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Continued fractions for bi-periodic Fibonacci sequence*

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Abstract. In this paper, we study generalized continued fractions for the expression of bi-periodic Fibonacci ratios.

Keywords: Bi-periodic Fibonacci sequence, bi-periodic Lucas sequence, continued fraction

AMS Subject Classification: 11A55, 11B39

1. Introduction

A continued fraction is an expression of the form

$$x = s_0 + \cfrac{r_1}{s_1 + \cfrac{r_2}{s_2 + \cfrac{r_3}{\ddots}}} = s_0 + \frac{r_1}{s_1} + \frac{r_2}{s_2} + \cdots,$$

where s_n and r_n are real or complex numbers with $r_n \neq 0$. The $r_1, r_2, r_3, \ldots$ in this context will usually be referred to as the "partial numerators" of the continued fraction and the terms $s_0, s_1, s_2, \ldots$ are the "partial denominators". The most common restriction imposed on continued fractions is to have $r_n = 1$ and then call the expression a *simple continued fraction*, denoted by $[s_0; s_1, s_2, \ldots]$. A periodic continued fraction is one that repeats and has the form $[s_0; s_1, \ldots, s_m, \overline{s_{m+1}, \ldots, s_n}]$.

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Let F_n and L_n denote the Fibonacci and Lucas numbers, defined, respectively, by $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$.

Several generalizations of the Fibonacci sequence have been presented in the literature [1–8]. One of them was given by Edson and Yayenie in [6], called the bi-periodic Fibonacci sequence defined for any nonzero real numbers a, b, and any integer $n \geq 2$, as follows

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{for } n \text{ even,} \\ bq_{n-1} + q_{n-2}, & \text{for } n \text{ odd,} \end{cases}$$

with initial values $q_0 = 0$ and $q_1 = 1$. Note that for $a = b = 1$, we get the classical Fibonacci sequence. Similarly, Bilgici [5] introduced the bi-periodic Lucas sequence, for $n \geq 2$, as follows

$$l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{for } n \text{ even,} \\ al_{n-1} + l_{n-2}, & \text{for } n \text{ odd,} \end{cases}$$

with initial conditions $l_0 = 2$ and $l_1 = a$. It gives the classical Lucas sequence for $a = b = 1$.

Also, the bi-periodic Fibonacci and Lucas sequences satisfy, for $n \geq 4$, the same recurrence relation

$$w_n = (ab + 2)w_{n-2} - w_{n-4}.$$

The Binet's formulas of the bi-periodic Fibonacci and Lucas sequences are given by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor n/2 \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \tag{1.1}$$

$$l_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor (n+1)/2 \rfloor}} (\alpha^n + \beta^n), \tag{1.2}$$

where $\lfloor x \rfloor$ is the floor function of x, $\xi(n) = n - 2\lfloor n/2 \rfloor$ is the parity function and α, β are the roots of the characteristic equation $x^2 - abx - ab = 0$ given by

$$\alpha = \frac{ab + \sqrt{ab(ab+4)}}{2} \quad \text{and} \quad \beta = \frac{ab - \sqrt{ab(ab+4)}}{2}.$$

It is well known that the limit of the ratios of consecutive Fibonacci numbers is the golden ratio ϕ. Thus

$$\phi = [\overline{1}] = [1; 1, 1, \cdots] = \frac{1 + \sqrt{5}}{2}.$$

For more details on continued fractions and their connection to the Fibonacci sequence, the reader is referred to [9–12].

Consider the bi-periodic Fibonacci sequence $\{q_n\}_{n\geq 0}$ with a and b nonnegative integers. For $a \neq b$, we have

$$\frac{q_n}{q_{n-1}} = a^{\xi(n-1)}b^{\xi(n)} + \frac{1}{\frac{q_{n-1}}{q_{n-2}}}.$$

So the ratios of the successive terms do not converge (see [6]). Therefore

$$\lim_{n\to\infty}\frac{q_{2n}}{q_{2n-1}} = \frac{\alpha}{b}, \quad \lim_{n\to\infty}\frac{q_{2n+1}}{q_{2n}} = \frac{\alpha}{a}, \quad \text{and} \quad \lim_{n\to\infty}\frac{q_n}{q_{n-2}} = \alpha + 1.$$

2. Main results

Let n, s, r, and t be positive integers with $r < s$ and $n \geq 2t$. Our aim is to derive closed form expressions for the continued fractions of $\lim\limits_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-t)+r}}$. Let's start with $t = 1$.

This theorem gives the recurrence satisfied by a subsequence of arithmetic progression.

Theorem 2.1. *For $n \geq 2$ and fixed s and r, we have the following relation*

$$q_{sn+r} = \left(\frac{b}{a}\right)^{\xi(s)\xi(n+r)} l_s q_{s(n-1)+r} + (-1)^{s+1} q_{s(n-2)+r}. \tag{2.1}$$

Proof. From Binet's formulas (1.1), (1.2), and since $\alpha\beta = -ab$, $\xi(n+m) = \xi(n) + \xi(m) - 2\xi(n)\xi(m)$, and $\lfloor n/2 \rfloor = (n - \xi(n))/2$, we can write

$$\begin{aligned}
&\left(\frac{b}{a}\right)^{\xi(s)\xi(sn+r)} l_s q_{s(n-1)+r} + (-1)^{s+1} q_{s(n-2)+r} \\
&= \left(\frac{b}{a}\right)^{\xi(s)\xi(sn+r)} \frac{a^{\xi(s)}}{(ab)^{\lfloor (s+1)/2 \rfloor}}(\alpha^s + \beta^s)\frac{a^{\xi(s(n-1)+r+1)}}{(ab)^{\lfloor (s(n-1)+r)/2 \rfloor}} \\
&\quad \times \left(\frac{\alpha^{s(n-1)+r} - \beta^{s(n-1)+r}}{\alpha - \beta}\right) \\
&\quad + (-1)^{s+1}\frac{a^{\xi(s(n-2)+r+1)}}{(ab)^{\lfloor (s(n-2)+r)/2 \rfloor}}\left(\frac{\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}}{\alpha - \beta}\right) \\
&= \left(\frac{b}{a}\right)^{\xi(s)\xi(sn+r)} \frac{a^{\xi(sn+r+1)+2\xi(s)\xi(sn+r)}}{(ab)^{\lfloor (sn+r)/2 \rfloor + \xi(s)\xi(sn+r)}} \\
&\quad \times \left(\frac{\alpha^{sn+r} - \beta^{sn+r} + (\alpha\beta)^s\left(\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}\right)}{\alpha - \beta}\right) \\
&\quad + (-1)^{s+1}\frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (s(n-2)+r)/2 \rfloor}}\left(\frac{\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}}{\alpha - \beta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (sn+r)/2 \rfloor}} \left(\frac{\alpha^{sn+r} - \beta^{sn+r} + (-ab)^s \left(\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}\right)}{\alpha - \beta} \right) \\
&\quad - (-1)^s \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (s(n-2)+r)/2 \rfloor}} \left(\frac{\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}}{\alpha - \beta} \right) \\
&= \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (sn+r)/2 \rfloor}} \left(\frac{\alpha^{sn+r} - \beta^{sn+r}}{\alpha - \beta} \right) \\
&= q_{sn+r}. \qquad \square
\end{aligned}
$$

The following theorem derives the closed form for continued fraction expressions of $\lim\limits_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-1)+r}}$, which is the ratio of two consecutive terms of the cited subsequence.

Theorem 2.2. *For $r < s$, we have*

$$
\lim_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-1)+r}} = \begin{cases} \left[l_s - 1; \overline{1, l_s - 2}\right] = \dfrac{\alpha^s}{(ab)^{s/2}}, & \text{for } s \text{ even}, \\ \left[l_s; \overline{l_s}\right] = \dfrac{a\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ odd and } n + r \text{ even}, \\ \left[\dfrac{b}{a} l_s; \overline{\dfrac{b}{a} l_s}\right] = \dfrac{b\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ and } n + r \text{ odd}. \end{cases}
$$

Proof. From (2.1), we have

- For s even

$$
\begin{aligned}
\frac{q_{sn+r}}{q_{s(n-1)+r}} &= l_s - \frac{q_{s(n-2)+r}}{q_{s(n-1)+r}} = l_s - 1 + \frac{q_{s(n-1)+r} - q_{s(n-2)+r}}{q_{s(n-1)+r}} \\
&= l_s - 1 + \cfrac{1}{\cfrac{q_{s(n-1)+r}}{q_{s(n-1)+r} - q_{s(n-2)+r}}} \\
&= l_s - 1 + \cfrac{1}{1 + \cfrac{q_{s(n-2)+r}}{q_{s(n-1)+r} - q_{s(n-2)+r}}} \\
&= l_s - 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{q_{s(n-1)+r}}{q_{s(n-2)+r}} - 1}} \\
&= l_s - 1 + \cfrac{1}{1 + \cfrac{1}{l_s - 2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}.
\end{aligned}
$$

- For s odd and $n+r$ even

$$\frac{q_{sn+r}}{q_{s(n-1)+r}} = l_s + \frac{q_{s(n-2)+r}}{q_{s(n-1)+r}} = l_s + \cfrac{1}{l_s + \cfrac{1}{l_s + \cfrac{1}{\ddots}}}.$$

- For s and $n+r$ odd

$$\frac{q_{sn+r}}{q_{s(n-1)+r}} = \frac{b}{a}l_s + \frac{q_{s(n-2)+r}}{q_{s(n-1)+r}} = \frac{b}{a}l_s + \cfrac{1}{\frac{b}{a}l_s + \cfrac{1}{\frac{b}{a}l_s + \cfrac{1}{\ddots}}}.$$

Using Binet's formula of the bi-periodic Fibonacci sequence, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-1)+r}} &= \lim_{n\to\infty}\frac{a^{\xi(sn+r-1)}(ab)^{\lfloor (s(n-1)+r)/2\rfloor}}{a^{\xi(s(n-1)+r-1)}(ab)^{\lfloor (sn+r)/2\rfloor}}\alpha^s\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{sn+r}}{1-\left(\frac{\beta}{\alpha}\right)^{s(n-1)+r}}\right)\\ &= \begin{cases}\dfrac{\alpha^s}{(ab)^{s/2}}, & \text{for } s \text{ even},\\ \dfrac{a\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ odd and } n+r \text{ even},\\ \dfrac{b\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ odd and } n+r \text{ odd}.\end{cases}\end{aligned}$$ □

For the classical Fibonacci sequence, when $a=b=1$, Theorem 2.2 gives the following result.

Corollary 2.3.

$$\lim_{n\to\infty}\frac{F_{sn+r}}{F_{s(n-1)+r}} = \phi^s = \begin{cases}\left[L_s-1;\overline{1, L_s-2}\right], & \textit{for } s \textit{ even},\\ \left[L_s;\overline{L_s}\right], & \textit{for } s \textit{ odd}.\end{cases}$$

The following theorem extends Theorem 2.1 in the sense that it considers the t-periodic terms in the arithmetic progression subsequence.

Theorem 2.4. *For $n \geq 2t$, we have*

$$q_{sn+r} = \left(\frac{b}{a}\right)^{\xi(st)\xi(sn+r)} l_{st}q_{s(n-t)+r} + (-1)^{st+1}q_{s(n-2t)+r}. \tag{2.2}$$

Proof. Using Binet's formula, we get

$$q_{sn+r} = C_1\chi_{sn+r}\alpha^{sn+r} + C_2\chi_{sn+r}\beta^{sn+r},$$

with $\chi_{sn+r} = \frac{a^{\xi(sn+r-1)}}{(ab)^{\lfloor (sn+r)/2 \rfloor}}$ and $C_1 = -C_2 = 1/(\alpha - \beta)$. The matrix form gives

$$\begin{pmatrix} q_{s(n-t)+r} \\ q_{s(n-2t)+r} \end{pmatrix} = \begin{pmatrix} \chi_{s(n-t)+r}\alpha^{s(n-t)+r} & \chi_{s(n-t)+r}\beta^{s(n-t)+r} \\ \chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} & \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} \chi_{s(n-t)+r}\alpha^{s(n-t)+r} & \chi_{s(n-t)+r}\beta^{s(n-t)+r} \\ \chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} & \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} \end{pmatrix}^{-1} \begin{pmatrix} q_{s(n-t)+r} \\ q_{s(n-2t)+r} \end{pmatrix} \\ &= \frac{1}{D} \begin{pmatrix} \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} & -\chi_{s(n-t)+r}\beta^{s(n-t)+r} \\ -\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} & \chi_{s(n-t)+r}\alpha^{s(n-t)+r} \end{pmatrix} \begin{pmatrix} q_{s(n-t)+r} \\ q_{s(n-2t)+r} \end{pmatrix} \\ &= \frac{1}{D} \begin{pmatrix} \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} q_{s(n-t)+r} & -\chi_{s(n-t)+r}\beta^{s(n-t)+r} q_{s(n-2t)+r} \\ -\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} q_{s(n-t)+r} & \chi_{s(n-t)+r}\alpha^{s(n-t)+r} q_{s(n-2t)+r} \end{pmatrix} \end{aligned}$$

where $D = \chi_{s(n-t)+r}\chi_{s(n-2t)+r}\left(\alpha^{s(n-t)+r}\beta^{s(n-2t)+r} - \alpha^{s(n-2t)+r}\beta^{s(n-t)+r}\right)$.

Therefore

$$C_1 = \frac{\chi_{s(n-2t)+r} q_{s(n-t)+r} - \chi_{s(n-t)+r}\beta^{st} q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r}(\alpha^{st} - \beta^{st})}$$

and

$$C_2 = -\frac{\chi_{s(n-2t)+r} q_{s(n-t)+r} - \chi_{s(n-t)+r}\alpha^{st} q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\beta^{s(n-2t)+r}(\alpha^{st} - \beta^{st})}.$$

Plugging into $q_{sn+r} = C_1\chi_{sn+r}\alpha^{sn+r} + C_2\chi_{sn+r}\beta^{sn+r}$, we get

$$\begin{aligned} q_{sn+r} &= \chi_{sn+r}\frac{\chi_{s(n-2t)+r} q_{s(n-t)+r} - \chi_{s(n-t)+r}\beta^{st} q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r}(\alpha^{st} - \beta^{st})}\alpha^{sn+r} \\ &\quad - \chi_{sn+r}\frac{\chi_{s(n-2t)+r} q_{s(n-t)+r} - \chi_{s(n-t)+r}\alpha^{st} q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\beta^{s(n-2t)+r}(\alpha^{st} - \beta^{st})}\beta^{sn+r} \\ &= \frac{\chi_{sn+r} q_{s(n-t)+r}\left(\alpha^{2st} - \beta^{2st}\right)}{\chi_{s(n-t)+r}(\alpha^{st} - \beta^{st})} - \frac{\chi_{sn+r} q_{s(n-2t)+r}(\alpha\beta)^{st}(\alpha^{st} - \beta^{st})}{\chi_{s(n-2t)+r}(\alpha^{st} - \beta^{st})} \\ &= \frac{a^{\xi(sn+r-1)}(ab)^{\lfloor (s(n-t)+r)/2 \rfloor}}{a^{\xi(s(n-t)+r-1)}(ab)^{\lfloor (sn+r)/2 \rfloor}} q_{s(n-t)+r}\left(\alpha^{st} + \beta^{st}\right) \\ &\quad - \frac{a^{\xi(sn+r-1)}(ab)^{\lfloor (s(n-2t)+r)/2 \rfloor}}{a^{\xi(s(n-2t)+r-1)}(ab)^{\lfloor (sn+r)/2 \rfloor}} (-ab)^{st} q_{s(n-2t)+r} \\ &= \left(\frac{b}{a}\right)^{\xi(st)\xi(sn+r)} \frac{a^{\xi(st)}}{(ab)^{\lfloor (st+1)/2 \rfloor}}(\alpha^{st} + \beta^{st}) q_{s(n-t)+r} + (-1)^{st+1} q_{s(n-2t)+r} \\ &= \left(\frac{b}{a}\right)^{\xi(st)\xi(sn+r)} l_{st} q_{s(n-t)+r} + (-1)^{st+1} q_{s(n-2t)+r}. \end{aligned}$$

□

We will now calculate $\lim\limits_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-t)+r}}$ explicitly, the ratio of two consecutive terms in the t-periodic subsequence.

Theorem 2.5. *For $r < s$, we have*

$$\lim_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-t)+r}}=\begin{cases}[l_{st}-1;\overline{1,l_{st}-2}]=\dfrac{\alpha^{st}}{(ab)^{st/2}}, & \text{for } st \text{ even},\\ [l_{st};\overline{l_{st}}]=\dfrac{a\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } st \text{ odd and } n+r \text{ even},\\ \left[\dfrac{b}{a}l_{st};\overline{\dfrac{b}{a}l_{st}}\right]=\dfrac{b\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } s,\ t, \text{ and } n+r \text{ odd}.\end{cases}$$

Proof. Using (2.2), we have

- For st even

$$\frac{q_{sn+r}}{q_{s(n-t)+r}}=l_{st}-\frac{q_{s(n-2t)+r}}{q_{s(n-t)+r}}=l_{st}-1+\frac{q_{s(n-t)+r}-q_{s(n-2t)+r}}{q_{s(n-t)+r}}$$
$$=l_{st}-1+\cfrac{1}{1+\cfrac{q_{s(n-2t)+r}}{q_{s(n-t)+r}-q_{s(n-2t)+r}}}=l_{st}-1+\cfrac{1}{1+\cfrac{1}{\cfrac{q_{s(n-t)+r}}{q_{s(n-2t)+r}}-1}}.$$

- For st odd and $n+r$ even

$$\frac{q_{sn+r}}{q_{s(n-t)+r}}=l_{st}+\frac{q_{s(n-2t)+r}}{q_{s(n-t)+r}}=l_{st}+\cfrac{1}{\cfrac{q_{s(n-t)+r}}{q_{s(n-2t)+r}}}.$$

- For s, t, and $n+r$ odd

$$\frac{q_{sn+r}}{q_{s(n-t)+r}}=\frac{b}{a}l_{st}+\frac{q_{s(n-2t)+r}}{q_{s(n-t)+r}}=\frac{b}{a}l_{st}+\cfrac{1}{\cfrac{q_{s(n-t)+r}}{q_{s(n-2t)+r}}}.$$

Using Binet's formula of the bi-periodic Fibonacci sequence, we obtain

$$\lim_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-t)+r}}=\lim_{n\to\infty}\frac{a^{\xi(sn+r-1)}(ab)^{\lfloor(s(n-t)+r)/2\rfloor}}{a^{\xi(s(n-t)+r-1)}(ab)^{\lfloor(sn+r)/2\rfloor}}\alpha^{st}\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{sn+r}}{1-\left(\frac{\beta}{\alpha}\right)^{s(n-t)+r}}\right)$$
$$=\begin{cases}\dfrac{\alpha^{st}}{(ab)^{st/2}}, & \text{for } st \text{ even},\\ \dfrac{a\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } st \text{ odd and } n+r \text{ even},\\ \dfrac{b\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } s,\ t, \text{ and } n+r \text{ odd}.\end{cases}\qquad\square$$

Note that if we take $t=2$ in Theorem 2.5, we get the following result.

Corollary 2.6.

$$\lim_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-2)+r}}=\left(\frac{\alpha^2}{ab}\right)^s=(\alpha+1)^s=[l_{2s}-1;\overline{1,l_{2s}-2}].$$

As a consequence of Theorem 2.5, for $a = b = 1$, we have the following result.

Corollary 2.7.

$$\lim_{n\to\infty} \frac{F_{sn+r}}{F_{s(n-t)+r}} = \phi^{st} = \begin{cases} \left[L_{st} - 1; \overline{1, L_{st} - 2}\right], & \textit{for } st \textit{ even}, \\ \left[L_{st}; \overline{L_{st}}\right], & \textit{for } st \textit{ odds}. \end{cases}$$

In the following theorem, we give a relation between some bi-periodic Fibonacci and Lucas sequences.

Theorem 2.8. *For $n \geq 1$, we obtain*

$$q_s l_{sn+r} = \left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} q_{s(n+1)+r} + (-1)^{s+1} \left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} q_{s(n-1)+r}.$$

Proof. From Binet's formulas (1.1), (1.2), and since $\alpha\beta = -ab$, we write

$$\begin{aligned}
&\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} \left(q_{s(n+1)+r} + (-1)^{s+1} q_{s(n-1)+r}\right) \\
&= \left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} \frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}} \left(\frac{\alpha^{s(n+1)+r} - \beta^{s(n+1)+r}}{\alpha - \beta}\right) \\
&\quad + (-1)^{s+1} \left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} \frac{a^{\xi(s(n-1)+r+1)}}{(ab)^{\lfloor (s(n-1)+r)/2 \rfloor}} \left(\frac{\alpha^{s(n-1)+r} - \beta^{s(n-1)+r}}{\alpha - \beta}\right) \\
&= \left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} \frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor (s(n+1)+r)/2 \rfloor} (\alpha - \beta)} \\
&\quad \times \left(\alpha^{sn+r}\left(\alpha^s - \left(\frac{-ab}{\alpha}\right)^s\right) - \beta^{sn+r}\left(\beta^s - \left(\frac{-ab}{\beta}\right)^s\right)\right) \\
&= \left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)} \frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}} \left(\frac{\alpha^s - \beta^s}{\alpha - \beta}\right) \left(\alpha^{sn+r} + \beta^{sn+r}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}} &= \frac{a^{\xi(sn+r)+\xi(s+1)-2\xi(s+1)\xi(sn+r)}}{(ab)^{\lfloor (sn+r+1)/2 \rfloor + \lfloor s/2 \rfloor + \xi(s)\xi(sn+r) - \xi(sn+r)}} \\
&= \frac{a^{\xi(sn+r)+\xi(s+1)-2\xi(s+1)\xi(r)}}{(ab)^{\lfloor (sn+r+1)/2 \rfloor + \lfloor s/2 \rfloor - \xi(s+1)\xi(sn+r)}} \\
&= \frac{a^{\xi(sn+r)+\xi(s+1)-2\xi(s+1)\xi(r)}}{(ab)^{\lfloor (sn+r+1)/2 \rfloor + \lfloor s/2 \rfloor - \xi(s+1)\xi(r)}} \\
&= \left(\frac{b}{a}\right)^{\xi(s+1)\xi(r)} \frac{a^{\xi(s+1)}}{(ab)^{\lfloor s/2 \rfloor}} \frac{a^{\xi(sn+r)}}{(ab)^{\lfloor (sn+r+1)/2 \rfloor}},
\end{aligned}$$

we obtain the result. $\square$

Note that for $s = 1$ and $r = 0$ in Theorem 2.8, we get the following result.

