGA of the groups $SL(2, 8)$ and $SL(2, 9)$, respectively. We notice that the difficulty of exactly identifying the unit group of the SGA increases as the size of the group increases. One may refer [15, 17] for some of the recent works in this area. Our first goal in determining the unit group is to infer the Wedderburn decomposition (WD) of $F_q SL(2, 8)$ and $F_q SL(2, 9)$, respectively. Further, it is easy to derive the unit group from the WD. The rest of this paper is structured as follows. The prerequisites for the article are covered in Section 2. In sections 3 and 4, we deduce the unit group of the group algebras $F_q SL(2, 8)$ and $F_q SL(2, 9)$ in the form of theorems 3.1 and 4.1, respectively. Section 5 concludes the paper.

2. Preliminaries

Throughout this paper, $SL(n, r)$ denotes the special linear group of $n \times n$ matrices with determinant 1 over the finite field of order $r$. The order of $SL(n, r)$ is given by

$$(r^n - 1)(r^n - r) \cdots (r^n - r^{n-1})(r - 1)^{-1}.$$ 

Next, we discuss some notations and results from [7]. Let $J(F_q G)$ denote the Jacobson radical of $F_q G$. Let $s$ be the least common multiple of the orders of $p$-regular elements of group $G$ and let $\eta$ be the primitive $s^{th}$ root of unity over a finite field $F$. Let $T_{G,F} = \{t : \eta \rightarrow \eta^t \text{ is an automorphism of } F(\eta) \text{ over } F\}$. Since the Galois group Gal($F(\eta) : F$) is cyclic, for any $\sigma \in$ Gal($F(\eta) : F$), there exists a positive integer $s$ such that $\sigma(\eta) = \eta^s$. For any $p$-regular element $g \in G$ (i.e., $p$ does not divide order of $g$), we define $\gamma_g = \sum h$, where $h$ runs over all the elements in the conjugacy class $C_g$ of $g$. The cyclotomic $F$-class of $\gamma_g$ is defined as $SF(\gamma_g) = \{\gamma_g^t \mid t \in T_{G,F}\}$. The following theorem characterizes the set $T_{G,F}$.

**Theorem 2.1** ([16, Theorem 2.3]). Let $F$ be a finite field with prime power order $d$ such that gcd($d, s) = 1$ and $c = \text{order}_s(d)$ is the multiplicative order of $d$ modulo $s$, then $T_{G,F} = \{1, d, \ldots, d^{c-1}\} \mod s$.

To uniquely identify the Wedderburn decomposition (WD) of the group algebra, the following six results will play an important role.
Proposition 2.2 ([7, Proposition 1.2]). The number of non isomorphic simple components of $\mathcal{F}G/J(\mathcal{F}G)$ is equal to the number of cyclotomic $\mathcal{F}$-classes in $G$.

Theorem 2.3 ([7, Theorem 1.3]). Assume that $G$ has $t$ cyclotomic $\mathcal{F}$-classes and $\operatorname{Gal}(\mathcal{F}(\eta) : \mathcal{F})$ is a cyclic group, then $|S_i| = [\mathcal{F}_i : \mathcal{F}]$ with appropriate index ordering if $S_1, S_2, \ldots, S_t$ are the cyclotomic $\mathcal{F}$-classes of $G$ and $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$ are the simple components of $Z(\mathcal{F}G/J(\mathcal{F}G))$.

Proposition 2.4 ([14, Proposition 3.6.11]). Let $G'$ be the commutator subgroup of $G$ and let $\mathcal{F}G$ be a semisimple group algebra, then

$$\mathcal{F}G \simeq \mathcal{F}(G/G') \oplus \Delta(G,G'),$$

where $\mathcal{F}(G/G')$ is the sum of all commutative simple components of $\mathcal{F}G$ and $\Delta(G,G')$ is the sum of all others.

Proposition 2.5 ([14, Proposition 3.6.7]). Let $N$ be a normal subgroup of $G$ and let $\mathcal{F}G$ be a semisimple group algebra (SGA), then

$$\mathcal{F}G \simeq \mathcal{F}(G/N) \oplus \Delta(G,N),$$

where $\Delta(G,N)$ is an ideal of $\mathcal{F}G$ generated by the set $\{n - 1 : n \in N\}$.

Proposition 2.6 ([6, Proposition 1]). Let $\mathcal{F}G$ be a finite SGA, where characteristics of $\mathcal{F}$ is $p$. Let $\mathcal{F}G \simeq \bigoplus_{i=1}^{t} M_{n_i}(\mathcal{F}_i)$, where $\mathcal{F}_i$ are finite extensions of $\mathcal{F}$ and $r$ is a positive integer. Then $p$ does not divide any of the $n_i$.