Corollary 2.9. *For $n \geq 1$, we obtain*

$$l_n = q_{n+1} + q_{n-1}.$$

In the following theorems, we give two continued fractions involving the bi-periodic Fibonacci and Lucas sequences.

Theorem 2.10.

$$\lim_{n\to\infty} \frac{l_n}{q_{n+1}} = [1; ab+1, \overline{1, ab}] = \beta + 2$$

and

$$\lim_{n\to\infty} \frac{q_{n+1}}{l_n} = [0; 1, ab+1, \overline{1, ab}] = \frac{\alpha+2}{ab+4}.$$

Proof. The bi-periodic Fibonacci and Lucas sequences satisfy the equation

$$l_n = q_{n+1} + q_{n-1}.$$

Thus, we obtain

$$\frac{l_n}{q_{n+1}} = \frac{q_{n+1} + q_{n-1}}{q_{n+1}} = 1 + \frac{1}{\frac{q_{n+1}}{q_{n-1}}}.$$

Using Corollary 2.6 by taking $s = 1$ and $r = 1$, we get the result.

Furthermore,

$$\lim_{n\to\infty} \frac{l_n}{q_{n+1}} = \lim_{n\to\infty} (\alpha - \beta) \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \lim_{n\to\infty} \frac{\alpha-\beta}{\alpha} \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^{n+1}} = \beta + 2.$$

Taking the reciprocal of this value, we get

$$\frac{q_{n+1}}{l_n} = \frac{1}{\frac{l_n}{q_{n+1}}} \quad \text{and} \quad \lim_{n\to\infty} \frac{q_{n+1}}{l_n} = \frac{1}{\beta+2} = \frac{\alpha+2}{ab+4}. \qquad \square$$

For the arithmetic progression situation, we get

Theorem 2.11. *For $r < s$, we have*

$$\lim_{n\to\infty} \frac{l_{sn+r}}{q_{s(n+1)+r}} =$$

$$\begin{cases} \frac{1}{q_s} + \frac{1/q_s}{l_{2s}-1} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} \cdots = \frac{(ab)^{(s-1)/2}(\alpha-\beta)}{\alpha^s}, & \textit{for } s \textit{ odd}, \\ 0 + \frac{1/q_s}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} \cdots = \frac{b(ab)^{(s-2)/2}(\alpha-\beta)}{\alpha^s}, & \textit{for } s \textit{ and } r \textit{ even}, \\ 0 + \frac{a/(bq_s)}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} \cdots = \frac{a(ab)^{(s-2)/2}(\alpha-\beta)}{\alpha^s}, & \textit{for } s \textit{ even and } r \textit{ odd}. \end{cases}$$

Proof. By taking $r \to r+s$ in Corollary 2.6 and using Theorem 2.8, we have

- For s odd

$$\frac{l_{sn+r}}{q_{s(n+1)+r}} = \frac{1}{q_s} + \frac{1}{q_s}\frac{q_{s(n-1)+r}}{q_{s(n+1)+r}} = 1/q_s + \frac{1/q_s}{\frac{q_{s(n+1)+r}}{q_{s(n-1)+r}}}.$$

- For s and r even

$$\frac{l_{sn+r}}{q_{s(n+1)+r}} = \frac{1}{q_s} - \frac{1}{q_s}\frac{q_{s(n-1)+r}}{q_{s(n+1)+r}} = \frac{1}{q_s}\frac{q_{s(n+1)+r} - q_{s(n-1)+r}}{q_{s(n+1)+r}} = \frac{1/q_s}{1 + \frac{1}{\frac{q_{s(n+1)+r}}{q_{s(n-1)+r}} - 1}}.$$

- For s even and r odd

$$\frac{l_{sn+r}}{q_{s(n+1)+r}} = \frac{a}{bq_s} - \frac{a}{bq_s}\frac{q_{s(n-1)+r}}{q_{s(n+1)+r}} = \frac{a}{bq_s}\frac{\left(q_{s(n+1)+r} - q_{s(n-1)+r}\right)}{q_{s(n+1)+r}} = \frac{a/(bq_s)}{1 + \frac{1}{\frac{q_{s(n+1)+r}}{q_{s(n-1)+r}} - 1}}.$$

Using Binet's formulas for the bi-periodic Fibonacci and Lucas sequences, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{l_{sn+r}}{q_{s(n+1)+r}} \\
&= \lim_{n\to\infty} \frac{a^{\xi(sn+r)}(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}}{a^{\xi(s(n+1)+r-1)}(ab)^{\lfloor (sn+r+1)/2 \rfloor}} (\alpha-\beta) \frac{\alpha^{sn+r} - \beta^{sn+r}}{\alpha^{s(n+1)+r} - \beta^{s(n+1)+r}} \\
&= \lim_{n\to\infty} \frac{a^{\xi(sn+r)}(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}}{a^{\xi(s(n+1)+r-1)}(ab)^{\lfloor (sn+r+1)/2 \rfloor}} \frac{\alpha-\beta}{\alpha^s} \frac{1 - \left(\frac{\beta}{\alpha}\right)^{sn+r}}{1 - \left(\frac{\beta}{\alpha}\right)^{s(n+1)+r}} \\
&= \begin{cases} (ab)^{(s-1)/2}(\alpha-\beta)/\alpha^s, & \text{for } s \text{ odd}, \\ b(ab)^{(s-2)/2}(\alpha-\beta)/\alpha^s, & \text{for } s \text{ and } r \text{ even}, \\ a(ab)^{(s-2)/2}(\alpha-\beta)/\alpha^s, & \text{for } s \text{ even and } r \text{ odd}. \end{cases}
\end{aligned}$$ □

Taking the reciprocal of Theorem 2.11, the next result follows.

Theorem 2.12. *For $r < s$, we have*

$$\lim_{n\to\infty} \frac{q_{s(n+1)+r}}{l_{sn+r}} =$$
$$\begin{cases} 0 + \frac{1}{1/q_s} + \frac{1/q_s}{l_{2s}-1} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \cdots = \frac{\alpha^s}{(ab)^{(s-1)/2}(\alpha-\beta)}, & \textit{for } s \textit{ odd}, \\ 0 + \frac{1}{0} + \frac{1/q_s}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \cdots = \frac{a\alpha^s}{(ab)^{s/2}(\alpha-\beta)}, & \textit{for } s \textit{ and } r \textit{ even}, \\ 0 + \frac{1}{0} + \frac{a/(bq_s)}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \cdots = \frac{b\alpha^s}{(ab)^{s/2}(\alpha-\beta)}, & \textit{for } s \textit{ even and } r \textit{ odd}. \end{cases}$$

Proof. Knowing that

$$\frac{q_{s(n+1)+r}}{l_{sn+r}} = \frac{1}{\dfrac{l_{sn+r}}{q_{s(n+1)+r}}} \quad \text{and} \quad \lim_{n\to\infty} \frac{q_{s(n+1)+r}}{l_{sn+r}} = \lim_{n\to\infty} \frac{1}{\dfrac{l_{sn+r}}{q_{s(n+1)+r}}},$$

using Theorem 2.11, we get the result. □

Note that, for $a = b = 1$, we get the following result.

Corollary 2.13.

$$\lim_{n\to\infty} \frac{L_{sn+r}}{F_{s(n+1)+r}} = \frac{\alpha-\beta}{\alpha^s} = \begin{cases} \dfrac{1}{F_s} + \dfrac{1/F_s}{L_{2s}-1} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \textit{for } s \textit{ odd},\\ 0 + \dfrac{1/F_s}{1} + \dfrac{1}{L_{2s}-2} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \textit{for } s \textit{ even}, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{F_{s(n+1)+r}}{L_{sn+r}} = \frac{\alpha^s}{\alpha-\beta} = \begin{cases} 0 + \dfrac{1}{1/F_s} + \dfrac{1/F_s}{l_{2s}-1} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \textit{for } s \textit{ odd},\\ 0 + \dfrac{1}{0} + \dfrac{1/F_s}{1} + \dfrac{1}{L_{2s}-2} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \textit{for } s \textit{ even}. \end{cases}$$

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Short remark to the Rimán's Theorem

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Abstract. In this paper the general solution of the functional equation $f(x+y) = g(x) + h(y)$ $((x,y) \in D)$ is given with unknown functions $f\colon D_{x+y} \to Y$, $g\colon D_x \to Y$, $h\colon D_y \to Y$ where $D \subseteq \mathbb{G}^2$ is a nonempty, open set, $(\mathbb{G}, \leqslant)$ is an ordered, dense, Abelian group, the topology on $\mathbb{G}$ is generated by the open intervals of $\mathbb{G}$, the sets D_x, D_y, D_{x+y} are defined by $D_x := \{u \in \mathbb{G} \mid \exists v \in \mathbb{G} : (u,v) \in D\}$, $D_y := \{v \in \mathbb{G} \mid \exists u \in \mathbb{G} : (u,v) \in D\}$, $D_{x+y} := \{z \in \mathbb{G} \mid \exists (u,v) \in D : z = u+v\}$, and $Y(+)$ is an Abelian group.

The main result of the article is a common generalization of similar results by L. Székelyhidi and J. Rimán. Analogous theorem concerning logarithmic functions is also shown.

Keywords: additive functional equations, logarithmic functional equations, Pexider generalizations, restricted functional equations, Archimedean ordered Abelian groups, dense ordered groups, general solution of functional equations

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1. Introduction

The main purpose of this article is to prove the generalization of J. Rimán's Extension Theorem [21]. Now, we give a non-exhaustive overview of the most important steps of the theory of Extension and Uniqueness Theorems concerning restricted Pexider additive functional equations.

In the sequel we will use the notations

$$\begin{aligned} D_x &:= \{u \in X \mid \exists v \in \mathbb{G} : (u,v) \in D\}, \\ D_y &:= \{v \in Y \mid \exists u \in \mathbb{G} : (u,v) \in D\}, \\ D_{x+y} &:= \{z \in X \mid \exists (u,v) \in D : z = u+v\} \end{aligned}$$

where $D \subseteq \mathbb{G}^2 := \mathbb{G} \times \mathbb{G}$ and $\mathbb{G}(+)$ is a grupoid.

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The early results were grouped around the following problem. Let be $D \subseteq \mathbb{R}^2$, $f\colon D_x \cup D_y \cup D_{x+y} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) + f(y) \quad ((x,y) \in D). \tag{RestAdd}$$

The functional equation (RestAdd) is said to be restricted additive functional equation. The problem is to find a function $F\colon \mathbb{R} \to \mathbb{R}$ such that

$$F(x+y) = F(x) + F(y) \quad (x, y \in \mathbb{R}), \tag{1.1}$$

and $F(x) = f(x)$ for all $x \in \mathcal{D}_f$ (see [14] Part IV. Geometry, Section Extension of Functional equation p. 447–460). The function F is said to be additive extension of the function f from the set D to the $\mathbb{R}^2$.

If a function $F\colon \mathbb{R} \to \mathbb{R}$ satisfies the equation (1.1), then the function F is said to be Cauchy-additive function (see A. E. Legendre [18], C. F. Gauss [9]). A. L. Cauchy first found the continuous solutions of equation (1.1) [5].

In [4] $D = (\mathbb{R}_+ \cup \{0\})^2$ ($\mathbb{R}_+ := \{x \in \mathbb{R}_+ \mid x > 0\}$). The solution of equation (RestAdd) is $f(x) = F(x)$ for all $x \in \mathbb{F}_+$ where the function F is a Cauchy-additive function.

In [2] the concept of quasi-extension can be found. The situation is that $D \subseteq \mathbb{R}^2$ is a nonempty connected open set, and the functions f satisfies the functional equation (RestAdd) for all $(x,y) \in D$ then there exists an additive function F and exist constants C_1, $C_2 \in \mathbb{R}$ such that

$$\begin{aligned} f(z) &= F(z) + C_1 + C_2 && (z \in D_{x+y}),\\ f(u) &= F(u) + C_1 && (u \in D_x),\\ f(v) &= F(v) + C_2 && (v \in D_y). \end{aligned} \tag{1.2}$$

If the function f and the additive function F is in the form of (1.2), then the function F is said to be quasi extension of the function f.

In [6] $D = \mathbb{R}_+^2$ or D is circle neighbourhood of the point $(0,0) \in \mathbb{R}^2$. In these case f has additive extension.

In [23] D is an open subset of $\mathbb{R}^2$. The author of this paper has shown that the set D is a countable disjoint union of connected open sets, that is $D = \bigcup_i D^i$. The sets D_i is said to be components of the set D. For all i there exists an additive function $F_i\colon \mathbb{R} \to \mathbb{R}$ and constants C_1^i, $C_2^i \in \mathbb{R}$ such that

$$\begin{aligned} f(z) &= F_i(z) + C_1^i + C_2^i && (z \in D_{x+y}^i),\\ f(u) &= F_i(u) + C_1^i && (u \in D_x^i),\\ f(v) &= F_i(v) + C_2^i && (v \in D_y^i). \end{aligned} \tag{1.3}$$

If $i \neq j$, then the obtained functions F_i, and F_j, as well as the obtained constants C_1^i, and C_1^j, or C_2^i, and C_2^j are not necessarily different depending on whether $D_x^i \cap D_x^j \neq \emptyset$ or $D_y^i \cap D_y^j \neq \emptyset$ or $D_{x+y}^i \cap D_{x+y}^j \neq \emptyset$.

It is also worth mentioning that if $D_x^i \cap D_{x+y}^i \neq \emptyset$ then $C_2^i = 0$; if $D_y^i \cap D_{x+y}^i \neq \emptyset$ then $C_1^i = 0$; if $D_x^i \cap D_y^i \neq \emptyset$ then $Ci_1 = Ci_2$ for all component D^i. If the point $(0,0)$ is an inner point of a component D^i then $D_x^i \cap D_y^i \cap D_{x+y}^i \neq \emptyset$ thus $C_1^i = C_2^i = 0$.

In [21] J. Rimán studied restricted Pexider additive functional equations in the form

$$f(x+y) = g(x) + h(y) \quad ((x,y) \in D) \tag{RestPexAdd}$$

where the set D is a connected open subset of the set $\mathbb{R}^2$, $E = E(+)$ is an Abelian group, and the unknown functions $f\colon D_{x+y} \to E$, $g\colon D_x \to E$, $h\colon D_y \to E$ satisfy the equation (RestPexAdd) for all $(x,y) \in D$. The solution of equation (1.4) is

$$\begin{aligned} f(z) &= F(z) + C_1 + C_2 && (z \in D_{x+y}),\\ g(u) &= F(u) + C_1 && (u \in D_x),\\ h(v) &= F(v) C_2 && (v \in D_y), \end{aligned} \tag{1.4}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is an additive function, C_1, $C_2 \in E$ are constants.

In [1] $D = H(I)$ where I is a nonempty open interval of the real line and the set $H(I)$ is defined by

$$H(I) := \{(x,y) \in \mathbb{R}^2 \mid x, y, x+y \in I\}.$$

The set $H(I)$ is a hexagon, sometimes a triangle or the emptyset.

M. Kuczma in his book [16] investigated both of Pexider type functional equations and additive functional equations, but did not consider restricted Pexider-additive functional equations. He used Jensen functions for his Extension Theorem and gave the solution of equation (RestAdd) (Theorem 13.6.1), where D is a nonempty, connected, open subset of $\mathbb{R}^{2N} := \mathbb{R}^N \times \mathbb{R}^N$ and $D_x \cup D_y \cup D_{x+y} \subseteq \mathcal{D}_f$. He showed that the solution of equation (RestAdd) is in the form of (1.2) where $F\colon \mathbb{R}^N \to \mathbb{R}^N$ is an additive function, C_1, $C_2 \in \mathbb{R}^N$ are constants. The extension was brought back to the theory of Jensen functions.

An $X = X(+)$ Abelian group is said to be uniquely 2-divisible, if for all $x \in X$ there uniquely exists an $y \in X$ such that $y + y := 2y = x$. This element $y \in X$ is denoted by $y = \frac{1}{2}x$. A nonempty set $A \subseteq X$ is said to be midconvexe, if $\frac{x+y}{2} \in A$ for all $x, y \in A$. Let $Y = Y(+)$ be also a uniquely 2-divisible Abelian group. A function $j\colon A \to Y$ is said to be Jensen [7, 15, 16] if

$$j\left(\frac{x+y}{2}\right) = \frac{j(x) + j(y)}{2} \quad (x, y \in A).$$

The way outlined by M. Kuczma is not suitable for us, since we do not want to deal with either 2-divisible or p-divisible groups, and we do not think that the vector space structure is necessary for an additive extension theorem.

In the article [20] an extension theorem for restricted Pexider additive functional equation can be found, where $D \subseteq (\mathbb{R}^N)^2$ is a nonempty, connected, open set.

In the book [3] several functional equations can be found in more general abstract algebraic settings .

Concerning the Extension Theorems see also [8, 13, 17].

2. Some necessary concepts and results

Now, we review the concepts and results which will be used in the sequel.

- If $\mathbb{G}(+,\leqslant)$ is an ordered group, α, $\beta \in \mathbb{G}$ such that $\alpha < \beta$ then the set $]\alpha,\beta[:= \{x \in \mathbb{G} \mid \alpha < x < \beta\}$ is said to be open interval.
- An ordered group $\mathbb{G}(+\leqslant)$ is said to be dense (in itself) if $]\alpha,\beta[\neq \emptyset$ for all α, $\beta \in \mathbb{G}$ with $\alpha < \beta$.
- An ordered group $\mathbb{G}(+,\leqslant)$ is said to be Archimedean ordered if for all x, $y \in \mathbb{G}_+$ there exists a positive integer n such that $y < nx := x + \cdots + x$.
- An ordered field $\mathbb{F}(+,\cdot,\leqslant)$ is said to be Archimedean ordered if $\mathbb{F}(+,\leqslant)$ is an Archimedean ordered group.

Now, we review some properties of open intervals ([10, 12]). The open intervals are

- translation invariant, that is, if $\mathbb{G}(+,\leqslant)$ is an ordered, dense, Abelian group, then $\gamma +]\alpha,\beta[=]\gamma+\alpha,\gamma+\beta[$ for all α, β, $\gamma \in \mathbb{G}$ such that $\alpha < \beta$.
- additive, that is, if $\mathbb{G}(+,\leqslant)$ is an ordered, dense, Abelian group, then $]\alpha,\beta[+]\gamma,\delta[=]\alpha+\gamma,\beta+\delta[$ for all α, β, γ, $\delta \in \mathbb{G}$ with $\alpha < \beta$ and $\gamma < \delta$.
- homothety invariant, that is, if $\mathbb{F}(+,\cdot,\leqslant)$ is an ordered field, then $\gamma \cdot]\alpha,\beta[=]\gamma\alpha,\gamma\beta[$ for all α, β, $\gamma \in \mathbb{F}$ with $\alpha < \beta$ and $\gamma > 0$.
- multiplicative, that is, if $\mathbb{F}(+,\cdot,\leqslant)$ is an ordered field, then $]\alpha,\beta[\cdot]\gamma,\delta[=]\alpha\gamma,\beta\delta[$ for all α, β, γ, $\delta \in \mathbb{F}$ with $0 < \alpha < \beta$ and $0 < \gamma < \delta$.

If $\mathbb{G}(+,\leqslant)$ is an ordered group, $x \in \mathbb{G}$, (or $x := (x_1,x_2) \in \mathbb{G}^2$), $\varepsilon \in \mathbb{G}_+$, then define the set $B(x,\varepsilon)$ by $B(x,\varepsilon) :=]x-\varepsilon, x+\varepsilon[$, $(B(x,\varepsilon) :=]x_1-\varepsilon, x_1+\varepsilon[\times]x_2-\varepsilon, x_2+\varepsilon[)$ respectively. The set $B(x,\varepsilon)$ is said to be open neighbourhood of the point x with radius ε.

A function $a\colon X \to Y$ is said to be additive if $X(+)$ and $Y(+)$ are algebraic structures, and

$$a(x+y) = a(x) + a(y) \quad (x, y \in X).$$

A function $l\colon X \to Y$ is said to be logarithmic if $X(\cdot)$ and $Y(+)$ are algebraic structures, and

$$l(xy) = l(x) + l(y) \quad (x, y \in X).$$

Concerning the additive and logarithmic functions see [3, 16].