Lemma 2.7 ([24]). Let $p_1$ and $p_2$ be two primes. Let $\mathcal{F}_{q_1}$ be a field with $q_1 = p_1^{k_1}$ elements and let $\mathcal{F}_{q_2}$ be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $\mathcal{F}_{q_1}G, \mathcal{F}_{q_2}G$ be semisimple. Suppose that

$$\mathcal{F}_{q_1}G \cong \bigoplus_{i=1}^{t} M(n_i, \mathcal{F}_{q_1}), \ n_i \geq 1$$

and $M(n, \mathcal{F}_{q_2}^r)$ is a Wedderburn component of the group algebra $\mathcal{F}_{q_2}G$ for some $r \geq 1$ and any positive integer $n$, i.e.,

$$\mathcal{F}_{q_2}G \cong \bigoplus_{i=1}^{s} M(m_i, \mathcal{F}_{q_2,i}) \oplus M(n, \mathcal{F}_{q_2}^r), \ m_i \geq 1.$$

Here $\mathcal{F}_{q_2,i}$ is a field extension of $\mathcal{F}_{q_2}$. Then $M(n, \mathcal{F}_{q_1})$ must be a Wedderburn component of the group algebra $\mathcal{F}_{q_1}G$ and it appears at least $r$ times in the WD of $\mathcal{F}_{q_1}G$.

Proposition 2.8 ([1, Corollary 3.8]). Let $\mathcal{F}G$ be a finite SGA, where characteristics of $\mathcal{F}$ is $p$. If there exists an irreducible representations of degree $n$ over $\mathcal{F}$, then one of the Wedderburn component of $\mathcal{F}G$ is $M(n, \mathcal{F})$. 

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3. Unit group of $\mathcal{F}_q SL(2, 8)$

Let $G_1 := SL(2, 8)$. Clearly, the order of $G_1$ is 504. The group algebra $\mathcal{F}_q G_1$ is semi simple for $p \neq 2, 3, 7$ by Maschke’s theorem [14]. Also, one can note that the degrees of irreducible representations of $G_1$ are 1, 7, 8, and 9 whenever $|\mathcal{S}\mathcal{F}_q(\gamma g)| = 1, \forall g \in G_1$ (see [23]). The group $G_1$ has 9 conjugacy classes (let the representative of these classes be denoted by $g_i$ for $i = 1, \ldots, 9$). The representatives (R) of the conjugacy classes, sizes (S) and the orders (O) of representatives are tabulated below.

<table>
<thead>
<tr>
<th>R</th>
<th>$I_2$</th>
<th>[0 1]</th>
<th>[0 1]</th>
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<th>[0 1]</th>
<th>[x 0]</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>1</td>
<td>63</td>
<td>56</td>
<td>56</td>
<td>56</td>
<td>72</td>
</tr>
<tr>
<td>O</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>9</td>
<td>9</td>
<td>7</td>
</tr>
</tbody>
</table>

Here $I_2$ is $2 \times 2$ identity matrix, $x$ is the generator of multiplicative group of finite field of order 8, $\alpha = x^2 + x$ and $\beta = x^2 + 1$. Also, $G_1$ can be generated by two elements $a$ and $b$, where

$$a = \begin{bmatrix} x & 0 \\ 0 & x^2 + 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$  

The exponent of $G_1$ is 126. In this section, we characterize the unit group of the group algebra $\mathcal{F}_q G_1$ for $p \neq 2, 3, 7$ such that the group algebra $\mathcal{F}_q G_1$ is semisimple and $q = p^k$. In the following theorems, $\mathcal{F}_i$ denotes the finite extensions of $\mathcal{F}_q$ and $n_i, r$ are positive integers.

**Theorem 3.1.** The unit group of $\mathcal{F}_q G_1$, where $q = p^k$ and $p \neq 2, 3, 7$ is given as follows:

1. for $p^k \equiv \{1, 55, 71, 125\}$ mod 126, we have

$$U(\mathcal{F}_q G_1) \simeq \mathcal{F}_q^* \oplus GL(7, \mathcal{F}_q)^4 \oplus GL(8, \mathcal{F}_q) \oplus GL(9, \mathcal{F}_q)^3.$$  

2. for $p^k \equiv \{13, 29, 41, 43, 83, 85, 97, 113\}$ mod 126, we have

$$U(\mathcal{F}_q G_1) \simeq \mathcal{F}_q^* \oplus GL(7, \mathcal{F}_q) \oplus GL(8, \mathcal{F}_q) \oplus GL(9, \mathcal{F}_q)^3 \oplus GL(7, \mathcal{F}_{q^3}).$$  

3. $p^k \equiv \{17, 19, 37, 53, 73, 89, 107, 109\}$ mod 126, we have

$$U(\mathcal{F}_q G_1) \simeq \mathcal{F}_q^* \oplus GL(7, \mathcal{F}_q)^4 \oplus GL(8, \mathcal{F}_q) \oplus GL(9, \mathcal{F}_{q^3}).$$  

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Proof. The group algebra $F_q G_1$ is semi simple, it follows from the Wedderburn decomposition theorem [14] that $F_q G_1 \simeq \bigoplus_{i=1}^{r-1} M(n_i, F_q)$. The derived subgroup $G'_1$ of $G_1$ is $G_1$ itself (i.e., $G_1$ is a perfect one). This accompanying with Proposition 2.2 gives

$$F_q G_1 \simeq F_q \bigoplus_{i=1}^{r-1} M(n_i, F_q), \quad n_i \geq 2. \quad (3.2)$$

Using Theorem 2.1, we construct the set $T_{G, F}$ of group $G_1$ and divide the proof into the following 4 cases.

Case 1: $p^k \equiv \{1, 55, 71, 125\}$ mod 126. In this case, we note that the cardinality of cyclotomic $F_q$-class of $\gamma$ is 1, for all $g$ in $G_1$. By employing this with Proposition 2.2 and Theorem 2.3, we further rewrite (3.2) as

$$F_q G_1 \simeq F_q \bigoplus_{i=1}^{8} M(n_i, F_q) \implies 503 = \sum_{i=1}^{8} n_i^2, \quad n_i \geq 2. \quad (3.3)$$

We have discussed earlier in this section that the degrees of irreducible representations of $G_1$ are 1, 7, 8 and 9, whenever $|SF_q(\gamma_g)| = 1, \forall g \in G_1$. We note that there are 158 choices of $n_i$'s fulfilling (4.3). The only choice that only contains 7, 8 and 9 is $(7^4, 8^3, 9^3)$. Hence, the Wedderburn decomposition (WD) is

$$F_q G_1 \simeq F_q \bigoplus M(7, F_q)^4 \bigoplus M(8, F_q) \bigoplus M(9, F_q)^3.$$ 

Case 2: $p^k \equiv \{13, 29, 41, 43, 83, 85, 97, 113\}$ mod 126. In this case, the cyclotomic $F_q$ classes of $\gamma$ are

$$SF_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \quad \text{for } i = 1, 2, 3, 7, 8, 9, \quad SF_q(\gamma_{g_4}) = \{\gamma_{g_4}, \gamma_{g_5}, \gamma_{g_6}\}.$$

By incorporating Proposition 2.2, we derive from (3.2) that

$$F_q G_1 \simeq F_q \bigoplus_{i=1}^{5} M(n_i, F_q) \bigoplus M(n_6, F_q)^3 \implies 503 = \sum_{i=1}^{5} n_i^2 + 3n_6^2, \quad n_i \geq 2. \quad (3.4)$$

Due to Lemma 2.7, it follows from (3.4) that $n_i \geq 7$. Consequently, the possible choices of $n_i$'s fulfilling (3.4) are $(7^4, 8, 9)$, $(7^3, 8, 10, 8)$, $(7, 8, 9^3, 7)$ and $(8^4, 10, 7)$. Again, Lemma 2.7 implies that $M(10, F_q)$ can not be a Wedderburn component. Therefore, we are only remaining with two choices of $n_i$'s given by $(7^4, 8, 9)$ and $(7, 8, 9^3, 7)$. Next, to uniquely identify the correct choice, we show that $M(9, F_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 13$. 

$$\begin{align*}
\text{(4) } p^k & \equiv \{5, 11, 23, 25, 31, 47, 59, 61, 65, 67, 79, 95, 101, 103, 115, 121\} \mod 126, \quad \text{we have} \\
U(F_q G_1) & \simeq F_q^* \oplus GL(7, F_q) \oplus GL(8, F_q) \oplus GL(7, F_q^3) \oplus GL(9, F_q^3). 
\end{align*}$$
and consider the following mapping from $G_1$ to $GL(9, \mathbb{F}_{13})$:

\[
a \rightarrow \begin{bmatrix}
5 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 2 & 4 & 8 & 3 & 0 & 0 & 0 & 0 \\
6 & 12 & 11 & 2 & 12 & 11 & 0 & 0 & 0 \\
0 & 6 & 2 & 11 & 9 & 4 & 2 & 4 & 0 \\
10 & 5 & 10 & 7 & 7 & 6 & 12 & 7 & 11 \\
3 & 2 & 2 & 4 & 9 & 10 & 12 & 9 & 8 \\
11 & 1 & 3 & 8 & 3 & 5 & 11 & 8 & 9 \\
4 & 5 & 3 & 12 & 6 & 10 & 11 & 6 & 1
\end{bmatrix},
\]

\[
b \rightarrow \begin{bmatrix}
8 & 9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 12 & 2 & 4 & 3 & 1 & 12 & 0 & 0 \\
3 & 10 & 1 & 4 & 6 & 0 & 8 & 1 & 11 \\
2 & 5 & 3 & 8 & 1 & 10 & 0 & 10 & 2 \\
9 & 8 & 5 & 6 & 2 & 8 & 1 & 5 & 11 \\
6 & 1 & 5 & 0 & 9 & 2 & 8 & 8 & 0 \\
12 & 4 & 0 & 10 & 8 & 12 & 6 & 7 & 10
\end{bmatrix}.
\]