3. Extension Theorem for Pexider additive functional equation

We shall use the Existence Theorem for additive functions [10] according to which if $\mathbb{G}(+,\leqslant)$ is an Archimedean ordered, dense, Abelian group, $Y(+)$ is a group, $\varepsilon \in \mathbb{G}_+$, and the function satisfy the equation (RestAdd) where $D :=]0,\varepsilon[^2$ then there exists an additive function $a\colon \mathbb{G} \to Y$ which extends the function f from $]-2\varepsilon, 2\varepsilon[$ to $\mathbb{G}$.

Theorem 3.1. *If $\mathbb{G}(+,\leqslant)$ is an Archimedean ordered, dense, Abelian group, $Y(+)$ is an Abelian group, x_0, $y_0 \in \mathbb{G}$, $\varepsilon \in \mathbb{G}_+$, and the functions $f\colon B(x_0+y_0, 2\varepsilon) \to Y$, $g\colon B(x_0,\varepsilon) \to Y$, $h\colon B(y_0,\varepsilon) \to Y$ satisfies the functional equation (RestPexAdd) then there exists an additive function $a\colon \mathbb{G} \to Y$ and exist constants C_1, $C_2 \in Y$ such that the functions f, g, h are in the form of (1.4).*

Proof. By the translation invariant property of the open intervals we have that

$$\begin{aligned} B(x_0,\varepsilon) &= x_0 + B(0,\varepsilon),\\ B(y_0,\varepsilon) &= y_0 + B(0,\varepsilon),\\ B(x_0+y_0,2\varepsilon) &= x_0+y_0+B(0,2\varepsilon). \end{aligned}$$

Define the functions $F\colon B(0,2\varepsilon) \to Y$, $G\colon B(0,\varepsilon) \to Y$, $H\colon B(0,\varepsilon) \to Y$ by

$$\begin{aligned} F(w) &= f(x_0+y_0+w) && (w \in B(0,2\varepsilon)),\\ G(u) &= g(x_0+u) && (u \in B(0,\varepsilon)),\\ H(v) &= h(y_0+v) && (v \in B(0,\varepsilon)). \end{aligned} \tag{3.1}$$

Then $F(0) = f(x_0+y_0)$, $G(0) = g(x_0)$, $H(0) = h(y_0)$ and

$$F(u+v) = G(u) + H(v) \quad (u,v \in B(0,\varepsilon)).$$

Thus we obtain that

$$\begin{aligned} F(u) &= G(u) + H(0) = G(u) + h(y_0) && (u \in B(0,\varepsilon)),\\ F(v) &= G(0) + H(v) = g(x_0) + H(v) && (v \in B(0,\varepsilon)), \end{aligned} \tag{3.2}$$

whence we obtain that

$$\begin{aligned} F(u) + F(v) &= G(u) + H(v) + g(x_0) + h(y_0)\\ &= F(u+v) + g(x_0) + h(y_0) \quad (u,v \in B(0,\varepsilon)). \end{aligned}$$

Define the function $\varphi\colon B(0,\varepsilon) \to Y$ by

$$\varphi(x) := F(x) - (g(x_0) + h(y_0)) \quad (x \in B(0,2\varepsilon)) \tag{3.3}$$

Then

$$\varphi(x+y) = \varphi(x) + \varphi(y) \quad (x,y \in B(0,\varepsilon)),$$

whence by the Extension Theorem [10] we obtain that there exists an additive function $a\colon \mathbb{G} \to Y$ such that

$$\varphi(x) = a(x) \quad (x \in B(0, 2\varepsilon)). \tag{3.4}$$

Then by equations (3.1), (3.3), and (3.4) we have that

$$\begin{aligned} f(x_0 + y_0 + w) &\overset{(3.1)}{=} F(w) \overset{(3.3)}{=} \varphi(w) + (g(x_0) + h(y_0)) \\ &\overset{(3.4)}{=} a(w) + (g(x_0) + h(y_0)) \quad (w \in B(0, 2\varepsilon)). \end{aligned} \tag{3.5}$$

By equations (3.1), (3.2), (3.3) and (3.4) we have that

$$\begin{aligned} g(x_0 + u) &\overset{(3.1)}{=} G(u) \overset{(3.2)}{=} F(u) - h(y_0) \overset{(3.3)}{=} \varphi(u) + g(x_0) \\ &\overset{(3.4)}{=} a(w) + g(x_0) \quad (w \in B(0, \varepsilon)). \end{aligned} \tag{3.6}$$

By equations (3.1), (3.2), (3.3) and (3.4) we have that

$$\begin{aligned} h(y_0 + v) &\overset{(3.1)}{=} H(v) \overset{(3.2)}{=} F(u) - g(x_0) \overset{(3.3)}{=} \varphi(v) + h(y_0) \\ &\overset{(3.4)}{=} a(u) + h(y_0) \quad (w \in B(0, \varepsilon)). \end{aligned} \tag{3.7}$$

Take the substitutions: $w \longleftarrow w - (x_0 + y_0)$ in (3.5), $u \longleftarrow u - x_0$ in (3.6), $v \longleftarrow v - y_0$ in (3.7), and define the constants $c\ d \in Y$ by $c := g(x_0) - a(x_0)$, $d := h(y_0) - a(y_0)$ thus the translation invariant property of the intervals we obtain equation (1.4) which was to be prooved. □

We shall use the Existence Theorem for logarithmic functions in [10] according to which if $\mathbb{F}(+, \cdot, \leqslant)$ is an Archimedean ordered field, $Y(+)$ is a group, $\varepsilon \in \mathbb{F}$ such that $\varepsilon > 1$, and the function $f\colon]\varepsilon^{-2}, \varepsilon^2[\to Y$ satisfies the equation

$$f(xy) = f(x) + f(y) \quad (x, y \in]\varepsilon^{-1}, \varepsilon[),$$

then there exists a logarithmic function $l\colon \mathbb{F}_+ \to Y$ which extends the function f from $]\varepsilon^{-2}, \varepsilon^2[$ to the $\mathbb{F}_+^2$.

Theorem 3.2. *If $\mathbb{F}(+, \cdot, \leqslant)$ is an Archimedean ordered field, $Y(+)$ is an Abelian group, x_0, $y_0 \in \mathbb{F}_+$, $\varepsilon \in \mathbb{F}_+$, and $f\colon]x_0y_0\varepsilon^{-2}, x_0y_0\varepsilon^2[\to Y$, $g\colon]x_0\varepsilon^{-1}, x_0\varepsilon[\to Y$, $h\colon]y_0\varepsilon^{-1}, y_0\varepsilon[\to Y$ are functions such that*

$$f(xy) = g(x) + h(y) \quad (x \in]x_0\varepsilon^{-1}, x_0\varepsilon[,\ y \in]y_0\varepsilon^{-1}, y_0\varepsilon[),$$

then there exists a logarithmic function $l\colon \mathbb{F}_+ \to Y$ and exist constants C_1, $C_2 \in Y$ such that

$$\begin{aligned} f(w) &= l(w) + C_1 + C_2 && (w \in]x_0y_0\varepsilon^{-2}, x_0y_0\varepsilon^2[), \\ g(u) &= l(u) + C_1 && (u \in]x_0\varepsilon^{-1}, x_0\varepsilon[), \\ h(v) &= l(v) + C_2 && (v \in]y_0\varepsilon^{-1}, y_0\varepsilon[). \end{aligned}$$

Proof. The proof is analogues to the proof of the Theorem 3.1. □

4. Topology generated by the open intervals of an Archimedean ordered Abelian group

Let $\mathbb{G} = \mathbb{G}(+, \leqslant)$ be an ordered group, $X \in \{\mathbb{G}, \mathbb{G}^2\}$ and $D \subseteq X$. The set D is said to be open if for every point x in D there exists an $\varepsilon \in \mathbb{G}_+$ such that $B(x, \varepsilon) \subseteq D$.

A subset $D \subseteq X$ is said to be well-chained, if for all $x, y \in D$ there exists a finite sequence $B_i := B(x_i, \varepsilon_i)$ $(i = 0, 1, \ldots, n)$ such that

- $B_i \subseteq D$ for all $i = 0, 1, \ldots, n$,
- $x \in B_0$, $y \in B_n$,
- $B_{i-1} \cap B_i \neq \emptyset$ for all $i = 1, \ldots, n$.

A subset C of a nonempty, open set $D \subseteq X$ is a component of D if C is a maximal (with respect the inclusion) well-chained, open subset of D.

A topological space $X(\mathcal{T})$ is said to be separable if there exists a subset $Y \subseteq X$ which is countable, infinite, and dense (in X).

Theorem 4.1. *If $\mathbb{G} = \mathbb{G}(+, \leqslant)$ is an ordered group, $X \in \{\mathbb{G}, \mathbb{G}^2\}$ and $D \subseteq X$ is a nonempty, well-chained, open set, then*

1. *D is a disjoint union of its components;*
2. *If X is separable then D has countable components.*

Proof. 1. Define the family $\mathcal{B}$ by

$$\mathcal{B} := \{B(x, \varepsilon) \subseteq D \mid x \in D, \varepsilon \in \mathbb{G}_+\} := \{B_\alpha \mid \alpha \in \Gamma\}.$$

Define the equivalence relation on $\mathcal{B}$ by $B_\alpha \sim B_\beta$ if and only if there exists a finite sequence B_{α_i} $(i = 0, 1, \ldots, n)$ such that $B_{\alpha_0} = B_\alpha$, $B_{\alpha_n} = B_\beta$ and $B_{\alpha_{i-1}} \cap B_{\alpha_i} \neq \emptyset$ for all $i = 1, 2, \ldots, n$. The set $\mathcal{B}$ is a disjoint union of its equivalence classes. The components of the set D are the union of all balls B_α that belong to the same equivalence class.

2. Let Y be a countable, dense subset of the set X, and let $B \subseteq X$ be a nonempty, open subset with components $\{D^i\}_{i \in I}$. Then for all $i \in I$ there exists a ball $B_i := B(x_i, \varepsilon_i)$ such that $B_i \subseteq D^i$. If $i \neq j$ then $B_i \cap B_j = \emptyset$. Since Y is dense in $\mathbb{G}$ thus for all $i \in I$ there exists an $y_i \in Y$ such that $y_i \in B_i$. Define the function $\varphi \colon \{D^i\}_{i \in I} \to Y$ by $\varphi(D^i) := y_i$. Since the function φ is injective thus the set $\{D^i\}_{i \in I}$ is countable. $\square$

Example 4.2. If $\mathbb{G}(+, \leqslant)$ is a p- divisible, Archimedean ordered, Abelean group for a prime number p, then $\mathbb{G}$ is separable.

Example 4.3. Let $a \colon \mathbb{R} \to \mathbb{R}$ is a noncontinuous additive function. As it is well-known that the graph of a is dense in $\mathbb{R}^2$ (with respect to usual topology on $\mathbb{R}^2$), but the restriction of the function a to the set $\mathbb{Q}$ (where $\mathbb{Q}$ denotes the set of all rationals) is continuous with respect to the topology on the set $\mathbb{Q}(+)$ defined above, and the usual topology on the real line [11].

5. The generalization of Rimán's Extension Theorem

We shall use the Uniqueness Theorem for additive functions [10], according to which if $\mathbb{G}(+,\leqslant)$ is an Archimedean ordered, Abelian group, $a\colon \mathbb{G}\to Y$ is an additive function, $C\in Y$, and $]\alpha,\beta[\subseteq Y$ is a nonempty interval such that $a(x)=C$ for all $x\in\,]\alpha,\beta[$ then $a(x)=0$ for all $x\in\mathbb{G}$ (and thus $C=0$).

Now, we give the generalization of Rimán's Extension Theorem:

Theorem 5.1. *If $\mathbb{G}(+,\leqslant)$ be an Archimedean ordered, dense, Abelian group, and $D\subseteq\mathbb{G}^2$ is an open set with components $\{D^i\mid i\in I\}$ and Y is an Abelian group then the functions $f\colon D_{x+y}\to Y$, $g\colon D_x\to Y$, $h\colon D_y\to Y$ if and only if are solutions of the functional equation (RestPexAdd) then there exists a family of additive functions $a_i\colon\mathbb{G}\to Y$ $(i\in I)$ and exist families of constants C_1^i, $C_2^i\in Y$ $(i\in I)$ such that*

$$\begin{aligned} f(z)&=a_i(z)+C_1^i+C_2^i && (z\in D^i_{x+y}),\\ g(u)&=a_i(u)+C_1^i && (u\in D^i_x),\\ h(v)&=a_i(v)+C_2^i && (v\in D^i_y)\end{aligned}\tag{5.1}$$

with

1. *if $D^i_{x+y}\cap D^j_{x+y}\neq\emptyset$, then $a_i=a_j$, and $C_1^i+C_2^i=C_1^i+C_2^i$;*
2. *if $D^i_x\cap D^j_x\neq\emptyset$, then $a_i=a_j$, and $C_1^i=C_1^j$;*
3. *if $D^i_y\cap D^j_y\neq\emptyset$, then $a_i=a_j$, and $C_2^i=C_2^j$*

for all i, $j\in I$, $i\neq j$.

Proof. Let us assume that the functions f, g, h satisfy the functional equation (RestPexAdd). By Theorem 3.1 we obtain that they are in the form of (5.1), and by Uniqueness Theorem [10] properties 1., 2., and 3. are fulfilled.

Conversely, let us assume that the functions f, g, h are defined by equation (5.1), and the properties 1., 2., and 3. are fulfilled. These functions are well-defined, and they satisfy the functional equation (RestPexAdd). □

Theorem 5.2. *If $\mathbb{G}(+,\leqslant)$ be an Archimedean ordered, dense, Abelian group, and $D\subseteq\mathbb{G}^2$ is an open set with components $\{D^i\mid i\in I\}$ and Y is an Abelian group. Define the set $D_0:=D_x\cup D_y\cup D_{x+x}$. The function $f\colon D_0\to Y$ is satisfies functional equation (RestAdd) if and only if then there exists a family of additive functions $a_i\colon\mathbb{G}\to Y$ $(i\in I)$ and exist families of constants C_1^i, $C_2^i\in Y$ for all $i\in I$ such that*

$$\begin{aligned} f(z)&=a_i(z)+C_1^i+C_2^i && (z\in D^i_{x+y}),\\ g(u)&=a_i(u)+C_1^i && (u\in D^i_x),\\ h(v)&=a_i(v)+C_2^i && (v\in D^i_y)\end{aligned}\tag{5.2}$$

with

1. *If* $D^i_{x+y} \cap D^j_{x+y} \neq \emptyset$*, then* $a_i = a_j$*, and* $C^i_1 + C^i_2 = C^i_1 + C^i_2$*;*
2. *If* $D^i_x \cap D^j_x \neq \emptyset$*, then* $a_i = a_j$*, and* $C^i_1 = C^j_1$*;*
3. *If* $D^i_y \cap D^j_y \neq \emptyset$*, then* $a_i = a_j$*, and* $C^i_2 = C^j_2$*;*

for all i*,* $j \in I$*,* $i \neq j$*, moreover,*

4. *If* $D^i_{x+y} \cap D^i_x \neq \emptyset$*, then* $C^i_2 = 0$*;*
5. *If* $D^i_{x+y} \cap D^i_y \neq \emptyset$*, then* $C^i_1 = 0$*;*
6. *If* $D^i_y \cap D^i_y \neq \emptyset$*,* $C^i_2 = C^j_2$

for all $i \in I$*.*

Proof. The proof can be easily obtained by Theorem 5.1 and the Uniqueness Theorem [10]. □

6. An application

Now we show a version of the well-known Rado-Baker functional equation [20].

It is worth mentioning that if $\mathbb{F}(+,\cdot,\leqslant)$ is an ordered field then $\mathbb{F}^2$ is a two-dimensional vector space over the ordered field $\mathbb{F}$ with the usual point-wise definition of vector operations. The set $C \subseteq \mathbb{F}^2$ is said to be

- convex if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$, and $\lambda \in]0,1[$;
- cone if $\lambda x \in C$ for all $\lambda \in \mathbb{F}_+$, and $x \in C$;
- convex cone if $\lambda x + \mu y \in C$ for all $x, y \in C$, and $\lambda, \mu \geqslant 0$ with $\lambda^2 + \mu^2 > 0$,

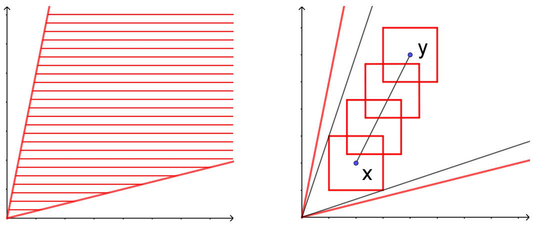
see Leonard Lewis [19], Rockafellar [22].

Let $\mathbb{F}(+,\cdot,\leqslant)$ be an Archimedean ordered field, $\alpha \in \mathbb{F}_+ \cup \{0\}$, $\beta \in \mathbb{F}_+ \cup \{+\infty\}$ such that $0 \leqslant \alpha < \beta \leqslant +\infty$. Define the set $C := C_{\alpha,\beta}$ by

$$C_{\alpha,\beta} \doteq \begin{cases} \{(x,y) \in \mathbb{F}^2_+ | \alpha x < y < \beta x\}, & \text{if } \alpha \in \mathbb{F}_+ \cup \{0\},\ \beta \in \mathbb{F}_+; \\ \{(x,y) \in \mathbb{F}^2_+ | \alpha x < y\}, & \text{if } \beta = +\infty. \end{cases}.$$

Proposition 6.1. *The set* $C = C_{\alpha,\beta}$ *is a nonempty, open, well-chained set.*

Proof. Since $C_{\alpha,\beta}$ is a nonempty, open, convex cone thus it is a nonempty, well-chained, open set.



□

Proposition 6.2. *If $\mathbb{F}(+,\leqslant)$ is an Archimedean ordered field, $Y(+)$ is an abelian group, the functions $P,Q,R\colon \mathbb{F}_+ \to Y$ are solutions of functional equation*

$$P(x+y) = Q(x) + R(y) \quad (x,y \in C(\alpha,\beta)) \tag{6.1}$$

then there exists an additive function $a\colon \mathbb{F} \to Y$ and constants C_1, $C_2 \in Y$ such that

$$\begin{aligned} P(x) &= a(x) + C_1 + C_2 && (x \in \mathbb{F}_+), \\ Q(x) &= a(x) + C_1 && (x \in \mathbb{F}_+), \\ R(x) &= a(x) + C_2 && (x \in \mathbb{F}_+). \end{aligned} \tag{6.2}$$

Proof. Let $D := C_{\alpha,\beta}$. Since $D_x = D_y = D_{x+y} = \mathbb{F}_+$ thus by Theorem 5.1 we obtain the statement. □

The following Theorem is a generalization of Rado–Baker Theorem [4], and if can be obtained from Proposition 6.2 as a simple consequence.

Theorem 6.3. *Let $\mathbb{F}(+,\cdot,\leqslant)$ be an Archimedean ordered field, $Y(+)$ be an Abelian group, $\alpha,\beta,\gamma,\delta \in \mathbb{F}$ such that $\alpha\delta - \beta\gamma \neq 0$. The functions $P,Q,R\colon \mathbb{F}_+ \to Y$ if and only if satisfy the functional equation*

$$P((\alpha+\gamma)x + (\beta+\delta)y) = Q(\alpha x + \beta y) + R(\gamma x + \delta y), \quad (x,y \in \mathbb{F}_+) \tag{6.3}$$

if they are of the form of (6.2) where $a\colon \mathbb{F} \to Y$ is an additive function, $C_1, C_2 \in Y$ are constants.

Proof. Let us assume that the functions $P,Q,R\colon \mathbb{F}_+ \to Y$ satisfy the functional equation (6.3) where az $\alpha,\beta,\gamma,\delta \in \mathbb{F}$ such that $\alpha\delta - \beta\gamma \neq 0$. Take the following substitution in (6.3):

$$\begin{gathered} P((\alpha+\gamma)x + (\beta+\delta)y) = Q(\underbrace{\alpha x + \beta y}_{u}) + R(\underbrace{\gamma x + \delta y}_{v}), \\ x \leftarrow \frac{\delta u - \beta v}{\alpha\beta - \beta\gamma} > 0 \quad y \leftarrow \frac{\alpha v - \gamma u}{\alpha\delta - \beta\gamma} > 0. \end{gathered} \tag{6.4}$$

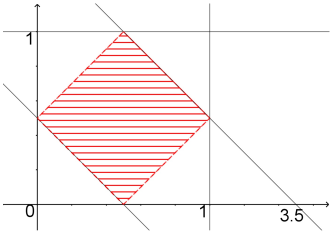
Thus we obtain that the functions P,Q,R satisfy the equation (6.1) where the constants $\alpha \in \mathbb{F}_+ \cup \{0\}$ and $\beta \in \mathbb{F}_+ \cup \{+\infty\}$ are defined by

- $\alpha := \frac{\gamma}{\alpha}$, $\beta := \frac{\delta}{\beta}$ if $\alpha\delta - \beta\gamma > 0$ and $\beta \neq 0$;
- $\alpha := \frac{\gamma}{\alpha}$, $\beta := +\infty$ if $\alpha\delta - \beta\gamma > 0$ and $\beta = 0$;
- $\alpha := \frac{\delta}{\beta}$, $\beta := \frac{\gamma}{\alpha}$ if $\alpha\delta - \beta\gamma < 0$ and $\alpha \neq 0$;
- $\alpha := \frac{\delta}{\beta}$, $\beta := +\infty$ if $\alpha\delta - \beta\gamma < 0$ and $\alpha = 0$.

By Proposition 6.1 we obtain the statement. The converse statement is evident. □

7. Examples and problems

Example 7.1. Let $D := \{(x,y) \in \mathbb{R}^2 \mid |x-0.5| + |y-0.5| < 0.5\}$.



$$D_x = D_y =]0,1[,$$
$$D_{x+y} =]0.5, 1.5[.$$

Define the set D_0 by $D_0 := D_x \cup D_y \cup D_{x+y}$. By Theorem 5.1 we obtain that the general solution of the functional equation is

$$f(x) = a(x) \quad (x \in D_0 =]0, 1.5[)$$

where $a \colon \mathbb{R} \to \mathbb{R}$ is a Cauchy additive function.

Let $D \subseteq \mathbb{R}^2$ be a well-chained open set with components D^1, D^2. By Theorem 5.1 we obtain that the general solution of functional equation (RestPexAdd) is in the form of (1.3).

$$f(z) = \begin{cases} a_1(z) + C_1^1 + C_2^1, & \text{if } z \in D_{x+y}^1; \\ a_2(z) + C_1^2 + C_2^2, & \text{if } z \in D_{x+y}^2; \end{cases}$$

$$g(u) = \begin{cases} a_1(u) + C_1^1, & \text{if } u \in D_x^1; \\ a_2(u) + C_1^2, & \text{if } u \in D_x^2; \end{cases}$$

$$h(v) = \begin{cases} a_1(v) + C_2^1, & \text{if } v \in D_y^1; \\ a_2(v) + C_2^2, & \text{if } v \in D_y^2, \end{cases}$$

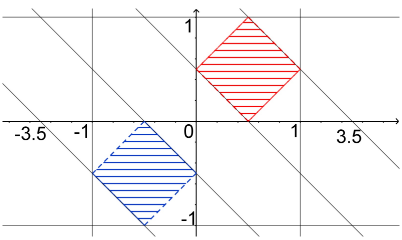
where a_i is an additive function, C_1^i, C_2^i are constants for all $i = 1, 2$.

The following two examples show how the structure of the general solution depends on the geometry of the sets D^1 and D^2.

Example 7.2. Let

$$D^1 := \{(x,y) \in \mathbb{R}^2 \mid |x-0.5| + |y-0.5| < 0.5\},$$
$$D^2 := \{(x,y) \in \mathbb{R}^2 \mid |x+0.5| + |y+0.5| < 0.5\},$$

and let $D := D_1 \cup D_2$.



$$D_x^1 = D_y^1 =]0,1[,$$
$$D_{x+y}^1 =]0.5, 1.5[,$$
$$D_x^2 = D_y^2 =]-1, 0[,$$
$$D_{x+y}^2 =]-1.5, -0.5[.$$

By Theorem 5.2 we have that since $D^1_{x+y} \cap D^1_x \neq \emptyset$ thus $C^1_2 = 0$. Since $D^1_{x+y} \cap D^1_y \neq \emptyset$ thus $C^1_1 = 0$. Since $D^2_{x+y} \cap D^2_x \neq \emptyset$ thus $C^2_2 = 0$. Since $D^2_{x+y} \cap D^2_y \neq \emptyset$ thus $C^2_1 = 0$. Whence we obtain that the general solution of equation (RestAdd) in this case is

$$f(z) = \begin{cases} a_1(z), & \text{if } z \in D^1_{x+y}; \\ a_2(z), & \text{if } z \in D^2_{x+y}; \end{cases}$$

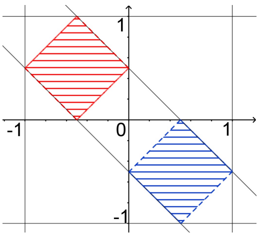
$$g(u) = \begin{cases} a_1(u), & \text{if } u \in D^1_x; \\ a_2(u), & \text{if } u \in D^2_x; \end{cases}$$

$$h(v) = \begin{cases} a_1(v), & \text{if } v \in D^1_y; \\ a_2(v), & \text{if } v \in D^2_y, \end{cases}$$

where a_i is additive function for all $i = 1, 2$.

Example 7.3. Define the sets

$$\begin{aligned} D_1 &:= \{(x, y) \in \mathbb{R}^2 \mid |x + 0.5| + |y - 0.5| < 0.5\}, \\ D_2 &:= \{(x, y) \in \mathbb{R}^2 \mid |x - 0.5| + |y + 0.5| < 0.5\}, \\ D &:= D_1 \cup D_2 \end{aligned}$$



$$\begin{aligned} D^1_{x+y} &= D^2_{x+y} =]-0.5, 0.5[, \\ D^1_x &= D^2_y =]-1, 0[, \\ D^1_y &= D^2_x =]0, 1[. \end{aligned}$$

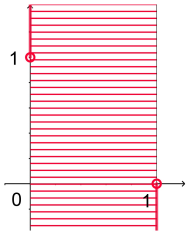
Since $D^1_{x+y} = D^2_{x+y}$ thus $a_1 = a_2$ and $C^1_1 + C^1_2 = C^2_1 + C^2_2$. Since $D^1_x = D^2_y$ thus $C^1_1 = C^2_2$, and $C^1_2 = C^2_1$. Since $D^1_{x+y} \cap D^1_x \neq \emptyset$ thus $C^1_1 + C^2_1 = C^1_1$. Since $D^1_{x+y} \cap D^1_y \neq \emptyset$ thus $C^1_1 + C^1_2 = C^1_2$. Consequently $C^i_1 = C^i_2 = 0$ for all $i = 1, 2$.

Whence we obtain that the general solution of equation (RestAdd) in this case is

$$\begin{aligned} f(z) &= a(z), \text{ if } z \in D^1_{x+y} = D^2_{x+y}; \\ g(u) &= a(u), \text{ if } u \in D^1_x = D^2_x; \\ h(v) &= a(v), \text{ if } v \in D^1_y = D^2_y. \end{aligned}$$

where $a \colon \mathbb{R} \to \mathbb{R}$ is an additive function.

Example 7.4. If $\mathbb{G} := (\mathbb{R}^2, +, \leqslant)$ where the addition is defined by the usual componentwise addition, and the ordering is the usual lexicographic ordering, that is, $(a_1, a_2) \leqslant (b_1, b_2)$ if and only if that either $a_1 < b_1$ or $a_1 = b_1$ and $b_1 \leqslant b_2$. Thus the group $\mathbb{G}(+, \leqslant)$ is an ordered Abelian group, but it is not an Archimedean ordered, because, for example, $(0,1) < (1,0)$, but there is no positive integer n with $n(0,1) > (1,0)$.



This is the open interval $](0,1),(1,0)[$ in $\mathbb{G}$.

Problem A. Preserve the notations of Example 7.4, and let $Y(+)$ be an Abelian group. We want to know the general solution of functional equation (RestAdd) where $D :=](0,1),(1,0)[^2$.

Problem B. Preserve the notations of Example 7.4, and $Y(+)$ be an Abelian group. We also want to know the general solution of functional equation (RestPex-Add) where $D :=](0,1),(1,0)[^2$.

Problem C. In general, we also want to know the general solution of equation (RestAdd), or equation (RestPexAdd) in the case when $\mathbb{G}(+,\leqslant)$ is a nonarchimedean ordered Abelian group, $Y(+)$ be an Abelian group, $D \subseteq \mathbb{G}^2$ is a nonempty, well-chained, open set. The topology on $\mathbb{G}$ (or on $\mathbb{G}^2$) is generated by the open interval of $\mathbb{G}$ (or by the open rectangles of $\mathbb{G}^2$).

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A Hosszú type functional equation with operation $x \circ y = (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c$

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Abstract. The aim of this paper is to give the general solution of the Hosszú type functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = g(x) + g(y) \quad (x, y \in \mathbb{R}_+)$$

with unknown functions $f, g\colon \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\} \to \mathbb{R}$, and binary operation defined by $x \circ y := (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c$ for all $x, y \in \mathbb{R}_+$ where $c \in \mathbb{R}$, $c \in \{2, 3\}$ is a fixed constant.

Keywords: functional equations, additive functions, Hosszú type functional equations, Hosszú functional equation, Hosszú cycle

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1. Introduction

At the International Symposium on Functional Equation conference held on Zakopane (Poland) in 1967 the functional equation

$$f(x + y - xy) + f(xy) = f(x) + f(y) \tag{1.1}$$

where the unknown function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the equation for all $x, y \in \mathbb{R}$ was proposed to investigate in the first time by M. Hosszú. This equation is known as Hosszú functional equation.

Z. Daróczy [6, 8] D. Blanusa [4], and H. Swiatak [25, 26] proved that the general solution of equation (1.1) is in the form $f(x) = A(x) + C$ for all $x \in \mathbb{R}$

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where $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function (that is, $A(x+y) = A(x)+A(y)$ is fulfilled for all $x, y \in \mathbb{R}$ [2, 3, 18]), and C is a real constant.

Since 1969, many researchers have investigated the Hosszú equation and its generalizations. In papers [9–12] the equation (1.1) was investigated on various abstract structures. In paper [13] a Pexider version (that is functional equation with more unknown function [1, 3, 15, 18]) of equation (1.1) and its locally integrable function solutions can be found. See also [20, 21].

The general solution of equations

$$f(xy) + g(x + y - xy) = f(x) + f(y) \tag{1.2}$$
$$f(xy) + g(x + y - xy) = h(x) + h(y) \tag{1.3}$$

was given by K. Lajkó [19] in the following cases:

Problem A. If the unknown functions $f, g\colon]0,1[\to \mathbb{R}$ satisfy the equation (1.2) for all $x, y \in]0, 1[$, then there exist additive functions A_1, A_2, and a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} f(x) &= A_1(x) + A_2(\log x) + C, && (x \in]0,1[),\\ g(x) &= A_1(x) + C, && (x \in]0,1[). \end{aligned}$$

(where log denotes the natural logarithm function).

Problem B. If the unknown functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ satisfy the equation (1.3) for all $x, y \in \mathbb{R}$, then there exist an additive function A and constants C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned} f(x) &= A(x) + C_2 && (x \in \mathbb{R}),\\ g(x) &= A(x) + C_3, && (x \in \mathbb{R}),\\ h(x) &= A(x) + C_1, && (x \in \mathbb{R}), \end{aligned}$$

where $2C_1 = C_2 + C_3$. If the unknown functions $f, h\colon \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \to \mathbb{R}$, and $g\colon \mathbb{R} \to \mathbb{R}$ satisfy the equation (1.3) for all $x, y \in \mathbb{R}_0$, then there exist additive functions A_1, A_2 and constants C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned} f(x) &= A_1(x) + A_2(\log|x|) + C_3 && (x \in \mathbb{R}_0),\\ g(x) &= A_1(x) + C_2, && (x \in \mathbb{R}),\\ h(x) &= A_1(x) + A_2(\log|x|) + C_1, && (x \in \mathbb{R}_0). \end{aligned}$$

If the unknown functions $f\colon \mathbb{R} \to \mathbb{R}$, and $g, h\colon \mathbb{R}_1 := \mathbb{R} \setminus \{1\} \to \mathbb{R}$ satisfy the equation (1.3) for all $x, y \in \mathbb{R}_1$, then there exist additive functions A_1, A_2 and constants C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned} f(x) &= A_1(x) + C_3 && (x \in \mathbb{R}),\\ g(x) &= A_1(x) + A_2(\log|1-x|) + C_2, && (x \in \mathbb{R}_1).\\ h(x) &= A_1(x) + A_2(\log|1-x|) + C_1, && (x \in \mathbb{R}_1). \end{aligned}$$

In his paper [7] Z. Daróczy investigate the functional equation

$$f(x + y - x \circ y) + f(x \circ y) = f(x) + f(y) \quad (x, y \in \mathbb{R}_+) \tag{1.4}$$

with unknown function $f\colon \mathbb{R} \to \mathbb{R}$ where the binary operation $\circ$ is defined by $x \circ y := \ln(e^x + e^y)$ for all $x, y \in \mathbb{R}$. The general solution of this equation is $f(x) = A(x) + C$ for all $x \in \mathbb{R}$ where A is an additive function, C is a real constant.

The main purpose of our present paper is to give the general solution of the functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = g(x) + g(y) \quad (x, y \in \mathbb{R}_+) \tag{1.5}$$

with unknown functions $f, g\colon \mathbb{R}_+ \to \mathbb{R}$. The binary operation is defined by

$$x \circ y := (x^{\frac{1}{c}} + y^{\frac{1}{c}})^c \quad (x, y \in \mathbb{R}_+), \tag{1.6}$$

where $c \in \mathbb{R} \setminus \{0, 1\}$ is a fixed constant.

We also consider the functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = h(x) + h(y) \quad (x, y \in \mathbb{R}_+) \tag{1.7}$$

with unknown functions $f, g, h\colon \mathbb{R}_+ \to \mathbb{R}$ and binary operation $\circ$ is defined by (1.6). The equation (1.5) is a common generalization of equations (1.2) and (1.4). The equation (1.7) is a common generalization of equations (1.3) and (1.4).

The devices needed to the problems we set out, and to the earlier problems are the theorems giving the solutions of the restricted Pexider additive functional equations (in the rest briefly Additive Extension Theorems) and the application of the Hosszú cycle.

Let $D \subseteq \mathbb{R}^2$ be a non-empty connected set. Define the sets

$$\begin{aligned} D_x &:= \{u \in \mathbb{R} \mid \exists v \in \mathbb{R} : (u, v) \in D\}, \\ D_y &:= \{v \in \mathbb{R} \mid \exists u \in \mathbb{R} : (u, v) \in D\}, \\ D_{x+y} &:= \{z \in \mathbb{R} \mid \exists (u, v) \in D : z = u + v\}. \end{aligned}$$

The functional equation

$$f(x + y) = g(x) + h(y) \quad ((x, y) \in D)$$

with unknown functions $f\colon D_{x+y} \to \mathbb{R}$, $g\colon D_x \to \mathbb{R}$, $h\colon D_y \to \mathbb{R}$ is a restricted Pexider additive functional equation. According to Rimán's Extension Theorem [24], there exist an additive function $A\colon \mathbb{R} \to \mathbb{R}$ and constants C_i $(i = 1, 2)$ such that

$$\begin{aligned} f(u) &= A(u) + C_1 + C_2 && (u \in D_{x+y}), \\ g(v) &= A(v) + C_1 && (v \in D_x), \\ h(z) &= A(z) + C_2 && (z \in D_y). \end{aligned}$$

Concerning the Additive Extension Theorems see also [1, 5, 14, 16, 18, 23].

A Hosszú cycle is a functional equation

$$F(x+y,z)+F(x,y)=F(x,y+z)+F(y,z) \quad (x,y,z\in D)$$

with unknown function $F\colon A^2\to D$, where $A=A(+)$ is a semi-group, $D=D(+)$ is an Abelian semi-group. The first appearance of this equation was in [17], although many researchers use the Hosszú Cycle to solve functional equations for example equations (1.1), and (1.2).

2. The decomposition of equation (1.5) by Hosszú cycle

Theorem 2.1. *If the functions $f,g\colon \mathbb{R}_+\to\mathbb{R}$ satisfy the functional equation (1.5) where the binary operation $\circ$ is defined by (1.6), then f satisfies the functional equation*

$$\begin{aligned} f(|(x+y+z)^c-((x+y)^c+z^c)|)+f(|(x+y)^c-((x^c+y^c)|)\\ =f(|(x+y+z)^c-(x^c+(y+z)^c)|)\\ +f(|(y+z)^c-(y^c+z^c)|) \quad (x,y\in\mathbb{R}_+). \end{aligned} \tag{2.1}$$

Proof. For the proof we shall use the well-known Hosszú Cycle (see [7]). Define the function $F\colon\mathbb{R}_+^2\to\mathbb{R}$ by

$$F(x,y):=g(x)+g(y)-g(x\circ y) \quad (x,y\in\mathbb{R}_+).$$

Since the operation $\circ$ is associated thus we have that

$$F(x\circ y,z)+F(x,y)=F(x,y\circ z)+F(y,z) \quad (x,y,z\in\mathbb{R}_+). \tag{2.2}$$

By equations (1.5) we have that

$$F(x,y)=f(|x+y-(x\circ y)|) \quad (x,y\in\mathbb{R}_+). \tag{2.3}$$

From equations (2.2) and (2.3) we have the equation (2.1). □

The following proposition is well known (and will be used later).

Proposition 2.2. *Let $c\in\mathbb{R}\setminus\{0,1\}$ and define the function $\varphi_c\colon\mathbb{R}_+\to\mathbb{R}_+$ by*

$$\varphi_c(x):=x^c \quad (x\in\mathbb{R}_+).$$

a. *If $c>1$, then the function φ_c is strictly superadditive in the sense that*

$$(x+y)^c>x^c+y^c \quad (x,y\in\mathbb{R}_+). \tag{2.4}$$

b. *If* $0 < c < 1$*, or* $c < 0$ *then the function* φ_c *is strictly subadditive in the sense that*

$$x^c + y^c > (x+y)^c \quad (x, y \in \mathbb{R}_+) \tag{2.5}$$

Example 2.3. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ thus by Proposition 2.2 it is easy to see that $e^{x+y} + 1 > e^x + e^y$ for all $x, y \in \mathbb{R}_+$, and $\log(x+1) + \log(y+1) > \log(x+y+1)$ for all $x, y \in \mathbb{R}_+$. It is also easy to see that $\log(x+y) > \log(x) + \log(y)$ for all $x, y \in]0,1[$.

Define the function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ by $\varphi(x) := x + x^2 + x^3$ for all $x \in \mathbb{R}_+$. Then function φ is a strictly superadditive bijection.

Our references concerning subadditive, and superadditive functions is [22].

Corollary 2.4. *Preserving the notations of Theorem 2.1 by Proposition 2.2 it is easy to see that*

- *if* $c > 1$*, then the function* f *satisfies the equation*

$$\begin{aligned} f((x+y+z)^c - ((x+y)^c + z^c)) + f((x+y)^c - ((x^c + y^c)) \\ = f((x+y+z)^c - (x^c + (y+z)^c)) \\ + f((y+z)^c - (y^c + z^c)) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{2.6}$$

- *if* $0 < c < 1$ *or* $c < 0$*, then the function* f *satisfies the equation*

$$\begin{aligned} f((x+y)^c + z^c - (x+y+z)^c) + f((x^c + y^c - (x+y)^c) \\ = f(x^c + (y+z)^c - (x+y+z)^c) \\ + f(y^c + z^c - (y+z)^c) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{2.7}$$

Corollary 2.5. *If the function* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy the functional equation* (2.6) *or equation* (2.7) *for all* $x, y, z \in \mathbb{R}_+$ *thus it is also satisfies the equation*

$$\begin{aligned} f\left(\frac{(x+y+z)^c - ((x+y)^c + z^c)}{(x+y+z)^c - (x^c + (y+z)^c)}\right) + f\left(\frac{(x+y)^c - (x^c + y^c)}{(x+y+z)^c - (x^c + (y+z)^c)}\right) \\ = f(1) + f\left(\frac{(y+z)^c - (y^c + z^c)}{(x+y+z)^c - (x^c + (y+z)^c)}\right) \quad (x, y, z \in \mathbb{R}_+). \end{aligned} \tag{2.8}$$

3. On the equation (2.8) in the cases $c = 2, 3$

Proposition 3.1. *If the function* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfies the equation*

$$f(u) + f(v) = f(1) + f(u + v - 1) \quad ((u, v) \in D) \tag{3.1}$$

where the set $D \subseteq \mathbb{R}^2$ *is defined by*

$$D := \{(u, v) \in \mathbb{R}^2 \mid v > -u + 1, v < 1\}, \tag{3.2}$$

then there exists an additive function $A\colon \mathbb{R} \to \mathbb{R}$ *and a constant* $C \in \mathbb{R}$ *such that.*

$$f(x) = A(x) + C \quad (x \in \mathbb{R}_+).$$

Proof. Define the function $g\colon]1,\infty[\to \mathbb{R}$ by

$$g(z) = f(z-1) + f(1) \quad (z \in]1,\infty[).$$

Then the functions f and g satisfy the equation

$$g(u+v) = f(u) + f(v) \quad ((u,v) \in D)$$

where the set D is defined by (3.2). Thus by Rimán's Extension Theorem (see [14, 24]) we obtain that there exist an additive function $A\colon \mathbb{R} \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ with

$$f(u) = A(u) + C \quad (u \in D_x = \mathbb{R}_+).$$

□

Theorem 3.2. *If the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ satisfy equation (2.8) for all $x,y,z \in \mathbb{R}_+$ with constant $c \in \{2,3\}$, there exist an additive function $A_1\colon \mathbb{R} \to \mathbb{R}$ and a constant $C_1 \in \mathbb{R}$ such that*

$$f(x) = A_1(x) + C_1 \quad (x \in \mathbb{R}_+). \tag{3.3}$$

Proof. **Case 1.** Let $c = 2$. By equation (2.8) the function f satisfies the equation

$$f\left(\frac{(x+y)z}{x(y+z)}\right) + f\left(\frac{y}{y+z}\right) = f(1) + f\left(\frac{yz}{x(y+z)}\right) \quad (x,y,z \in \mathbb{R}_+). \tag{3.4}$$

Take the substitution in equation (3.4)

$$y \longleftarrow \frac{x(u+v-1)}{1-v} \qquad z \longleftarrow \frac{x(u+v-1)}{v}$$

thus we have that the function f satisfies the equation (3.1) where the set $D \subseteq \mathbb{R}^2$ is defined by (3.2). By Proposition 3.1 we have that the function f is in the form of (3.3).