This mapping is a homomorphism from $G_1$ to $GL(9, \mathbb{F}_{13})$ (as $a$ and $b$ given in (3.1) generates $G_1$). It should be noted that this map is an irreducible representation of $G_1$ over $\mathbb{F}_{13}$. Therefore, according to Proposition 2.8, $M(9, \mathbb{F}_{13})$ will always be a Wedderburn component of $\mathbb{F}_qG_1$. Hence, it follows that $(7, 8, 9^3, 7)$ is the only possible value for $n_i$. Hence, the WD is

\[\mathbb{F}_qG \simeq \mathbb{F}_q \oplus M(7, \mathbb{F}_q) \oplus M(8, \mathbb{F}_q) \oplus M(9, \mathbb{F}_q)^3 \oplus M(7, \mathbb{F}_q^3).\]

Case 3: $p^k \equiv \{17, 19, 37, 53, 73, 89, 107, 109\}$ mod 124. The cyclotomic $\mathbb{F}_q$ classes of $\gamma_g$ are

\[SF_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } 1, 2, 3, 4, 5, 6, \quad SF_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_8}, \gamma_{g_9}\}.\]

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (3.2) that

\[\mathbb{F}_qG_1 \simeq \mathbb{F}_q \bigoplus_{i=1}^{5} M(n_i, \mathbb{F}_q) \oplus M(n_6, \mathbb{F}_q^3) \implies 503 = \sum_{i=1}^{5} n_i^2 + 3n_6^2, \quad n_i \geq 2. \quad (3.5)\]

By proceeding on the similar lines of case 2, one can show that we need to deduce the unique choices among the 2 choices $(7^4, 8, 9)$ and $(7, 8, 9^3, 7)$. For this, we take
$p = 17$ and show that there are two distinct homomorphisms from $G_1$ to $GL(7, \mathcal{F}_{17})$. We consider the following mappings:

$$a \rightarrow \begin{bmatrix} 13 & 10 & 0 & 0 & 0 & 0 & 0 \\ 1 & 11 & 10 & 4 & 0 & 0 & 0 \\ 0 & 8 & 6 & 11 & 7 & 9 & 0 \\ 0 & 1 & 14 & 12 & 14 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 & 15 & 4 \\ 0 & 0 & 1 & 1 & 4 & 3 & 10 \\ 0 & 0 & 0 & 12 & 12 & 3 & 12 \end{bmatrix}, \quad b \rightarrow \begin{bmatrix} 8 & 14 & 16 & 0 & 0 & 0 & 0 \\ 9 & 2 & 6 & 13 & 4 & 0 & 0 \\ 0 & 5 & 14 & 1 & 6 & 15 & 4 \\ 0 & 2 & 3 & 11 & 1 & 13 & 15 \\ 0 & 1 & 12 & 14 & 4 & 3 & 12 \\ 0 & 0 & 12 & 8 & 10 & 14 & 10 \\ 0 & 0 & 1 & 9 & 1 & 11 & 16 \end{bmatrix},$$

$$a \rightarrow \begin{bmatrix} 14 & 14 & 0 & 0 & 0 & 0 & 0 \\ 13 & 16 & 0 & 4 & 0 & 0 & 0 \\ 16 & 12 & 12 & 15 & 0 & 0 & 0 \\ 5 & 8 & 9 & 15 & 7 & 2 & 2 \\ 10 & 9 & 13 & 13 & 9 & 1 & 12 \\ 9 & 1 & 6 & 3 & 13 & 15 & 0 \\ 1 & 5 & 11 & 8 & 2 & 14 & 2 \end{bmatrix}, \quad b \rightarrow \begin{bmatrix} 6 & 8 & 16 & 0 & 0 & 0 & 0 \\ 12 & 1 & 16 & 14 & 0 & 16 & 0 \\ 11 & 12 & 13 & 7 & 4 & 12 & 0 \\ 0 & 15 & 9 & 6 & 0 & 13 & 0 \\ 9 & 7 & 2 & 9 & 13 & 13 & 14 \\ 12 & 14 & 10 & 6 & 9 & 9 & 4 \\ 5 & 9 & 14 & 13 & 4 & 16 & 4 \end{bmatrix}.$$

These mappings are 2 irreducible representations of $G_1$ over $\mathcal{F}_{17}$. Therefore, Proposition 2.8 derives that $M(7, \mathcal{F}_{17})^2$ is a summand of the group algebra $\mathcal{F}_{17}G_1$. Thus, the required choices of $n_i$’s fulfilling (3.5) is $(7, 4, 8, 9)$. Hence, the WD is

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus M(7, \mathcal{F}_q)^4 \oplus M(8, \mathcal{F}_q) \oplus M(9, \mathcal{F}_q^3).$$