Case 2. Let $c = 3$. By equation (2.8) the function f satisfies the equation

$$\begin{aligned} f\left(\frac{(x+y)z}{x(y+z)}\right) + f\left(\frac{y(x+y)}{(y+z)(x+y+z)}\right) \\ = f(1) + f\left(\frac{yz}{x(x+y+z)}\right) \quad (x,y,z \in \mathbb{R}_+). \end{aligned} \tag{3.5}$$

Take the substitution in equation (3.5)

$$y \longleftarrow \frac{\sqrt{uvx^2(u+v-1)} + x(u+v-1)}{1-v}, \quad z \longleftarrow \frac{\sqrt{uvx^2(u+v-1)}}{v}$$

whence we have that the function f satisfy the equation (3.1) where the set D is defined by (3.2). The rest of the proof of this case is analogous to the case $c = 3$.

□

4. Results and open problems

Theorem 4.1. *If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly superadditive bijection, the binary operation $\circ$ is defined by*

$$x \circ y := \varphi(\varphi^{-1}(x) + \varphi^{-1}(y)) \quad (x, y \in \mathbb{R}_+), \tag{4.1}$$

$f\colon \mathbb{R}_+ \to \mathbb{R}$ is a function such that that exist an additive function $A_1\colon \mathbb{R}_+ \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ with

$$f(x) = A_1(x) + C \quad (x \in \mathbb{R}_+), \tag{4.2}$$

$g, h\colon \mathbb{R}_+ \to \mathbb{R}$ are functions such that

$$f(|x + y - (x \circ y)|) + g(x \circ y) = h(x) + h(y) \quad (x, y \in \mathbb{R}_+), \tag{4.3}$$

then there exist an additive function $A_2\colon \mathbb{R} \to \mathbb{R}$ and constant C_2 with

$$\begin{aligned} g(x) &= -A_1(x) + A_2(\varphi^{-1}(x)) + 2C_2 - C_1 && (x \in \mathbb{R}_+), \\ h(x) &= -A_1(x) + A_2(\varphi^{-1}(x)) + C_2 && (x \in \mathbb{R}_+). \end{aligned} \tag{4.4}$$

Proof. During the proof, $\circ$ will denote the usual function composition (for example $(g \circ \varphi)(x) := g(\varphi(x))$). Since $\varphi(x + y) > \varphi(x) + \varphi(y)$ thus from equation (4.3) we have that

$$\begin{aligned} &f(\varphi(x + y) - (\varphi(x) + \varphi(y))) + g(\varphi(x + y)) \\ &\quad = h(\varphi(x)) + h(\varphi(x)) \quad (x, y \in \mathbb{R}_+). \end{aligned} \tag{4.5}$$

From equations (4.2) and (4.5) we have that

$$\begin{aligned} &((A_1 \circ \varphi) + (g \circ \varphi) + C_1))(x + y) \\ &\quad = ((A_1 \circ \varphi) + (h \circ \varphi))(x) + ((A_1 \circ \varphi) + (h \circ \varphi))(y) \quad (x, y \in \mathbb{R}_+). \end{aligned} \tag{4.6}$$

Define de functions $F, G\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$F(x) : = ((A_1 \circ \varphi) + (g \circ \varphi) + C_1)(x) \quad (x \in \mathbb{R}_+), \tag{4.7}$$

$$G(x) : = ((A_1 \circ \varphi) + (h \circ \varphi))(x) \quad (x \in \mathbb{R}_+). \tag{4.8}$$

From equation (4.6) we have that

$$F(x + y) = G(x) + G(y) \quad (x, y \in \mathbb{R}_+). \tag{4.9}$$

From equation (4.9) by Rimán's Extension Theorem we have that there exist an additive function $A_2\colon \mathbb{R} \to \mathbb{R}$ and a constant $C_2 \in \mathbb{R}$ such that

$$F(x) = A_2(x) + 2C_2 \quad (x \in \mathbb{R}_+), \tag{4.10}$$

$$G(x) = A_2(x) + C_2 \quad (x \in \mathbb{R}_+). \tag{4.11}$$

From equations (4.8), and (4.11) we obtain equation (4.4). $\square$

Theorem 4.2. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly subadditive bijection. Preserving the notation of Theorem 4.1 the functions g, h is in the form*

$$g(x) = A_1(x) + A_2(\varphi^{-1}(x)) + 2C_2 - C_1 \qquad (x \in \mathbb{R}_+), \tag{4.12}$$

$$h(x) = A_1(x) + A_2(\varphi^{-1}(x)) + C_2 \qquad (x \in \mathbb{R}_+). \tag{4.13}$$

Proof. The proof is analogous to the proof of Theorem 4.1. □

Theorem 4.3. *If the functions $f, g\colon \mathbb{R}_+ \to \mathbb{R}$ satisfy the functional equation (1.5) where the operation $\circ$ is defined by (1.6) where $c \in \{2, 3\}$, then there exist an additive functions $A_1, A_2\colon \mathbb{R} \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ such that*

$$f(x) = A_1(x) + C \qquad (x \in \mathbb{R}_+), \tag{4.14}$$

$$g(x) = A_1(x) + A_2(x^{\frac{1}{c}}) + C \qquad (x \in \mathbb{R}_+). \tag{4.15}$$

Proof. The proof is evident by Theorem 2.1, Corollary 2.4, Corollary 2.5, Theorem 3.2, and Theorem 4.1. □

Theorem 4.1, Theorem 4.2, and Theorem 4.3 suggest the following open problems.

Problem A. Does Theorem 4.3 hold for all $c \in \mathbb{R}_+ \setminus \{0, 1\}$?

Our conjecture is that it remains true.

Problem B. Find the general solution of equation (1.7) with unknown functions $f, g, h\colon \mathbb{R}_+ \to \mathbb{R}$ and binary operation $\circ$ is defined by (1.6) where $c \in \mathbb{R}_+ \setminus \{0, 1\}$.

Problem C. In particular, the authors of the present article would like to know the solution of Problem B in the case of $c = -1$, in more details, find the general solution of functional equation

$$f\left(x + y - \frac{xy}{x+y}\right) + g\left(\frac{xy}{x+y}\right) = h(x) + h(y) \quad (x, y \in \mathbb{R}_+)$$

with unknown functions $f, g, h\colon \mathbb{R}_+ \to \mathbb{R}$.

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A static analysis approach for modern iterator development

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Abstract. Programming languages evolve in the long term, new standards are specified in which new constructs appear, old elements may become deprecated. Standard library of programming languages also changes time by time.

The standard of the C++ programming language defines the elements of the C++ Standard Template Library (STL) that provides containers, algorithms, and iterators. According to the STL's generic programming approach, these sets can be extended in a convenient way. The `std::iterator` class template had been in the C++ since beginning and has been deprecated in the C++17 standard. This class template's purpose was to specify the traits of an iterator. Typically, it was a base class of many standard and non-standard iterator class to provide the necessary traits. However, the usage of iterator is straightforward and fits into the object-oriented programming paradigm. Many non-standard containers offer custom iterators because of the STL compatibility. Using this base class does not cause any weird effect, therefore usage of iterator can be found in code legacy.

In this paper, we present a static analysis approach to assist the development of iterator classes in a modern way in which the iterator class template is not taken advantage of. We utilize the Clang compiler infrastructure to look for how the deprecated iterator classes can be found in legacy code and present an approach how to modernize them.

Keywords: C++, static analysis, iterator, Clang

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1. Introduction

Every programming language evolves regularly. New standards of programming languages published, new compiler techniques and constructs become available. For instance, many different Fortran standards have been developed in the last sixty years and many constructs have evolved during the years [7]. In 2022, Oracle announced the nineteenth release of the Java standard [15].

This language evolution takes part in the history of the C++ programming language, as well. C++98, C++03, C++11, C++14, C++17, and C++20 are the official standards. C++23 is already done, but it is not offial yet. Therefore, no compiler supports the entire C++23 standard recently. New standards may affect the core language and its standard library.

In general, language standard updates introduce new language constructs and may deprecate older constructs [3]. For instance, C++11 made the template class `std::auto_ptr` deprecated and provides new standard smart pointers instead [2]. Later, `std::auto_ptr` has been removed from the C++ standard library.

New language constructs and standard libraries can require migration in code legacies with a method called *source code rejuvenation* that is not considered code refactoring [13].

The `std::iterator` class template had been in the C++ since beginning and has been deprecated in the C++17 standard [10]. This class template's purpose was to specify the traits of an iterator [11]. Typically, it was a base class of many standard and non-standard iterator class to provide the necessary traits [12]. However, the usage of `iterator` is straightforward and fits into the object-oriented programming paradigm. Many non-standard containers offer custom iterators because of the Standard Template Library compatibility [1]. Using this base class does not cause any weird effect, therefore usage of `iterator` can be found in code legacy.

In this paper, we present a static analysis approach to assist the development of iterator classes in a modern way in which the iterator class template is not taken advantage of. We utilize the Clang compiler infrastructure to look for how the deprecated iterator classes can be found in legacy code and present an approach how to rejuvenate them. Clang's checker approach is proper to detect and emit warning based on static analysis [8].

This paper is organized as follows. In Section 2, we give an overview about the C++ Standard Template Library (STL) and iterators. We detail our approach in Section 3 and we present its evaluation in Section 4. Section 5 provides possible ways of the future work. Finally, this paper concludes in Section 6.

2. Iterators

C++ Standard Template Library is an examplar library based on the generic programming paradigm [1]. STL provides containers (e.g. `std::vector`, `std::map`) and container-independent algorithms (e.g. `std::max_element`, `std::sort`) [14].

These components are separated and they can be extended simultaneously in a non-intrusive way. Iterators bridge the gap between containers and algorithms that are abstraction of pointers [9].

In the C++ language, it is possible to access or manage the memory directly from the source code. One of the tools provided by the language which can be used for such purposes are the pointers – these are a special kind of variables. They store an integer value which represents the memory address to which the variable is pointing to, therefore the information stored in that memory block can be retrieved or modified by using the pointer for it. In contrast to reference variables, the pointers can store a different value than the one which they were initialized with: the memory addresses they point to can be shifted in both directions (forward or backward) based on the allocated size of the type of the value which they hold the pointer for – this is done by using pointer arithmetics.

However, pointers are compound types, so they do not store much additional information or metadata about themselves, nor have we the ability to customize them – how dereferencing the variable, or shifting it should happen exactly. To solve these issues we could use the concept of iterators. An iterator is an object which can be used to maintain an element of a given range, using a set of operators. A special form of the iterators are the pointers, however, sometimes we do not need to have all the capabilities of a pointer implemented in our custom iterator: depending on the use-case, it might be enough to have an iterator which is only capable of stepping forward, or can only be written to but it does not have the ability to be read. To achieve this, iterators can be sorted into one of the five main iterator categories: *input*, *output*, *forward*, *bidirectional* or *random access iterators*.

On top of the customized methods iterators can – and in a lot of cases have to – define additional information about themselves. This information is available in the form of iterator traits: there should be five iterator traits defined in total. The `difference_type` should express the result of subtracting one iterator from another, `value_type` stores information about the type of the value which the iterator points to, `pointer` is the type of a pointer which can point to the value maintained by the iterator, so is `reference` but instead of pointers the reference type is described, `iterator_category` shows us into which one of the iterator categories does the iterator belong to. The metadata defined by the iterator traits will be used by several STL algorithms to provide the most optimal behavior, or to be able to check whether the instance of the iterator type provided to them is implementing all the operators or methods they require, so the iterator object has all the capabilities they need [14].

2.1. Defining custom iterators – legacy way

To check the capabilities of an iterator object, the `std::iterator_traits` class of the STL can be used – this wrapper class is needed when both pointers and iterator objects can be accepted. Based on the template parameter it receives, it can generate a proper definition through which all the needed information can be accessed e.g. by an algorithm, and accepts both pointers and iterator objects as a template parameter. Before C++17, to declare all the information needed to this specific wrapper class needed by the majority of the standard algorithms, the STL provided us a helper class named `std::iterator` [9]. When creating our own custom iterator class, by deriving from the `std::iterator` it was possible to define all the needed iterator traits by passing them to the parent iterator class as template arguments:

```
struct DummyIteratorDepr :
    std::iterator<std::forward_iterator_tag,  // iterator_category
                  int,                        // value_type
                  int,                        // difference_type
                  int*,                       // pointer
                  int &                       // reference
                  >
{
public:
    DummyIteratorDepr(pointer ptr) {n = ptr;}
    DummyIteratorDepr& operator++() {return *this;}
    DummyIteratorDepr operator++(int) {return *this;}
    reference operator*() {return *n;}
    pointer operator->() {return n;}
    bool operator==(const DummyIteratorDepr &rhs) {return true;}
    bool operator!=(const DummyIteratorDepr &rhs) {return true;}

private:
    int *n;
};
```

Another advantage of inheriting the `std::iterator` to have compatibility with the STL containers and algorithms is that – by making use of the default template arguments the parent class has – we do not even have to define all the attributes if they do not have to be specific ones, or we are sure they will not be needed at all: `difference_type`, `pointer` and `reference` all have default values, which can be deduced from the values we provided to the mandatory `iterator_category` and `value_type` fields.

Since this tool provided by the standard library seems to be extremely useful, it would be understandable to ask why did it become deprecated in C++17? It is worth to mention, that the concept of iterator traits for providing compatibility did not become deprecated, only the `std::iterator` class, and the main reason for that is its ambiguity. Consider the following example taken from the standard [5]:

```
template <class T,
          class charT = char,
          class traits = char_traits<charT> >
class ostream_iterator:
  public iterator<output_iterator_tag, void, void, void, void>;
```

In this example, it is hard to understand which `void` stands for which attribute. Declaring an iterator like this could be very confusing and hard to read. Another reason for deprecating the class is that if the custom iterator itself depends on a template argument which is then passed to the `std::iterator` class, finding the traits during name lookup could fail:

```
template <typename T>
struct MyIterator : std::iterator<std::random_access_iterator_tag,
                                  T>
{
  value_type data;  // Error: value_type is not found by name lookup
};
```

The result would be the same if we put `void` instead of `T` to the parent iterator class as second template argument.

2.2. Defining custom iterators – modern way

As the usage of `std::iterator` would be deprecated now, all the attributes which are needed to describe our custom iterator class have to be declared explicitly by using type aliases:

```
class DummyIterator
{
public:
    using iterator_category = std::forward_iterator_tag;
    using value_type = int;
    using difference_type = int;
    using pointer = int*;
    using reference = int&;

    DummyIterator (int* ptr) {n = ptr;}
    DummyIterator& operator++() {return *this;}
    DummyIterator operator++(int n) {return *this;}
    reference operator*() {return *n;}
    int* operator->() {return n;}
    bool operator==(const DummyIterator &rhs) {return true;}
    bool operator!=(const DummyIterator &rhs) {return true;}

private:
    int *n;
};
```

Note that for declaring type aliases both `typedef` and `using` keywords can be used, in this specific case they would be semantically equivalent, since we do not make the aliases depend on template parameters. Despite their semantic equivalence the syntax would differ a bit, consider the following example:

```
using value_type_u = int;
typedef int value_type_t;
```

The latter one is the method of creating type aliases in the legacy way, but since C++11 the first one is preferred, as being a more powerful tool compared to the second one. To ensure this, clang-tidy already implements several checkers which would warn in case of using the old approach instead of the modern one [6].

We can see that by using the modern iterator definition method, it is much more readable and much easier to understand what properties a given custom iterator has. This way the original concerns regarding the usage of the `std::iterator` class have been overcome, however, we have to face a new problem, which in some situations could be uncomfortable: we lost the ability the make use of the default template arguments – since now we do not have any. We have to declare every trait properly for our custom iterator to be compatible with the standard library, even if some of them (the ones which had default values earlier) would be trivial. In the following, we will try to create a tool, or to be more specific a set of tools to help the transition between the old and new way of defining a custom iterator, and to help to avoid the potential incompatibilities between the STL and our custom iterators created by using the modern approach.

3. Verifying custom iterators with static code analysis

We will define two major problem categories that can be divided into smaller problems, then we will address these smaller problems with static analysis tools provided by the Clang compiler infrastructure. The two major problems would be the handling of legacy custom iterators, and detecting potential custom iterators defined without using class `std::iterator`. We will then decompose the latter by ranking potential iterator findings based on how likely it is, that the class we found is meant to be used as an iterator, which has to be compatible with the standard library. To achieve this, we implemented a new clang-tidy checker named *modernize-replace-std-iterator*, as part of the *modernize* checker category. The exact behavior and logic behind the checker is described below.

3.1. Transforming legacy custom iterators

Since the usage of `std::iterator` has become deprecated, it is better to avoid using it when developing custom iterators. To help this, we developed a static analysis tool based on Clang. Clang supports developing new tools, thus the built abstract syntax tree (AST) can be utilized, queried and visited. We have implemented an AST matcher to find and warn for every class definition which derives from the class `std::iterator`. An example for these kind of classes could be `DummyIteratorDepr`. However, we have to consider the cases when the custom iterator class is derived not directly, but indirectly from the deprecated iterator.

This means one of the parents of our iterator in the inheritance chain would have `std::iterator` as a parent class (or at least as one of the parent classes, since in C++ it is allowed to have multiple inheritance, which means that it is possible to inherit from multiple base classes to create a common derived type) [14]. To handle these issues, it is not enough to simply check whether the class in question has the `std::iterator` as a base class, but we should also check if one of its parents at any level has it as a parent.

```
struct DummyIteratorDeprDesc : DummyIteratorDepr
{
    DummyIteratorDeprDesc(pointer ptr) : DummyIteratorDepr(ptr){}
};
```

To avoid unnecessary and redundant findings, the warning will only be triggered if our class has inherited the legacy iterator in a direct way. Since all class definitions will be checked, we will cover all the possible results, all the nodes of all inheritance chains. To modernize our iterators it is needed to update that one exact base class, which we get the warning for, since all the newly declared type aliases will be inherited by all the (directly or indirectly) deriving child classes (if we make sure that the access specifier of the traits is at least *protected* but considering that the purpose of them is to provide information about the iterator to the outside world, we should declare them as public aliases).

We have now clarified that our approach would be the detection of direct inheritances, which has an additional advantage on top of avoiding duplicate matches and redundant steps. The scope of the analysis will be the translation unit which we are currently analysing - narrowing this down to our problem we get, that the scope of the analysis would be the class definitions described in the given translation unit. However, we will have cases when the removal of the `std::iterator` class would be a valid step without modifying any of the iterator definitions we have in our translation unit, despite the fact that they had the standard iterator class as an indirect base class. This is the situation when we have a "custom" iterator class in the middle of our inheritance chain, but outside the unit which we are analysing right now. In this case, two explanations are possible: one of them is that we will take care of the custom class when analysing the translation unit introducing it – the other one is that the custom iterator is defined outside of our project. If we face the latter, we have to trust the project defining the iterator will solve the issues caused by the deprecation of the standard iterator.

To provide more information to the developer, our tool not only warns about the class definitions mentioned above, but also gives hints about how to update them. After analysing a proper iterator class, we will get the following warning:

```
test_iterator.cc:12:8: warning: Derived from std::iterator,
which is deprecated since C++17. From C++17 type aliases
should be declared:
 using iterator_category = std::forward_iterator_tag;
```

```
 using value_type = int;
 using difference_type = int;
 using pointer = int*;
 using reference = int&; [modernize-replace-std-iterator]
struct DummyIteratorDepr : std::iterator<std::forward_iterator_tag,
                                          ...>
```

This is done by querying the concrete type parameters of the template instantiation we defined when inheriting the base class. It is worth to mention, that we would get the same list of arguments if we relied only on the mandatory template parameters:

```
struct DummyIteratorDepr :
    std::iterator<std::forward_iterator_tag,  // iterator_category
                  int>                        // value_type
{
  // ...
}
```

We now have all the information which is needed to automatize the transformation, which would result in the class definition having the iterator traits declared. Currently according to the scope and the goal of our tool, only a warning would be triggered, but as a future improvement it would be easy to implement the transformation itself by using the fix-it hints of the clang-tidy tool.

3.2. Detecting custom iterators

3.2.1. Analysing potential iterators

As we have described earlier, detecting usages of the deprecated custom iterator defining method is only one part of our goal. Another part would be to detect all the existing iterators, or classes which seem to be iterators which can face compatibility issues when used with the Standard Template Library. Also, we try to keep in mind the motivation behind the deprecation of `std::iterator` – readability is an aspect which should be considered when analysing the code.

First, we try to focus on the classes which – apart from some extreme cases – can convince the analyser that they are iterators, and they are used as if they were one. To achieve this, we will define two key criteria: the custom iterator should define at least one of the mandatory iterator traits (or should derive from a class which defines one of them), and an instance of this custom class can be used as an argument for algorithms defined by the Standard Template Library. The library defines a wide range of methods operating on a given range of elements, for multiple purposes. These algorithms can be found in the *algorithm* header, and since they are analysing/modifying ranges, or a range of elements, the range itself should be determined when trying to execute them.