Case 4: $p^k \equiv \{5, 11, 23, 25, 31, 47, 59, 61, 67, 79, 101, 103, 115, 121, 65, 95\}$ mod 124. The cyclotomic $\mathcal{F}_q$ classes of $\gamma_g$ are

$$SF_q(\gamma_g) = \{\gamma_g\}, \text{ for } 1, 2, 3, \quad SF_q(\gamma_g) = \{\gamma_g, \gamma_g, \gamma_g\}, \quad SF_q(\gamma_g) = \{\gamma_g, \gamma_g, \gamma_g\}. $$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (3.2) that

$$\mathcal{F}_qG_1 \simeq (\mathcal{F}_q) \bigoplus_{i=1}^2 M(n_i, \mathcal{F}_q) \oplus M(n_3, \mathcal{F}_q^3) \oplus M(n_4, \mathcal{F}_q^3) \quad \implies 503 = \sum_{i=1}^2 n_i^2 + 3(n_3^2 + n_4^2), \quad n_i \geq 2. \quad (3.6)$$

By following the procedure as in case 1, we can show that the $n_i \geq 7$ in (3.6). Hence, the only possible choice of $n_i$’s is $(7, 8, 7, 9)$, which means that

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus M(7, \mathcal{F}_q) \oplus M(8, \mathcal{F}_q) \oplus M(7, \mathcal{F}_q^3) \oplus M(9, \mathcal{F}_q^3).$$

This completes the proof. □

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4. Unit group of $\mathcal{F}_q SL(2, 9)$

Let $G_2 = SL(2, 9)$. The order of $G_2$ is 720. Since $p > 5$, it does not divide the order of $G_2$, the group algebra $\mathcal{F}_q G_2$ is semisimple. Also, from [8] one can note that $G_2$ has irreducible representations of degrees 1, 4, 5, 8, 9 and 10 whenever $|S\mathcal{F}_q(\gamma g)| = 1, \forall g \in G_2$. The group $G_2$ has 13 conjugacy classes and it is represented as $g'_i$s. The representative of the conjugacy classes, size and the order of representatives are tabulated below:

<table>
<thead>
<tr>
<th>R</th>
<th>[1 0]</th>
<th>[0 2]</th>
<th>[0 2x+2]</th>
<th>[2 0]</th>
<th>[0 2]</th>
<th>[0 2x+2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
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<td>40</td>
<td>40</td>
<td>1</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>O</td>
<td>1</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
0 & 2 \\
1 & 2x+2
\end{bmatrix}
\begin{bmatrix}
0 & 2 \\
1 & x+2
\end{bmatrix}
\begin{bmatrix}
0 & 2 \\
1 & x+1
\end{bmatrix}
\begin{bmatrix}
0 & 2 \\
1 & 2x+1
\end{bmatrix}
\begin{bmatrix}
2x+2 & 0 \\
0 & 2x+1
\end{bmatrix}
\begin{bmatrix}
x & 0 \\
0 & 2x
\end{bmatrix}
\begin{bmatrix}
x+2 & 0 \\
0 & x+1
\end{bmatrix}
\]

\[
\begin{bmatrix}
x+1 & 0 \\
0 & x+2
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
2 & 0
\end{bmatrix}
\]

Here $x$ is the generator of multiplicative group of finite field of order 9. Also, $G_2$ can be generated by two elements $a$ and $b$, where

\[
a = \begin{bmatrix}
x+1 & 0 \\
0 & x+2
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
2 & 1 \\
2 & 0
\end{bmatrix}.
\]

In this section, we characterize the unit group of the group algebra $\mathcal{F}_q G_2$ for $p > 5$ such that the group algebra $\mathcal{F}_q G_2$ is semisimple and $q = p^k$. It is clear from the above table that the exponent of $G_2$ is 120.

**Theorem 4.1.** The unit group of $\mathcal{F}_q G_2$ is as follows:

1. For $p^k \equiv \{1, 31, 41, 49, 71, 79, 89, 119\}$ mod 120, we have

\[
U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4, \mathcal{F}_q)^2 \oplus GL(5, \mathcal{F}_q)^2 \oplus GL(8, \mathcal{F}_q)^4 \oplus GL(9, \mathcal{F}_q) \oplus GL(10, \mathcal{F}_q)^3.
\]

2. For $p^k \equiv \{7, 17, 23, 47, 73, 97, 103, 113\}$ mod 120, we have

\[
U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4, \mathcal{F}_q)^2 \oplus GL(5, \mathcal{F}_q)^2 \oplus GL(9, \mathcal{F}_q) \oplus GL(10, \mathcal{F}_q)^3 \oplus GL(8, \mathcal{F}_q^2)^2.
\]

3. For $p^k \equiv \{11, 19, 29, 59, 61, 91, 101, 109\}$ mod 120, we have

\[
U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4, \mathcal{F}_q)^2 \oplus GL(5, \mathcal{F}_q)^2 \oplus GL(9, \mathcal{F}_q) \oplus GL(8, \mathcal{F}_q)^4.
\]
Proof. It follows from the Wedderburn decomposition theorem that $\mathcal{F}_qG_2 \simeq \bigoplus_{i=1}^{r} M(n_i, \mathcal{F}_q)$. Also, $G_2$ is a perfect group. This and Proposition 2.2 imply that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{r-1} M(n_i, \mathcal{F}_q), \quad n_i \geq 2. \quad (4.2)$$

As in the previous theorem, we construct the set $T_{G, \mathcal{F}}$ of group $G_2$ and divide the proof into the following 4 cases.