This is done by passing iterators as arguments to them, which iterators can define the range by marking the start and the end of the range we want to use. When we create custom iterator classes, a typical usage would be to combine them with the powerful tools provided by the STL algorithms. Assuming that we have a custom iterator class named `I` declaring all the needed iterator traits properly, and the type of the value to which its instances are pointing to is `int`. In this case, we could find e.g. the value 2 in the following way (`i_begin` and `i_end` are instances of our iterator class, defining the range which we would like to analyse):

```
std::find(i_begin, i_end, 2)
```

In this example, if 2 is part of the range, the iterator pointing to the first occurrence will be returned, otherwise the result will be *i_end*, which points after the last element. Sticking to this example, this specific function will require all the iterator traits to be declared, otherwise compiling the code would result in an error. Consider the following example:

```
class CustomIterator {
public:
    using iterator_category = std::forward_iterator_tag;
    using value_type = int;
    using difference_type = int;
    ...
};
```

We have only three attributes defined, *pointer* and *reference* are missing. Because of this, a compilation error should happen, and we would get an error message similar to this:

```
error: no matching function for call to '__iterator_category'
...
substitution failure [with _Iter = CustomIterator]: no type
named 'iterator_category' in 'std::iterator_traits<CustomIterator>'
```

What interesting here is, that even if we had the trait `iterator_category` defined because of the template substitution failure of class `std::iterator_traits`, compiling a code like this will result in an error which can be misleading. Of course, if we would have used the legacy way for defining `CustomIterator`, the problem would not be present since the missing parameters could be determined by using the default template argument values of `std::iterator`:

```
struct CustomIteratorDepr :
    std::iterator<std::forward_iterator_tag,
                                       int,
                                       int
                                       >
```

```
{
  // ...
};
```

Now we have seen that despite its advantages, the modern approach prevents us to make use of the default template arguments of the legacy class. Using the legacy method, we do not have the possibility to skip any of the non-mandatory parameters, the order of the template arguments matters and should be considered to avoid failures during compilation, but it was a bit easier to define iterators which did not require all the iterator attributes being defined by their own.

However, not all algorithms make use of the iterator traits or require them to be present in the class defining the iterator they got as an argument. Let us take `std::swap_ranges` as an example. The function exchanges the elements of two ranges, and requires three parameters to achieve this: the first two parameters define the first range, the third one points to the beginning of the second range. Let us define our custom iterator in the following way:

```
class DummyIterator
{
public:
    DummyIterator (int* ptr) {n = ptr;}
    DummyIterator& operator++() {return *this;}
    DummyIterator operator++(int n) {return *this;}
    int& operator*() {return *n;}
    int* operator->() {return n;}
    bool operator==(const DummyIterator &rhs) {return true;}
    bool operator!=(const DummyIterator &rhs) {return true;}

private:
    int *n;
};
```

As we can see, we defined only the operators required by the `std::swap_ranges`, but none of the iterator traits. Based on the previous examples we have seen, using this iterator with for example `std::find` would lead to compilation error. This is not the case with this function:

```
std::swap_ranges(d1_begin, d1_end, d2_begin);
```

This example compiles just fine, if `d1_begin`, `d1_end` and `d2_begin` are all instances of `DummyIterator` we defined above. In case the method does not require the substitution of the template arguments defined by `std::iterator_traits`, it is possible to be compatible with the function by having only a number of traits defined (or defining none of them). At this point, we can divide the problem of being compatible with STL algorithms into two subcategories: in one case, the function call will compile just fine, in the other case a compilation error will happen. We have implemented our clang-tidy checker to address both problems at the same time.

Matching the AST Our concept here would be to help to avoid compatibility issues with the standard library while keeping the code as much readable as possible. The base concept of our AST matcher is to find all nodes, which belong to the class declaration defining the objects which are used as parameters when calling functions from the `std` namespace. Note that we mentioned earlier that we are interested in the calls of functions defined in the *algorithm* header. This approach would be much more strict than analysing all the function calls using methods from the `std` namespace, however, members of the *algorithm* library are also part of it. The reasoning behind it is that we would like to help the developer to avoid incompatibilities in the future, for use cases which might not be relevant right now. If a custom iterator is used with the Standard Template Library, it has the potential to be used later together with a method, with which it would have compatibility issues resulting in unexpected errors, mainly during compilation time.

In case of larger code bases and rather complex projects, it is not unusual to have a lot of legacy code in it. Due to this, we have to handle the cases which would be covered by using the matcher of our checker tool described previously (detecting legacy `std::iterator` usage). These are the cases when the custom iterator inherits its attributes from the `std::iterator` class – therefore we exclude these matches, since they are not part of the scope of the current analysis.

To determine if the arguments are meant to be used as iterators, we are looking for the explicitly declared type aliases representing the iterator traits, or to be more specific, we are looking for one of the mandatory ones: `iterator_category`. Since the legacy cases had been excluded, all our custom iterators should define the two mandatory attributes, which are not derived from a custom iterator class coming from a third party library. However, based on this logic the false-positive findings should be considered too: what happens, if a class declares the type alias `value_type`, and an instance of it is used as an argument of a standard function, but it is not an iterator?

```
struct A
{
    using value_type = int;
};
...
std::vector<A> v;
A a;
v.insert(v.begin(), a);
```

This example meets all of our conditions, so `A` could be considered as an iterator. To avoid this, we only look for the attribute `iterator_category`, which is less likely to be defined in a class representing a different concept then the iterators. Another thing which we have to deal with is the case, when the class definition does not contain the traits, but a base class of it does. In this case, the custom iterator has a more abstract iterator class from which it inherits the common attributes, which applies for this specific subtype too.

```
struct Iter_A
{
    using iterator_category = std::forward_iterator_tag;
    using value_type = int;
};

struct Iter_B : Iter_A
{
    using difference_type = int;
    using pointer = int*;
    using reference = int&;
};
```

In this example, the parent (`Iter_A`) only defines the mandatory attributes, the rest of them are present only in the derived class. To handle the problem of abstract iterators, the matcher would follow the following logic: if the definition of the parameter type contains the mandatory attribute, then this AST node should be matched, if not, the matcher will look for the node which implements it. If we have multiple matches, because the traits have been redefined multiple times in the inheritance chain, we will look for the first one, which declares the attribute (the top one). The motivation behind this is to find the first class definition which can potentially act as a standalone iterator itself. After we have found all the nodes which should be considered regarding a function call, the checker will analyse the class definitions and determine which iterator traits are missing. When we say "missing", it means that we are interested in what are the traits which are not declared in this exact class definition. The traits can be present without having to declare them: this is the case if the class inherits these attributes from a base class. After we have collected the missing type aliases, a warning will be triggered for the user to see what should be declared on top of the existing aliases. Using the class we declared earlier (`CustomIterator`) with `std::swap_ranges`, we will get the following warning:

```
test_iterator.cc:36:7: warning: Type seems to be an iterator used
by std::swap_ranges. The following type aliases should be
declared additionally within the class:

pointer
reference
```

In this case, the `pointer` and `reference` attributes were missing. By analysing the class definition further, it could be determined, or at least suggested how the iterator traits should be declared, but for now triggering a warning like the one above is a limitation of our tool. Now let us see, what it means regarding the separate problem categories we defined:

Matching the base or the derived class If the base class is missing some of these type aliases, it could be useful to add them, since once we have done it, all the newly defined deriving classes would inherit all the iterator traits, which means they could be handled as an iterator by themselves. If we have matched the deriving class, declaring the traits can be redundant: if the base class (or in case of multiple inheritance, or a longer inheritance chain one of the base classes) also defines these attributes, the type alias in the deriving class will hide all the previous declarations, avoiding name collisions. However, by doing so the class definition could be much more readable since we would have all the aliases declared in one place, and it could be understood easily without investigating the parent-child relationships further. Of course, if none of the attributes are missing either in the base, or in the derived class, no warning will be triggered.

Matching function calls which would not compile We have mentioned earlier that in a number of cases it is mandatory to have all the traits defined properly, since they are needed by `std::iterator_traits`. It is possible for a custom iterator class to possesses all the values required via inheritance: in this case, triggering the warning could be relevant to have all the aliases declared in one place (see the case of base-derived classes). Our warnings will have one more advantage in case of calls which could not compile at all: it can give a hint, which traits are missing. As we could see earlier, it is possible that the error message triggered by the compiler only tells us that the substitution of the template arguments failed. In this case, our warning would highlight which attributes seem to be missing. However, it will not consider the attributes inherited, but a warning like this could be a motivation to define the class in a more comprehensible, readable way – otherwise these warnings would count as false positives, but since readability is one of the main aspect we follow, these warnings could be relevant also.

3.2.2. Analysing possible iterators

The last problem category which we wanted to cover consists of custom iterator candidates, which have the possibility − based on our conditions − to be treated as iterators. We can not be as confident as we were in case of the previous cases, regarding the false-positive results, since our conditions are much less strict for this category. The goal here also would be to provide readability and compatibility with the standard library, but the scope of our matchers will be much wider. We do not limit our findings to classes defining type aliases which could be interpreted as iterator traits, or to classes whose instances are being used as parameters for functions of the `std` namespace.

Our matching logic here will be similar to what is known as *duck typing* [4]. We will find and mark classes which have similar structure to an STL compatible iterator. The similarity in this case will not be defined by the members, types, etc.

defined by the class, but the member methods and operators overloaded by it. As we have seen in our previous examples, an iterator has to overload a given set of operators to be treated as a valid one. The set is determined by the type of the iterator - this is the same type which is stored as the value of `iterator_category`. As we have described earlier, all iterators must belong to one of the five iterator categories, which defines all the capabilities expected from the iterator.

Table 1. Iterator categories and their operations.

Output	`*p=`, `++`
Input	`*p`, `++`, `->`, `==`, `!=`
Forward	`*p=`, *(+Input iterator)*
Bidirectional	`-`, *(+Forward iterator)*
Random Access	`[]`, `+`, `-`, `+=`, `-=`, `<`, `>`, `<=`, `>=` *(+Bidirectional iterator)*

In Table 1, we can see all the required methods for each iterator categories. Based on that, if we see a class which implements all the operations needed for a forward iterator for example, we can mark it as a potential forward iterator. After we have done that, we can generate a warning that this class could be a potential iterator, and we can give a hint what values could be used for the iterator traits (in this example, `iterator_category` would get the value `forward_iterator_tag`). Similarly to the previous case, if all the traits have been defined by the class or one of its parents, the warning will not be issued. The reason why we match for the operations defined instead of the type aliases declared is, that – as we have shown earlier – in several cases the iterator traits are not needed at all by the function which takes the pointer as an argument, but this is not the case with the operator overloads. Missing a mandatory operator would result in a compilation error, hence relying on them would be useful. Also, the operators to overload (for example the unary `*`) are specific enough to match for them:

```
struct IteratorCandidateA
{
    IteratorCandidateA& operator++() { ... }
    IteratorCandidateA operator++(int n) { ... }
    int& operator*() { ..}
    int* operator->() { ... }
    bool operator==(const IteratorCandidateA &rhs) { ... }
    bool operator!=(const IteratorCandidateA &rhs) { ... }
};
```

The example shows us a candidate for the category *forward iterator*. If we find a candidate which defines all the operations needed by a category, which is a superset of another one (regarding the operators overloaded), then we will warn for it using the iterator category which would provide the most features.

4. Evaluation

To verify our three different approaches we have defined so far, we have executed the checker with all the different AST matchers in it on the LLVM project. LLVM is an open source compiler infrastructure, including Clang and also clang-tidy, the tool which we implemented our checkers in. Because of its nature, LLVM implements a number of custom iterator classes, making it a good candidate for our analysis. After analysing the results, filtering out all the duplicates caused by the same findings in header files included in multiple translation units, we had proper matches for all three categories, without any false-positive results.

In total, we have found examples for deprecated `std::iterator` usage in case of 52 class definitions, 1 custom iterator class whose instance had been used as an argument of a function defined in the Standard Template Library (and whose definition does not contain all the possibly required iterator traits). For the last category, we have identified a total of 12 iterator-like classes – class definitions which seem to be iterators based on the operator overloads they implemented.

After analysing the results further, we came to the conclusion that all the findings are valid, there are no false-positive matches among them. Based on these facts, it is proven that our tool can be used as a tool for modernizing the source code, and for updating the code base in a way, that future incompatibilities with the standard library can be avoided.

5. Future work

We have shown that our checker can be a useful tool when modernizing the source code, however, the matcher logics could be further refined, to find more accurate results. A refinement like this would be in case of the third problem category to not only match for the operator overloads (by name), but to consider the parameters and return types of these overloads also. However, in case of asterisk (`*`) operators the `void` return values are checked even now, to avoid false-positive findings for output iterators.

We have mentioned earlier, that in a number of cases rejuvenation of the code could be done automatically – we have all the information available which is needed to insert all the type aliases which are required to define the iterator traits: we can extract the proper types from the template arguments of `std::iterator` when deriving from it (in case of the first problem category), or we could define the missing attributes by analysing the return values and parameter types of the overloaded operators in case of the second and third problem categories.

In our example run, we have not faced any false-positive matches. However, earlier we have shown that finding faulty results might be possible. One improvement to avoid these findings would be to filter only for the members of the *algorithm* part of the STL, instead of matching for calls of functions defined in the `std` namespace. Also, cases when class definitions are hidden to the static analyser by macro definitions should be considered too.

6. Conclusion

Languages and their standard libraries evolve over time. For instance, C++17 provides much more constructs than C++98. On the other hand, some of the language elements become deprecated sometimes, C++98's `auto_ptr` is a typical obsolete component of the C++ Standard Template Library.

C++17 standard made `iterator` base class deprecated. However, its usage was common and safe, there was some reasons to make this class template obsolete. This class template is widely used when one develops a new iterator to specify the traits.

We implemented a static analysis method to emit warning if the usage of `iterator` base class can be found. Moreover, we presented an approach how custom iterators can be found. Our approaches provide hint how to improve the source code. Our method gives feedback if any trait is missing from iterator-like class. We have implemented a tool for the approaches based on the Clang compiler infrastructure. We evaluated our solution with real-world software artifacts, the result is promising.

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Computation of the Wedderburn decomposition of semisimple group algebras of groups up to order 120

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Abstract. In this paper, we discuss the Wedderburn decompositions of the semisimple group algebras of all groups up to order 120. More precisely, we explicitly compute the Wedderburn decompositions of the semisimple group algebras of 26 non-metabelian groups.

Keywords: Unit group, Finite field, Wedderburn decomposition

AMS Subject Classification: 16U60, 20C05

1. Introduction

Let G be a finite group and $\mathbb{F}_p$ be a finite field for a prime p having characteristics p. Let p be such that $p \nmid |G|$. This means that the group algebra $\mathbb{F}_pG$ is semisimple (see [13]). Due to various applications of units of group algebras (for example, in cryptography [6, 14], in coding theory [7], in isomorphism problems and exploration of Lie properties of group algebras [2] etc.), the problem of computing the Wedderburn decompositions (or unit groups) of finite semisimple group algebras is an extensively studied problem (see [1, 3, 5, 9, 11, 12, 15, 19, 21] and the references therein).

One of the major steps in the direction of computation of Wedderburn decompositions (WDs) of finite semisimple group algebras was taken in [1]. The paper [1] gave an algorithm to compute the WDs of the semisimple group algebras of all

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metabelian groups. We recall that a finite group G is metabelian if its derived subgroup is abelian. Consequently, the entire research in this direction is shifted on to the computation of WDs of semisimple group algebras of non-metabelian groups. Mittal et al. [17] computed the WDs of semisimple group algebras of all non-metabelian groups up to order 72. Furthermore, Mittal et al. [16, 18, 22] also computed the WDs of all semisimple group algebras of all non-metabelian groups of order 108 and some non-metabelian groups of order 120. Since, the WDs of semisimple group algebras of the symmetric groups S_n can be easily computed by employing the representation theory (see [8]), the papers [18, 22] completed the task of computation of WDs of group algebras of non-metabelian groups of order 120.

Using [20] we note that the only non-metabelian groups of order less than 120 that are not yet studied in the literature are those of order 96. Hence, the main objective of this paper is to complete the task of computation of WDs of group algebras of 26 non-metabelian groups of order 96. Consequently, with this paper, the computation of the WDs of semisimple group algebras of all groups up to order 120 will be complete. From the WD, the unit group can be computed straightforwardly.

Organization of the paper. Section 2 contains certain preliminaries that play an important role in the computation of WDs. Our main results related to WDs of semisimple group algebras are discussed in Section 3. We give the complete details of computation of WDs only for a few groups among the 26 groups. This is because for the remaining groups, the details can be generated analogously. We conclude the paper in the last section.

2. Preliminaries

Let the exponent of the group G be denoted by e and let the primitive e^{th} root of unity be denoted by ε. In our work, we use the notations of [4]. Let $\mathbb{F}$ denote a finite field. Let us define

$$I_{\mathbb{F}} = \{\omega \mid \varepsilon \mapsto \varepsilon^{\omega} \text{ is an automorphism of } \mathbb{F}(\varepsilon) \text{ over } \mathbb{F}\}.$$

It can be noted that the Galois group $\mathrm{Gal}\big(\mathbb{F}(\varepsilon),\mathbb{F}\big)$ is a cyclic group. This guarantees the existence of an $s \in \mathbb{Z}_e^*$ fulfilling $\lambda(\varepsilon) = \varepsilon^s$ for any $\lambda \in \mathrm{Gal}\big(\mathbb{F}(\varepsilon),\mathbb{F}\big)$. More specifically, $I_{\mathbb{F}}$ is a subgroup of the group $\mathbb{Z}_e^*$ (multiplicative). Let g be a p-regular element of the group G. Let us define

$$\gamma_g = \sum_{h \in C(g)} h,$$

where $C(g)$ denotes the set of all those elements of G that are conjugate to the p-regular element g. For γ_g, let the cyclotomic $\mathbb{F}$-class of be represented by

$$S(\gamma_g) = \{\gamma_{g^{\omega}} \mid \omega \in I_{\mathbb{F}}\}.$$

Let $J(\mathbb{F}G)$ represent the Jacobson radical of the group algebra $\mathbb{F}G$.

Next, we discuss two important results of [4].

Theorem 2.1. *The number of cyclotomic $\mathbb{F}$-classes in G is equal to the number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$.*

Theorem 2.2. *Let the number of cyclotomic $\mathbb{F}$-classes in G be π and let ε be primitive e^{th} root of unity, where e is the exponent of G. Let $S_1, \dots, S_\pi$ be the simple components of the center of $\mathbb{F}G/J(\mathbb{F}G)$ and let $Y_1, \dots, Y_\pi$ be the cyclotomic $\mathbb{F}$-classes in G. Then, $|Y_i| = [S_i : \mathbb{F}]$ for each $1 \le i \le \pi$, after suitable ordering of the indices.*

We remark that both the Theorems 2.1 and 2.2 will be very crucial for our main results. Next, we discuss a significant result that shows that in the WD of a finite group algebra $\mathbb{F}G/J(\mathbb{F}G)$, $\mathbb{F}$ is always a Wedderburn component (see [17]).

Lemma 2.3. *Let Σ_1 and Σ_2 be two algebras over $\mathbb{F}$ having finite dimension. Let Σ_2 be semisimple and let $\varphi : \Sigma_1 \to \Sigma_2$ be a homomorphism that is also surjective. Then, there holds*

$$\Sigma_1/J(\Sigma_1) \cong \Sigma_2 + \Sigma_3,$$

where Σ_3 is an another semisimple $\mathbb{F}$-algebra.

Suppose that $J(\mathbb{F}G) = 0$. Then Lemma 2.3 confirms that $\mathbb{F}$ is always a simple component of $\mathbb{F}G$. Next, we recall a result from [10] that explicitly characterizes the set $I_{\mathbb{F}}$.

Theorem 2.4. *Let $q = p^r$ for a positive integer r and a prime p and let $\mathbb{F}_q$ be a finite field. Let e be such that $\gcd(e, q) = 1$ and let ε be the primitive e^{th} root of unity. Let $o(q)$ be the order of q modulo e. Then we have*

$$I_{\mathbb{F}_q} = \{1, q, \dots, q^{o(q)-1}\} \mod e.$$

Further, we recall two important theorems from [13].

Theorem 2.5. *Let R be a commutative ring and let RG be a semisimple group algebra. Then we have*

$$RG \cong R\big(G/G'\big) \oplus \Delta(G, G').$$

Here G' is the derived subgroup of G, $R\big(G/G'\big)$ is the sum of all commutative simple components and $\Delta(G, G')$ is the sum of all non-commutative simple components of RG.

Theorem 2.6. *Let RG be a semisimple group algebra and H be a normal subgroup of G. Then*

$$RG \cong R\big(G/H\big) \oplus \Delta(G, H).$$

Here $\Delta(G, H)$ represents the left ideal of RG and it is generated by the set $\{h - 1 : h \in H\}$.

We remark that through Theorem 2.5 one can obtain all the possible commutative simple components of the group algebra $\mathbb{F}_qG$. Further, Theorem 2.6 relates WD of the group algebra $\mathbb{F}_q(G/H)$ with that of $\mathbb{F}_qG$ for a normal subgroup H of G. Finally, we end this section by invoking an important result from [3]. This result will be very crucial in unique computation of the WD for any semisimple group algebra.