Case 1: $p^k \equiv \{1, 31, 41, 49, 71, 79, 89, 119\}$ mod 120. In this case, it can be verified that $|S\mathcal{F}_q(\gamma_q)| = 1, \forall g \in G_2$. By utilizing this along with Proposition 2.2, we further rewrite (4.2) as

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{12} M(n_i, \mathcal{F}_q) \Rightarrow 719 = \sum_{i=1}^{12} n_i^2, \quad n_i \geq 2. \quad (4.3)$$

Next, we consider the normal subgroup $N$ of $G_2$ generated by $[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}]$. One can observe that with $G_2/N \simeq A_6$. We recall from [2, Proposition 4.7] that

$$\mathcal{F}_qA_6 \simeq \mathcal{F}_q \oplus M(5, \mathcal{F}_q)^2 \oplus M(9, \mathcal{F}_q) \oplus M(10, \mathcal{F}_q) \oplus M(8, \mathcal{F}_q)^2. \quad (4.4)$$

Utilizing (4.4) and Proposition 2.5 in (4.3) to derive that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus M(5, \mathcal{F}_q)^2 \oplus M(9, \mathcal{F}_q) \oplus M(10, \mathcal{F}_q) \oplus M(8, \mathcal{F}_q)^2 \bigoplus_{i=1}^{6} M(n_i, \mathcal{F}_q) \quad (4.5)$$

with $360 = \sum_{i=1}^{6} n_i^2, \quad n_i \geq 2$. We note that $G_2$ has irreducible representations of degrees 1, 4, 5, 8, 9, and 10. This means $n_i$’s in (4.5) are among the set $\{4, 5, 8, 9, 10\}$. Among all the possible choices of $n_i$’s fulfilling $360 = \sum_{i=1}^{6} n_i^2$, the only choice that contains elements from the set $\{4, 5, 8, 9, 10\}$ is $(4^2, 8^2, 10^2)$. Hence, (4.5) implies that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus M(4, \mathcal{F}_q)^2 \oplus M(5, \mathcal{F}_q)^2 \oplus M(8, \mathcal{F}_q)^4 \oplus M(9, \mathcal{F}_q) \oplus M(10, \mathcal{F}_q)^3.$$

Case 2: $p^k \equiv \{7, 17, 23, 47, 73, 97, 103, 113\}$ mod 120. The cyclotomic $\mathcal{F}_q$ classes of $\gamma_q$ are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \quad \text{for } 1-6, 11-3, \quad S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_9}\}, \quad S\mathcal{F}_q(\gamma_{g_9}) = \{\gamma_{g_9}, \gamma_{g_{20}}\}. \quad (4.6)$$
By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that
\[ F_qG_2 \cong F_q \bigoplus_{i=1}^{8} M(n_i, F_q) + M(n_9, F_q^2) + M(n_{10}, F_q^2), \]
where \( n_i \geq 2 \). We observe the WD of \( F_qA_6 \) in this case is (see [2, Proposition 4.7])
\[ F_qA_6 \cong F_q + M(5, F_q)^2 + M(9, F_q) + M(10, F_q) + M(8, F_q^2). \] (4.7)
Using (4.7) and Proposition 2.5, we further obtain from (4.3) that
\[ F_qG_2 \cong F_q + M(5, F_q)^2 + M(9, F_q) + M(10, F_q) + M(8, F_q^2), \]
with
\[ 360 = \sum_{i=1}^{4} n_i^2 + 2n_5^2, \quad n_i \geq 2. \] (4.9)
According to Lemma 2.7 and Case 1, \( 4 \leq n_i \leq 10 \). Moreover, Proposition 2.6 confirms that \( n_i \neq 7 \) in this case. Thus, we are remaining with the three choices of \( n_i \)'s fulfilling (4.9) given by \((4^2, 8^2, 10), (4^2, 10^2, 8)\) and \((8^2, 10^2, 4)\). Next, to uniquely identify the correct choice, we show that \( M(4, F_q) \) will always be a Wedderburn component in this case. In particular, we take \( p = 7 \) and consider the following mapping from \( G_2 \) to \( GL(4, F_7) \):
\[
\begin{align*}
  a &\rightarrow \begin{bmatrix} 3 & 6 & 2 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 5 & 4 & 2 \\ 5 & 2 & 1 & 1 \end{bmatrix}, \\
  b &\rightarrow \begin{bmatrix} 4 & 2 & 5 & 2 \\ 4 & 5 & 4 & 0 \\ 4 & 6 & 3 & 5 \\ 0 & 3 & 4 & 0 \end{bmatrix}.
\end{align*}
\]
This mapping is a homomorphism from \( G_2 \) to \( GL(4, F_7) \) (as \( a \) and \( b \) given in (4.1) generates \( G_2 \)). It should be noted that this map is an irreducible representation of \( G_2 \) over \( F_7 \). Therefore, according to Proposition 2.8, \( M(4, F_q) \) will always be a Wedderburn component of \( F_qG_2 \). Consequently, we are left with two possible choices of \( n_i \)'s given by \((4^2, 8^2, 10)\) and \((4^2, 10^2, 8)\). Finally, we show that \( M(10, F_q)^2 \)
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is a summand of $F_q G_2$. For this, we define the following two maps:

$$a \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 6 & 0 & 1 & 6 & 0 & 1 & 0 \\ 3 & 3 & 0 & 5 & 1 & 1 & 3 & 3 & 4 & 0 \\ 3 & 5 & 6 & 0 & 4 & 1 & 2 & 5 & 0 & 3 \\ 6 & 0 & 4 & 4 & 2 & 3 & 6 & 6 & 1 & 5 \\ 6 & 6 & 1 & 4 & 4 & 1 & 2 & 3 & 5 & 1 \\ 3 & 2 & 5 & 1 & 1 & 0 & 4 & 1 & 4 & 4 \end{bmatrix}, \quad b \rightarrow \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 4 & 3 & 5 & 0 & 0 & 0 & 0 & 0 \\ 6 & 3 & 4 & 3 & 3 & 4 & 0 & 0 & 0 & 0 \\ 2 & 5 & 2 & 5 & 2 & 4 & 3 & 0 & 0 & 0 \\ 6 & 0 & 3 & 3 & 1 & 4 & 6 & 0 & 0 & 0 \\ 3 & 2 & 6 & 2 & 5 & 6 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 3 & 2 & 2 & 5 & 0 & 0 & 0 \\ 4 & 1 & 6 & 5 & 5 & 1 & 2 & 4 & 3 & 4 \\ 2 & 3 & 6 & 6 & 0 & 0 & 6 & 0 & 1 & 3 \\ 4 & 1 & 0 & 3 & 5 & 4 & 4 & 3 & 3 & 2 \end{bmatrix}$$

$$a \rightarrow \begin{bmatrix} 4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 2 & 4 & 6 & 0 & 4 & 0 & 0 & 0 \\ 4 & 1 & 4 & 5 & 1 & 3 & 1 & 6 & 0 & 0 \\ 4 & 3 & 0 & 3 & 4 & 3 & 4 & 0 & 4 & 5 \\ 3 & 6 & 2 & 2 & 5 & 0 & 4 & 4 & 6 & 5 \\ 2 & 0 & 5 & 3 & 3 & 3 & 4 & 1 & 0 & 5 \\ 3 & 2 & 3 & 1 & 0 & 3 & 0 & 0 & 0 & 6 \\ 5 & 4 & 0 & 1 & 6 & 2 & 2 & 5 & 3 & 0 \\ 5 & 1 & 0 & 4 & 1 & 6 & 3 & 1 & 0 & 5 \end{bmatrix}, \quad b \rightarrow \begin{bmatrix} 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 6 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 5 & 3 & 5 & 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 2 & 2 & 0 & 3 & 6 & 0 \\ 0 & 3 & 3 & 0 & 6 & 5 & 4 & 2 & 1 & 2 \\ 0 & 3 & 2 & 2 & 4 & 1 & 1 & 4 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 2 & 6 & 3 & 6 \\ 0 & 1 & 3 & 3 & 6 & 1 & 5 & 1 & 0 & 1 \\ 0 & 6 & 4 & 2 & 2 & 6 & 5 & 4 & 1 & 6 \\ 0 & 5 & 1 & 5 & 0 & 6 & 5 & 3 & 0 & 6 \end{bmatrix}$$

These mappings are 2 irreducible representations of $G_2$ over $F_7$. Therefore, Proposition 2.8 derives that $M(10, F_7)^2$ is a summand of the group algebra $F_7 G_2$. Consequently, the required choice of $n_i$ is $(4^2, 10^2, 8)$. Hence, using (4.8), we get

$$F_q G_2 \simeq F_q \oplus M(4, F_q)^2 \oplus M(5, F_q)^2 \oplus M(9, F_q) \oplus M(10, F_q)^3 \oplus M(8, F_q^2)^2.$$  