Theorem 2.7. *Let $\mathbb{F}$ be a finite field of characteristics p. Let $\Sigma = \oplus_{s=1}^{t} M_{n_s}(\mathbb{F}_s)$ be a summand of a semisimple group algebra $\mathbb{F}G$, where $\mathbb{F}_s$ denotes a finite extension of $\mathbb{F}$ for each s. Then $p \nmid n_s$ for every $1 \leq s \leq t$.*

3. WDs of non-metabelian groups of order 96

In this section, we discuss all the non-metabelian groups of order 96 along with their WDs. Up to isomorphism, we note that there are 231 groups of order 96 and 26 of them are non-metabelian. Among these 26 groups, 11 have exponent 24 and rest all have exponent 12.

3.1. Non-metabelian groups of order 96 having exponent 24

The non-metabelian groups of order 96 having exponent 24 are as follows:

1. $G_1 = A_4 \rtimes C_8$
2. $G_2 = SL(2,3) \rtimes C_4$
3. $G_3 = SL(2,3) \rtimes C_4$
4. $G_4 = C_2 \times (SL(2,3) \cdot C_2)$
5. $G_5 = C_2 \times GL(2,3)$
6. $G_6 = (C_2 \times SL(2,3)) \rtimes C_2$
7. $G_7 = (SL(2,3) \cdot C_2) \rtimes C_2$
8. $G_8 = (((C_4 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$
9. $G_9 = (((C_4 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$
10. $G_{10} = ((C_8 \times C_2) \rtimes C_2) \rtimes C_3$
11. $G_{11} = ((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$.

3.2. Wedderburn decomposition of $\mathbb{F}_qG_1$ and some other group algebras

The presentation of $G_1 = A_4 \rtimes C_8$ is as follows:

$$\langle x, y, z, w, t, u \mid x^2y^{-1},\ [y,x],\ [z,x],\ [w,x]w^{-1},\ [t,x]u^{-1}t^{-1},\ [u,x]u^{-1}t^{-1},\ y^2z^{-1},\ [z,y], [w,y],\ [t,y],\ [u,y],\ z^2,\ [w,z],\ [t,z],\ [u,z],\ w^3,\ [t,w]u^{-1}t^{-1},\ [u,w]t^{-1},\ t^2,\ [u,t],\ u^2 \rangle.$$

This group has 20 conjugacy classes as shown in the next table.

R	e	x	y	z	w	t	xy	xz	xt	yz	yw	yt	zw	zt	xyz
S	1	6	1	1	8	3	6	6	6	1	8	3	8	3	6
O	1	8	4	2	3	2	8	8	8	4	12	4	6	2	8

xyt	xzt	yzw	yzt	$xyzt$
6	6	8	3	6
8	8	12	4	8

where R, S and O represent representative, size and order of conjugacy classes, respectively. From the above discussion, we conclude that the exponent of G_1 is 24. Also $G_1' \cong A_4$ with $G_1/G_1' \cong C_8$. Since $p > 3$, we have $\gcd(|G_1|, p) = 1$, and so $J(\mathbb{F}_q G_1) = 0$.

Theorem 3.1. *The Wedderburn decomposition of $\mathbb{F}_q G_1$ for $q = p^k$, $p > 3$ is as follows:*

values of p and k	*Wedderburn decomposition*
k *even or* $p \equiv \{1, 17\} \mod 24$ *and* k *odd*	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$
$p^k \equiv \{5, 13\} \mod 24$ *and* k *odd*	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_3(\mathbb{F}_{q^2})^2$
$p^k \equiv \{7, 23\} \mod 24$ *and* k *odd or* $p^k \equiv \{11, 19\} \mod 24$ *and* k *odd*	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus$ $M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})^3$

Proof. As $\mathbb{F}_q G_1$ is semisimple, we have $\mathbb{F}_q G_1 \cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_r)$, $t \in \mathbb{Z}$, where for each r, $\mathbb{F}_r$ is a finite extension of $\mathbb{F}_q$, $n_r \geq 1$. Incorporating Lemma 2.3 in above to obtain

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.1}$$

For k even and any prime $p > 3$, $p^k \equiv 1 \mod 24$. This means $|S(\gamma_g)| = 1$ for each $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$ (see Theorem 2.4). Hence, (3.1) and Theorems 2.1, 2.2 imply that $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{19} M_{n_r}(\mathbb{F}_r)$. This with $G_1/G_1' \cong C_8$ and Theorem 2.5 leads to (with suitable rearrangement of indexes) $\mathbb{F}_q G_1 \cong \mathbb{F}_q^8 \oplus_{r=1}^{12} M_{n_r}(\mathbb{F}_r)$ with $88 = \sum_{r=1}^{12} n_r^2$, $n_r \geq 2$, which gives the only possible choice $(2^4, 3^8)$ (here a^b means $(a, a, \ldots, b$ times)) for values of n_r's. Therefore, the required WD is

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8. \tag{3.2}$$

Now, we assume that k is odd. We discuss this possibility in the following 4 cases:

Case 1. $p \equiv 1 \mod 24$ or $p^k \equiv 17 \mod 24$. In this case, we have $|S(\gamma_g)| = 1$ for each $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$ or $I_{\mathbb{F}} = \{1, 17\}$. Hence, WD is given by (3.2).

Case 2. $p^k \equiv 5 \mod 24$ or $p^k \equiv 13 \mod 24$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyz}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xzt}\}$, $S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xyzt}\}$, and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1 and 2.2 and (3.1), we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus_{r=12}^{15} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_1/G_1' \cong C_8$ and $\mathbb{F}_q C_8 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2$ to obtain

$$\begin{aligned} &\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{10} M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with } \; 88 = \sum_{r=1}^{8} n_r^2 + 2\sum_{r=9}^{10} n_r^2, \; n_r \geq 2, \end{aligned} \tag{3.3}$$

which gives 3 possibilities for values of n_r' namely $(3^8, 2^2), (2^2, 3^6, 2, 3)$ and $(2^4, 3^6)$. For uniqueness, consider a normal subgroup $H_1 = \langle t, u \rangle$ of G_1 with $K_1 = G_1/H_1 \cong C_3 \rtimes C_8$. It can be verified that K_1 has 12 conjugacy classes as shown in the table below.

R	e	x	y	z	w	xy	xz	yz	yw	zw	xyz	yzw
S	1	3	1	1	2	3	3	1	2	2	3	2
O	1	8	4	2	3	8	8	4	12	6	8	12

Also $K_1' \cong C_3$ with $K_1/K_1' \cong C_8$. For the representatives k of K_1, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyz}\}$, $S(\gamma_k) = \{\gamma_k\}$ for the remaining representatives. Therefore, employ Theorems 2.1, 2.2 and 2.5 to obtain $\mathbb{F}_q K_1 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus_{r=1}^4 M_{t_r}(\mathbb{F}_q)$ with $16 = \sum_{r=1}^4 t_r^2$. This gives us the only possibility (2^4) for value of t_r's. Next, incorporate Theorem 2.6 in (3.3) to deduce that $(2^4, 3^6)$ is the correct choice for n_r's and therefore, we have $\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_{q^2})^2$.

Case 3. $p^k \equiv 7 \mod 24$ or $p^k \equiv 23 \mod 24$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xyz}\}$, $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xz}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyzt}\}$, $S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xzt}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yzw}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yzt}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.1), we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^5 M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{12} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_1/G_1' \cong C_8$ and $\mathbb{F}_q C_8 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3$ in this to obtain

$$\begin{aligned} &\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus_{r=1}^4 M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^8 M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with } \ 88 = \sum_{r=1}^4 n_r^2 + 2\sum_{r=5}^8 n_r^2, \ n_r \geq 2 \end{aligned} \tag{3.4}$$

which gives three possibilities for values of n_r's namely $(3^4, 2^2, 3^2), (2^2, 3^2, 2, 3^3)$ and $(2^4, 3^4)$. Further, we can verify that for the representatives k of K_1, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xyz}\}$, $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xz}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yzw}\}$ and $S(\gamma_k) = \{\gamma_k\}$ for the remaining representatives. This with Theorems 2.1, 2.2 and 2.5 leads to $\mathbb{F}_q K_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus_{t=1}^2 M_{t_r}(\mathbb{F}_q) \oplus M_{t_3}(\mathbb{F}_{q^2})$, $t_r \geq 2$, $t_r \in \mathbb{Z}$ with $16 = \sum_{r=1}^2 t_r^2 + 2t_3^2$, which gives the only choice (2^3) for t_r's. Therefore, (3.4) and Theorem 2.6 imply that $(2^2, 3^2, 2, 3^3)$ is the correct choice for n_r's. So, we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^3 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})^3$.

Case 4. $p^k \equiv 11 \mod 24$ or $p^k \equiv 19 \mod 24$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$, $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xyz}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\}$, $S(\gamma_{xzt}) = \{\gamma_{xzt}, \gamma_{xyzt}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yzw}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yzt}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g. Using Theorems 2.1 and 2.2 and (3.1), we get $\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^5 M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{12} M_{n_r}(\mathbb{F}_{q^2})$. Further, we can easily see that rest part of this case is similar to Case 3. □

Next, we remark that for the groups G_i, where $2 \leq i \leq 8$ and $i = 10$, the Wedderburn decomposition of their group algebras can be computed by following the steps of Theorem 3.1 (see Tables 1–8). Hence, we are omitting their proofs from the paper.

Table 1. Wedderburn decomposition of $\mathbb{F}_qG_2$.

values of p and k	Wedderburn decomposition
k even or $p \equiv \{1,17\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{5,13\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2$
$p^k \equiv \{7,23\} \mod 24$ and k odd or $p^k \equiv \{11,19\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})$

Table 2. Wedderburn decomposition of $\mathbb{F}_qG_3$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1,5,13,17\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{7,11,19,23\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_4(\mathbb{F}_{q^2})$

Table 3. Wedderburn decomposition of $\mathbb{F}_qG_4$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1,7,17,23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{5,11,13,19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2$

Table 4. Wedderburn decomposition of $\mathbb{F}_qG_5$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1,11,17,19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \equiv \{5,7,13,23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2$

Table 5. Wedderburn decomposition of $\mathbb{F}_q G_6$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 7, 13, 19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \equiv \{5, 7, 13, 23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_4(\mathbb{F}_{q^2})$

Table 6. Wedderburn decomposition of $\mathbb{F}_q G_7$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 11, 13, 23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \equiv \{5, 7, 13, 23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_4(\mathbb{F}_{q^2})$

Table 7. Wedderburn decomposition of $\mathbb{F}_q G_8$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 11, 13, 23\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \equiv \{5, 7, 17, 19\} \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_4(\mathbb{F}_{q^2})$

Table 8. Wedderburn decomposition of $\mathbb{F}_q G_{10}$.

values of p and k	Wedderburn decomposition
k even or $p \equiv 1 \mod 24$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \in \{7, 19\} \mod 24$ and k odd	$\mathbb{F}_q^6 \oplus \mathbb{F}_{q^2}^3 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 13 \mod 24$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 17 \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^4$
$p^k \in \{11, 23\} \mod 24$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^5 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 5 \mod 24$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$

3.3. Wedderburn decomposition of $\mathbb{F}_q G_{11}$

It is to be noted that for the group algebra $\mathbb{F}_q G_{11}$, WD can not be uniquely characterize only by using Theorems 2.5 and 2.6. We also need Theorem 2.7 for its unique characterization. Consequently, we separately discuss the WD of $\mathbb{F}_qG_{11}$ in the following theorem. We have $G_{11} = ((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$. This group has 10 conjugacy classes.

R	e	x	y	z	t	xz	xt	zw	zt	xzt
S	1	12	32	3	3	12	12	3	6	12
O	1	2	3	4	2	8	4	4	4	8

Clearly, the exponent of G_{11} is 24 and $G'_{11} \cong (C_4 \times C_4) \rtimes C_3$ with $G_{11}/G'_{11} \cong C_2$.

Theorem 3.2. *The Wedderburn decomposition of $\mathbb{F}_qG_{11}$ for $q = p^k$, $p > 3$ is*

values of p and k	*Wedderburn decomposition*
k even or $p \in \{1, 5, 13, 17\}$ mod 24 and k odd	$\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^6 \oplus M_6(\mathbb{F}_q)$
$p^k \equiv \{7, 11, 19, 23\}$ mod 24 *and k odd*	$\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$ $\oplus M_3(\mathbb{F}_{q^2})^2$

Proof. For k even and any prime $p > 3$, $p^k \equiv 1 \mod 24$. This means $|S(\gamma_g)| = 1$ for each $g \in G_{11}$ and hence, (3.1) and Theorems 2.1, 2.2 imply that $\mathbb{F}_qG_{11} \cong \mathbb{F}_q \oplus_{r=1}^{9} M_{n_r}(\mathbb{F}_r)$. This with $G_{11}/G'_{11} \cong C_2$ and Theorem 2.5 leads to $\mathbb{F}_qG_{11} \cong \mathbb{F}_q^2 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_r)$ with $94 = \sum_{r=1}^{8} n_r^2$, $n_r \geq 2$ which gives four possible choices for n_r's as $(2^5, 3, 4, 7), (2^3, 3, 4^3, 5), (2^2, 3^4, 5^2)$ and $(2, 3^6, 6)$. In order to seek uniqueness, consider a normal subgroup $H_{11,1} = \langle t, u \rangle$ of G_{11} with $K_{11,1} = G_{11}/H_{11,1} \cong S_4$. From [9] and Theorem 2.6, we conclude that $(2^2, 3^4, 5^2)$ and $(2, 3^6, 6)$ are the only required possibility for n_r's. Further, using Theorem 2.7, we derive that the required choice for n_r's is $(2, 3^6, 6)$. Therefore, we have the result. Next, we assume that k is odd. We discuss this possibility in the following 2 cases:

Case 1. $p^k \equiv \{1, 5, 13, 17\} \mod 24$. In this case, WD is same as in the case of k even as $|S(\gamma_g)| = 1$ for each representative g of conjugacy classes.

Case 2. $p^k \equiv \{7, 11, 19, 23\} \mod 24$. In this case, we have $S(\gamma_z) = \{\gamma_z, \gamma_{zw}\}$, $S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xzt}\}$, $S(\gamma_g) = \{\gamma_g\}$, for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.1), we reach to $\mathbb{F}_qG_{11} \cong \mathbb{F}_q \oplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^2})$. Now incorporate Theorem 2.4 to obtain $\mathbb{F}_qG_{11} \cong \mathbb{F}_q^2 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2})$ with $94 = \sum_{r=1}^{4} n_r^2 + 2\sum_{r=5}^{6} n_r^2$, $n_r \geq 2$. Further, again consider the normal subgroup $H_{11,1}$. This with Theorem 2.6 yields $\mathbb{F}_qG_{11} \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_{n_1}(\mathbb{F}_q) \oplus_{r=2}^{3} M_{n_r}(\mathbb{F}_{q^2})$, $72 = n_1^2 + 2\sum_{r=2}^{3} n_r^2$, $n_r \geq 2$. The possible choices for n_r's satisfying this are $(2, 3, 5)$ and $(6, 3, 3)$. By the same logic given for the case when k is even, we derive that $(6, 3, 3)$ is the required choice. $\square$

3.4. Wedderburn decomposition of $\mathbb{F}_qG_9$

Next, we discuss the WD of $\mathbb{F}_qG_9$ (see Table 9). We mention that, unfortunately for this particular group, our theory is not enough to uniquely characterize the WD of its corresponding group algebra when $p^k \in \{5, 13\}$ mod 24. We obtain that WD is one of the following two possibilities: $\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^2})^2$; $\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2}) \oplus M_4(\mathbb{F}_{q^2})$. Consequently, we make use of computer package GAP in this case for uniquely determine WD.

Table 9. Wedderburn decomposition of $\mathbb{F}_q G_9$.

values of p and k	Wedderburn decomposition
k even or $p \in \{1, 17\}$ mod 24 and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^2$
$p^k \in \{5, 13\}$ mod 24 and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_{q^2})$ $\oplus M_2(\mathbb{F}_{q^2})^2$
$p^k \in \{7, 11, 19, 23\}$ mod 24 and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^2$ $\oplus M_4(\mathbb{F}_{q^2})$

This completes the computation of WDs of semisimple group algebras of non-metabelian groups of order 96 having exponent 24. Next, we proceed to compute the WDs of semisimple group algebras of non-metabelian groups of order 96 having exponent 12.

3.5. Non-metabelian groups of order 96 having exponent 12

1. $G_{12} = ((C_4 \times C_2) \rtimes C_4) \rtimes C_3$
2. $G_{13} = A_4 \rtimes Q_8$
3. $G_{14} = C_4 \times S_4$
4. $G_{15} = (C_4 \times A_4) \rtimes C_2$
5. $G_{16} = (C_2 \times C_2 \times A_4) \rtimes C_2$
6. $G_{17} = (C_2 \times C_2 \times Q_8) \rtimes C_3$
7. $G_{18} = ((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3$
8. $G_{19} = ((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3$
9. $G_{20} = ((C_2 \times Q_8) \rtimes C_2) \rtimes C_3$
10. $G_{21} = C_2 \times (A_4 \rtimes C_4)$
11. $G_{22} = C_2 \times C_2 \times S_4$
12. $G_{23} = ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$
13. $G_{24} = C_4 \times SL(2, 3)$
14. $G_{25} = C_2 \times C_2 \times SL(2, 3)$
15. $G_{26} = C_2 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_3)$.

3.6. Wedderburn decomposition of $\mathbb{F}_q G_{12}$ and some other group algebras

The presentation of $G_{12} = ((C_4 \times C_2) \rtimes C_4) \rtimes C_3$ is

$$\langle\, x, y, z, w, t, u \mid x^3,\ [y, x]t^{-1}w^{-1}z^{-1}y^{-1},\ [z, x]u^{-1}t^{-1}y^{-1},\ [w, x]u^{-1}t^{-1}w^{-1},$$
$$[t, x]u^{-1}w^{-1},\ [u, x],\ y^2t^{-1}w^{-1},\ [z, y]u^{-1},\ [w, y],\ [t, y],\ [u, y],\ z^2w^{-1},$$

$[w,z],\ [t,z],\ [u,z],\ w^2,\ [t,w],\ [u,w],\ t^2,\ [u,t],\ u^2\,\rangle.$

This group has 12 conjugacy classes as shown in the table below.

R	e	x	y	w	t	u	x^2	xw	yz	yw	yt	x^2y
S	1	16	6	3	3	1	16	16	6	6	6	16
O	1	3	4	2	2	2	3	6	4	4	4	6

From above discussion, we see that exponent of G_{12} is 12. Also $G'_{12} \cong (C_4 \times C_2) \rtimes C_4$ with $G_{12}/G'_{12} \cong C_3$. Since $p > 3$, we have $\gcd(|G_{12}|, p) = 1$, and so $J(\mathbb{F}_q G_{12}) = 0$. We are now ready to give the WD and unit group of $\mathbb{F}_q G_{12}$ for $p > 3$.

Theorem 3.3. *The WD with unit group of $\mathbb{F}_q G_{12}$ for $q = p^k,\ p > 3$ is as follows:*

values of p and k	*Wedderburn decomposition*
k even or $p \equiv 1 \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_6(\mathbb{F}_q)$
$p^k \equiv 5 \mod 12$ and k odd	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^5$ $\oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$
$p^k \equiv 7 \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)$ $\oplus M_6(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2$
$p^k \equiv 11 \mod 12$ and k odd	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)$ $\oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})^2$

Proof. As $\mathbb{F}_q G_{12}$ is semisimple, we have $\mathbb{F}_q G_{12} \cong \oplus_{r=1}^{t} M_{n_r}(\mathbb{F}_r),\ t \in \mathbb{Z}$, where for each r, $\mathbb{F}_r$ is a finite extension of $\mathbb{F}_q$, $n_r \geq 1$. As in Theorem 3.1, we can write

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.5}$$

For k even and any prime $p > 3$, $p^k \equiv 1 \mod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_{12}$ as $I_{\mathbb{F}} = \{1\}$. Hence, (3.5) and Theorems 2.1 and 2.2 imply that $\mathbb{F}_q G_{12} \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_r)$. This with $G_{12}/G'_{12} \cong C_3$ and Theorem 2.5 leads to (with suitable rearrangement of indexes)

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus_{r=1}^{9} M_{n_r}(\mathbb{F}_r) \ \text{ with } \ 93 = \sum_{r=1}^{9} n_r^2,\ n_r \geq 2 \tag{3.6}$$

which gives four possible choices for n_r's namely $(2,2,2,2,2,2,2,4,7)$, $(2,2,2,2,2,4,4,4,5)$, $(2,2,2,2,3,3,3,5,5)$, and $(2,2,2,3,3,3,3,3,6)$. We consider the normal subgroup $H_1 = \langle wu, t\rangle \cong C_2 \times C_2$ with $G_{12}/H_1 \cong SL(2,3)$. From [17], we know that WD of $\mathbb{F}_q G_{12}/H_1$ contains $M_2(\mathbb{F}_q)$ as well as $M_3(\mathbb{F}_q)$. So, Theorem 2.6 implies that the choices $(2,2,2,2,2,2,2,4,\ 7)$ and $(2,2,2,2,2,4,4,4,5)$ are no longer in race. For uniqueness, we consider another normal subgroup $H_2 = \langle u\rangle$ with $K_2 = G_{12}/H_2 \cong (C_4 \times C_4) \rtimes C_3$. Using [1], we note that $\mathbb{F}_q K_2 \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q)^5$. This with Theorem 2.6 imply that $(2,2,2,3,3,3,3,3,6)$ is the only possibility for n_r's. Therefore, we have

$$\mathbb{F}_q G_{12} \cong \mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_6(\mathbb{F}_q). \tag{3.7}$$

Next, we assume that k is odd. We discuss this possibility in the following 4 cases:

Case 1. $p \equiv 1 \mod 12$. In this case, we have $|S(\gamma_g)| = 1$ for each $g \in G_{12}$ as $I_{\mathbb{F}} = \{1\}$. Hence, Wedderburn decomposition is given by (3.7).