Case 3: $p^k \equiv \{11, 19, 29, 59, 61, 91, 101, 109\} \mod 120$. The cyclotomic $F_q$ classes of $\gamma_g$ are

$$SF_q(\gamma_{g_1}) = \{\gamma_{g_1}\}, \quad SF_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{13}}\}.$$  

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$F_q G_2 \simeq F_q \bigoplus_{i=1}^{10} M(n_i, F_q) \oplus M(n_{11}, F_q^2), \quad 719 = \sum_{i=1}^{10} n_i^2 + 2n_{11}^2, \quad n_i \geq 2. \quad (4.10)$$

We observe the WD of $F_q A_6$ in this case same as in Case 1. Using this and Proposition 2.5, we further obtain from (4.10) that

$$F_q G_2 \simeq F_q \oplus M(5, F_q)^2 \oplus M(8, F_q)^2 M(9, F_q) \oplus M(10, F_q)$$

$$\bigoplus_{i=1}^{4} M(n_i, F_q) \oplus M(n_5, F_q^2). \quad (4.11)$$

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By proceeding as in the previous case, we are remaining with the three choices of \( n \)'s fulfilling (4.9) given by \( (4^2, 8^2, 10), (4^2, 10^2, 8) \) and \( (8^2, 10^2, 4) \). Further, on the similar lines of the previous case, we can show that the final choice of \( n \)'s is \( (4, 4, 8, 8, 10) \). Hence, it follows from (4.11) that the WD is

\[
\mathcal{F}_q G_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4, \mathcal{F}_q)^2 \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q)^4 \\
\oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q^2).
\]

Case 4: \( p^k \equiv \{13, 37, 43, 53, 67, 77, 83, 107\} \mod 120 \). The cyclotomic \( \mathcal{F}_q \) classes of \( \gamma_q \) are

\[
\mathcal{S}_\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } i = 1-6, 12; \mathcal{S}_\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}, \gamma_{g_{i+1}}\} \text{ for } i = 7, 9,
\]

\[
\mathcal{S}_\mathcal{F}_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{13}}\}.
\]

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

\[
\mathcal{F}_q G_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{6} \mathbf{M}(n_i, \mathcal{F}_q) \bigoplus_{i=7}^{9} \mathbf{M}(n_i, \mathcal{F}_q^2)
\]

\[
\implies 719 = \sum_{i=1}^{6} n_i^2 + 2 \sum_{i=7}^{9} n_i^2, \quad n_i \geq 2.
\]

In this case, the WD of \( \mathcal{F}_q A_6 \) is given by (4.7). Using this and proposition 2.5, we further obtain from (4.12) that

\[
\mathcal{F}_q G_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q^2)
\]

\[
\bigoplus_{i=1}^{2} \mathbf{M}(n_i, \mathcal{F}_q) \bigoplus_{i=3}^{4} \mathbf{M}(n_i, \mathcal{F}_q^2),
\]

with \( 360 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2, \quad n_i \geq 2 \). According to lemma 2.7 and case 1, \( 4 \leq n_i \leq 10 \) for each \( i \). Furthermore, lemma 2.7 and case 1 implies that \( n_i \neq 7 \) for any \( i \). This leaves us with three possible values of \( n \)'s given by \( (4, 4, 8, 10), (8, 8, 4, 10) \) and \( (10, 10, 4, 8) \). Next, to uniquely identify the correct choice, we show that \( \mathbf{M}(4, \mathcal{F}_q) \) will always be a Wedderburn component in this case. In particular, we take \( p = 13 \) and consider the following mapping from \( G_2 \) to \( GL(4, \mathcal{F}_{13}) \):

\[
a \rightarrow \begin{bmatrix} 0 & 12 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 9 \\ 0 & 1 & 2 & 8 \end{bmatrix}, \quad b \rightarrow \begin{bmatrix} 0 & 2 & 7 & 0 \\ 9 & 3 & 3 & 3 \\ 1 & 10 & 0 & 1 \\ 0 & 10 & 1 & 8 \end{bmatrix}.
\]

This mapping is an irreducible representation of \( G_2 \) over \( \mathcal{F}_{13} \). Therefore, according to proposition 2.8, \( \mathbf{M}(4, \mathcal{F}_q) \) will always be a Wedderburn component of \( \mathcal{F}_q G_2 \). Hence, we get

\[
\mathcal{F}_q G_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4, \mathcal{F}_q)^2 \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q)
\]
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\[ \oplus M(10, \mathbb{F}_q^2) \oplus M(8, \mathbb{F}_q^2)^2. \]

This completes the proof. \qed

5. Conclusion

In this paper, we focused on deriving the unit groups of the semisimple group algebras of groups $SL(2, 8)$ and $SL(2, 9)$. In order to derive these, we computed the Wedderburn decomposition using the findings from the classical theory of group algebras. Having the wide range of possible Wedderburn components, it is evident that it becomes more and more challenging to characterize the Wedderburn decomposition with increasing group size. Finally, this paper further motivates to deduce the unit groups of special linear groups of higher order.

References