Case 2. $p^k \equiv 5 \mod 12$. In this case, we have $S(\gamma_x) = \{\gamma_x, \gamma_{x^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{x^2y}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.5), we get $\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_{12}/G_{12}' \cong C_3$ and $\mathbb{F}_qC_3 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}$ to obtain that

$$\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus M_{n_8}(\mathbb{F}_{q^2}) \text{ with } 93 = \sum_{r=1}^{7} n_r^2 + 2n_8^2,\ n_r \geq 2. \tag{3.8}$$

Further, we note using [1] that $\mathbb{F}_qK_2 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5$. Therefore, (3.8) and Theorem 2.6 imply that $\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5 \oplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})$ with $48 = \sum_{r=1}^{2} n_r^2 + 2n_3^2$, $n_r \geq 2$. This gives the only possibility $(2, 6, 2)$ for n_r's which means the required WD is

$$\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5 \oplus M_2(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}).$$

Case 3. $p^k \equiv 7 \mod 12$. In this case, we have $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yt}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Using Theorems 2.1, 2.2 and (3.5), we get $\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $G_{12}/G_{12}' \cong C_3$ and $\mathbb{F}_qC_3 \cong \mathbb{F}_q^3$ in above to obtain

$$\begin{aligned} &\mathbb{F}_qG_{12} \cong \mathbb{F}_q^3 \oplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with } 93 = \sum_{r=1}^{5} n_r^2 + 2\sum_{r=6}^{7} n_r^2,\ n_r \geq 2. \end{aligned} \tag{3.9}$$

Further, we observe that $\mathbb{F}_qK_2 \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q) \oplus M_{t_r}(\mathbb{F}_{q^2})^2$. Therefore, (3.9) and Theorem 2.6 imply that $\mathbb{F}_qG_{12} \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q) \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2$ with $48 = \sum_{r=1}^{4} n_r^2$, $n_r \geq 2$. This gives the only possibility $(2, 2, 2, 6)$ for n_r's which means that the WD is

$$\mathbb{F}_qG_{12} \cong \mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q) \oplus M_2(\mathbb{F}_q)^3 \oplus M_6(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2.$$

Case 4. $p^k \equiv 11 \mod 12$. In this case, we can verify that $S(\gamma_y) = \{\gamma_y, \gamma_{yz}\}$, $S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yt}\}$, $S(\gamma_x) = \{\gamma_x, \gamma_{x^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{x^2y}\}$, and $S(\gamma_g) = \{\gamma_g\}$ for the representatives e, w, t and u. Using Theorems 2.1, 2.2 and (3.5), we get $\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \oplus_{r=4}^{7} M_{n_r}(\mathbb{F}_{q^2})$. Applying Theorem 2.5 with $\mathbb{F}_qC_3 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}$ in above to obtain

$$\begin{aligned} &\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \oplus_{r=4}^{6} M_{n_r}(\mathbb{F}_{q^2}) \\ &\text{with } 93 = \sum_{r=1}^{3} n_r^2 + 2\sum_{r=4}^{6} n_r^2,\ n_r \geq 2. \end{aligned} \tag{3.10}$$

Further, we see that $\mathbb{F}_qK_2 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q) \oplus M_{t_r}(\mathbb{F}_{q^2})^2$. Therefore, (3.10) and Theorem 2.6 imply that $\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2 \oplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus M_{n_3}(\mathbb{F}_{q^2})$ with $48 = \sum_{r=1}^{2} n_r^2 + 2n_3^2$, which means the only possibility for n_r's is $(2, 6, 2)$. Thus, the required WD is

$$\mathbb{F}_qG_{12} \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2})^2 \oplus M_2(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}). \qquad \square$$

Next, we remark that for the groups G_i, where $13 \leq i \leq 26$, the WD of their group algebras can be computed by following the steps of Theorem 3.2 and Theorem 3.3 (see Tables 10–23). Hence, we are omitting their proofs from the paper.

Table 10. Wedderburn decomposition of $\mathbb{F}_qG_{13}$.

values of p and k	Wedderburn decomposition
k even or $p^k \equiv \pm 1 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$
$p^k \equiv \pm 5 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$ $\oplus M_2(\mathbb{F}_{q^2})$

Table 11. Wedderburn decomposition of $\mathbb{F}_qG_{14}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 5\} \mod 12$ and k odd	$\mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$
$p^k \in \{7, 11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})$

Table 12. Wedderburn decomposition of $\mathbb{F}_qG_{15}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 5\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$
$p^k \in \{7, 11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$ $\oplus M_2(\mathbb{F}_{q^2})$

Table 13. Wedderburn decomposition of $\mathbb{F}_qG_{16}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 7\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$
$p^k \in \{5, 11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)$ $\oplus M_2(\mathbb{F}_{q^2})$

Table 14. Wedderburn decomposition of $\mathbb{F}_q G_{17}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_6(\mathbb{F}_q)$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^5$ $\oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$

Table 15. Wedderburn decomposition of $\mathbb{F}_q G_{18}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^3 \oplus M_3(\mathbb{F}_q)^5 \oplus M_4(\mathbb{F}_q)^3$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus M_3(\mathbb{F}_q)^5 \oplus M_4(\mathbb{F}_q)$ $\oplus M_4(\mathbb{F}_{q^2})$

Table 16. Wedderburn decomposition of $\mathbb{F}_q G_{19}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)$ $\oplus M_4(\mathbb{F}_{q^2})$

Table 17. Wedderburn decomposition of $\mathbb{F}_q G_{20}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)^3$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q)$ $\oplus M_4(\mathbb{F}_{q^2})$

Table 18. Wedderburn decomposition of $\mathbb{F}_q G_{21}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1,7\} \mod 12$ and k odd	$\mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$
$p^k \in \{5,11\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_3(\mathbb{F}_{q^2})^2$

Table 19. Wedderburn decomposition of $\mathbb{F}_q G_{22}$.

values of p and k	Wedderburn decomposition
for any k and p	$\mathbb{F}_q^8 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^8$

Table 20. Wedderburn decomposition of $\mathbb{F}_q G_{23}$.

values of p and k	Wedderburn decomposition
for any k and p	$\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^6 \oplus M_6(\mathbb{F}_q)$

Table 21. Wedderburn decomposition of $\mathbb{F}_q G_{24}$.

values of p and k	Wedderburn decomposition
k even or $p \equiv 1 \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \equiv 5 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2})^4$
$p^k \equiv 7 \mod 12$ and k odd	$\mathbb{F}_q^6 \oplus \mathbb{F}_{q^2}^3 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^2$ $\oplus M_2(\mathbb{F}_{q^2})^3 \oplus M_3(\mathbb{F}_{q^2})$
$p^k \equiv 11 \mod 12$ and k odd	$\mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^5 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2$ $\oplus M_2(\mathbb{F}_{q^2})^5 \oplus M_3(\mathbb{F}_{q^2})$

Table 22. Wedderburn decomposition of $\mathbb{F}_q G_{25}$.

values of p and k	Wedderburn decomposition
k even or $p^k \in \{1, 11\} \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \in \{5, 7\} \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2})^4$

Table 23. Wedderburn decomposition of $\mathbb{F}_q G_{26}$.

values of p and k	Wedderburn decomposition
k even or $p \equiv 1 \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_2(\mathbb{F}_q)^{12} \oplus M_3(\mathbb{F}_q)^4$
$p^k \equiv 5 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^4$ $\oplus M_2(\mathbb{F}_{q^2})^4$
$p^k \equiv 7 \mod 12$ and k odd	$\mathbb{F}_q^{12} \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$
$p^k \equiv 11 \mod 12$ and k odd	$\mathbb{F}_q^4 \oplus \mathbb{F}_{q^2}^4 \oplus M_3(\mathbb{F}_q)^4 \oplus M_2(\mathbb{F}_{q^2})^6$

4. Conclusion

We have computed the WDs of semisimple group algebras of non-metabelian groups of order 96. Hence, this study completes the computation of WDs of semisimple group algebras of all groups up to order 120. In future, this paper motivates the study of unit groups of the group algebras of non-metabelian groups having order greater than 120.

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Conjugation of overpartitions and some applications of over q-binomial coefficients

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Abstract. We study the conjugation of overpartitions and give the generating function for the number of self-conjugate overpartitions of an integer. Following the recent introduction of over q-binomial coefficients, we obtain the over q-analogue of the Chu-Vandermonde identity. Consequently a new generating function for the number of overpartitions is proved. We also give a new over q-analogue of the Chu-Vandermonde identity.

Keywords: partition, overpartition, Durfee Square

AMS Subject Classification: 05A17, 11P84

1. Introduction

A partition of a positive integer n is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We call the summands λ_i *parts*. For example, there are 7 partitions of 5:

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

An overpartition of n is an integer partition of n in which the last occurrence of a part may be overlined [3]. The number of overpartitions of n is denoted by $\overline{p}(n)$. For example, $\overline{p}(4) = 14$, where the overpartitions of 4 are

$$(4),\ (\overline{4}),\ (3,1),\ (\overline{3},1),\ (3,\overline{1}),\ (\overline{3},\overline{1}),\ (2,2),\ (2,\overline{2}),\ (2,1,1),$$
$$(\overline{2},1,1),\ (2,1,\overline{1}),\ (\overline{2},1,\overline{1}),\ (1,1,1,1),\ (1,1,1,\overline{1}).$$

It is well known that

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{1.1}$$

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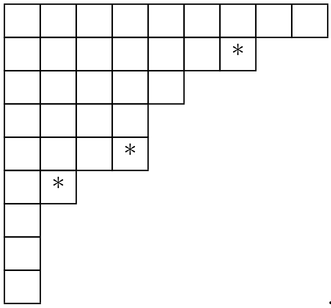
where $(A;q)_0 = 1$ and

$$(A;q)_n := (1-A)(1-Aq)\cdots(1-Aq^{n-1}) = \prod_{j=0}^{n-1}(1-Aq^j),$$

$$(A;q)_\infty := \lim_{n\to\infty}(A;q)_n = \prod_{j=0}^{\infty}(1-Aq^j).$$

The *Young diagram* (or Ferrers board) of an overpartition is the same as that of the underlying ordinary partition with the exception that the last block of an overlined part is marked. For example, the Young diagram of $\lambda = (9, \overline{7}, 5, 4, \overline{4}, \overline{2}, 1, 1, 1)$ is



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The *Durfee square* of an overpartition is the largest square that can fit into its Ferrers board.

We emphasize that a Durfee square of length s consists of $s \cdot s = s^2$ unit squares in the Young diagram of an overpartition such that s is maximal. So it's generating function is

$$q^{1+\cdots+1} \equiv q^{s\cdot s} = q^{s^2}. \tag{1.2}$$

Analogously one may consider *Durfee rectangles* of side lengths s and t, when necessary, and apply the generating function q^{st}.

The *conjugate* overpartition of λ is denoted by λ' and is obtained by reading the columns of the Ferrers board of λ. Thus for example, $\lambda' = (9, \overline{6}, 5, \overline{5}, 3, 2, \overline{2}, 1, 1)$.

A second method of obtaining the conjugate of an overpartition is as follows.

The conjugate of $\lambda = (p_1, p_2, \ldots, p_k)$ is given by $\lambda' = (q_1, q_2, \ldots, q_s)$, where

$$q_j = \begin{cases} \overline{|\{r : p_r \geq j\}|} & \text{if } j = p_i \text{ is overlined} \\ |\{r : p_r \geq j\}| & \text{otherwise.} \end{cases}$$

In other words, let the overlined parts of λ be $u_1 > u_2 > \cdots > u_t$, and let the underlying ordinary partition be $f(\lambda)$. Then λ' is obtained by overlining the parts of the partition $f(\lambda)$ that are in positions $u_1, u_2, \ldots, u_t$.

For instance, given $\lambda = (9, \overline{7}, 5, 4, \overline{4}, \overline{2}, 1, 1, 1)$, then to obtain λ', the 2nd, 4th and 7th parts of the conjugate of $f(\lambda) = (9, 7, 5, 4, 4, 2, 1, 1, 1)$ will be overlined. That is, $\lambda' = (9, \overline{6}, 5, \overline{5}, 3, 2, \overline{2}, 1, 1)$.

Definition 1.1. An overpartition is said to be *self-conjugate* if it is identical with its conjugate.

For example, it may be verified that $\lambda = (7, \overline{6}, 4, 4, 2, \overline{2}, 1)$ is self-conjugate.

One of our main results is the following:

Theorem 1.2. *Let $\overline{sc}(N)$ be the number of self-conjugate overpartitions of N. Then*

$$\sum_{N=0}^{\infty} \overline{sc}(N)q^N = 1 + \sum_{j=1}^{\infty} 2q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_j}.$$

The q-binomial coefficients (or Gaussian polynomials) are defined, for non-negative integers m, n, as

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \frac{(1-q^{m+n})(1-q^{m+n-1})\cdots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}. \tag{1.3}$$

These polynomials have many important applications in Combinatorics, Number Theory and Physics [1]. In partition theory, Eqn (1.3) is interpreted as the generating function for the number of partitions fitting inside an $m \times n$ rectangle, i.e., partitions that have parts of size $\leq m$ and a number of parts $\leq n$.

Recently, Dousse and Kim [5] introduced the *over q-binomial coefficient* which is an overpartition analogue of the q-binomial coefficients, defined by

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} := \sum_{k=0}^{\min\{m,n\}} q^{\binom{k+1}{2}} \frac{(q;q)_{m+n-k}}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}}. \tag{1.4}$$

This function is interpreted as the generating function for the number of overpartitions fitting inside an $m \times n$ rectangle. The over q-binomial coefficients have many properties similar to those of ordinary q-binomial coefficients [4, 5].

In 2003 Prellberg and Stanton [8] published a proof of the monotonicity conjecture which states the coefficients of the function

$$(1-q)\frac{1}{(q^n;q)_n} + q$$

are non-negative, for all positive integers n. This conjecture was originally formulated by Friedman et al. [6].

Subsequently, Dousse and Kim [4] formulated the following analogous conjecture based on the geometry of over q-binomial coefficients.

Conjecture 1.3. *For all positive integers n, the coefficients of*

$$(1-q)\frac{(-q^n;q)_n}{(q^n;q)_n} + q$$

are non-negative.

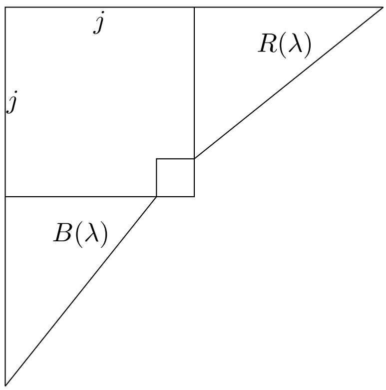
Conjecture 1.3 is an over q-analogue of the monotonicity conjecture. We will indicate a possible path to realizing a combinatorial proof of this conjecture in Section 4.

In Section 2, we give a proof of Theorem 1.2. In Section 3, we prove a certain over q-binomial coefficient identity, and establish an over q-analogue of the Chu-Vandermonde identity. We end the section with an alternative summative generating function for the number of overpartitions of n.

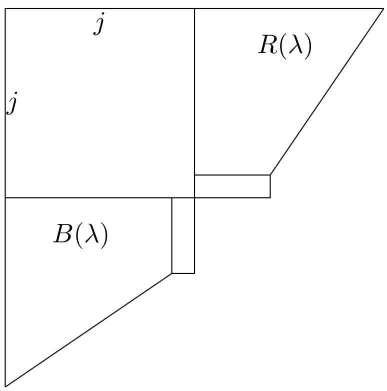
2. Proof of Theorem 1.2

Suppose we have a self-conjugate overpartition λ of N with a Durfee square of length $j > 0$. Then in the Ferrers graph of λ the j^{th} part of λ, i.e., the last part of the Durfee square, may be overlined or not. There are two cases to consider (see the diagrams below).

- Case I: when the j^{th} part is overlined (i.e., j^{th} part $= j$ and $(j+1)^{th}$ part $< j$);



- Case II: when the j^{th} part is not overlined (i.e., j^{th} part $\geq j$ and $(j+1)^{th}$ part $\leq j$).



In the first case, the Durfee square is generated by q^{j^2} (from (1.2)). Since λ is self-conjugate the overpartition $R(\lambda)$ represented by the boxes on the right of the Durfee square is the conjugate of the overpartition $B(\lambda)$ represented by the boxes below the Durfee square. Therefore each of these overpartitions is generated by $\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}$, which is the generating function for the number of overpartitions with at most $j-1$ parts (cf. Eqn (1.1)). Adding the rows of $B(\lambda)$ to the corresponding columns of $R(\lambda)$ gives an overpartition into even parts. So for each $j \geq 1$ we deduce that these overpartitions are generated by

$$q^{j^2}\prod_{r=1}^{j-1}\frac{1+q^{2r}}{1-q^{2r}} = q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}}.$$

Similarly in the second case, the Durfee square is generated by q^{j^2} while the overpartitions represented on the right of and below the Durfee square are generated by $\frac{(-q^2;q^2)_j}{(q^2;q^2)_j}$. Thus such overpartitions are generated, for all $j \geq 1$, by

$$q^{j^2}\prod_{r=1}^{j}\frac{1+q^{2r}}{1-q^{2r}} = q^{j^2}\frac{(-q^2;q^2)_{j}}{(q^2;q^2)_{j}}.$$

Hence, with 1 counting the empty (self-conjugate) overpartition, we have

$$\begin{aligned}
\sum_{N=0}^{\infty}\overline{sc}(N)q^N &= 1+\sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}} + \sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j}}{(q^2;q^2)_{j}}\\
&= 1+\sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}}\left(1+\frac{1+q^{2j}}{1-q^{2j}}\right)\\
&= 1+\sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j}}\left(1-q^{2j}+1+q^{2j}\right)\\
&= 1+\sum_{j=0}^{\infty}2q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j}}.
\end{aligned}$$

This completes the proof.

3. Over q-binomial coefficients

Basic properties of over q-binomial coefficients are given in [4, 5]. Several of these properties resemble those of ordinary q-binomial coefficients. For example, we have the symmetry property,

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} m+n \\ m \end{bmatrix}}. \tag{3.1}$$

We recall the following series-product identities, known respectively, as Cauchy's Identity and q-Binomial Theorem (see [2, 7]).

Theorem 3.1. *For* $|q|, |z| < 1$ *we have*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{3.2}$$

$$\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k = \prod_{k=1}^{n} (1 + zq^k). \tag{3.3}$$

The following theorem was proved combinatorially in [4]. Here we will give an algebraic proof.

Theorem 3.2 (Dousse-Kim [4])**.** *For every positive integer* m*, we have*

$$\sum_{k=0}^{\infty} \overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} z^k q^k = \frac{(-zq^2;q)_{m-1}}{(zq;q)_m}. \tag{3.4}$$

Algebraic Proof. We will use the fact that

$$(q;q)_{m+k} = (q;q)_m (q^{m+1};q)_k. \tag{3.5}$$

We simplify the left-hand side of (3.4) (with $m-1$ replaced by m). Note that in the second equality below we set $\min(m,k) = m$ since $0 \le j \le m$ but $j \le k \to \infty$.

$$\begin{aligned}
\sum_{k=0}^{\infty} \overline{\begin{bmatrix} m+k \\ k \end{bmatrix}} z^k q^k &= \sum_{k=0}^{\infty} \sum_{j=0}^{\min(m,k)} q^{\binom{j+1}{2}} \frac{(q;q)_{m+k-j}}{(q;q)_j (q;q)_{m-j} (q;q)_{k-j}} (zq)^k \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} \sum_{k=j}^{\infty} \frac{(q;q)_{m+k-j}}{(q;q)_{k-j}} (zq)^k \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} (zq)^j \sum_{k=0}^{\infty} \frac{(q;q)_{m+k}}{(q;q)_k} (zq)^k \\
&= \sum_{j=0}^{m} \frac{(zq)^j q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} (q;q)_m \sum_{k=0}^{\infty} \frac{(q^{m+1};q)_k}{(q;q)_k} (zq)^k \quad \text{(by Eqn (3.5))} \\
&= \sum_{j=0}^{m} \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}} (zq)^j q^{\binom{j+1}{2}} \cdot \frac{(zq^{m+2};q)_\infty}{(zq;q)_\infty} \quad \text{(by Eqn (3.2))} \\
&= \frac{(zq^{m+2};q)_\infty}{(zq;q)_\infty} (-zq \cdot q;q)_m \quad \text{(by Eqn (3.3))} \\
&= \frac{(-zq^2;q)_m}{(zq;q)_{m+1}}.
\end{aligned}$$

This completes the proof. □

3.1. An over q-analogue of Chu-Vandermonde identity

Consider the classical combinatorial identity,

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}, \quad n \geq 0. \tag{3.6}$$

It is known that this identity has the following q-analogue [2]:

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

We state a new over q-analogue of (3.6) using the over q-binomial coefficients.

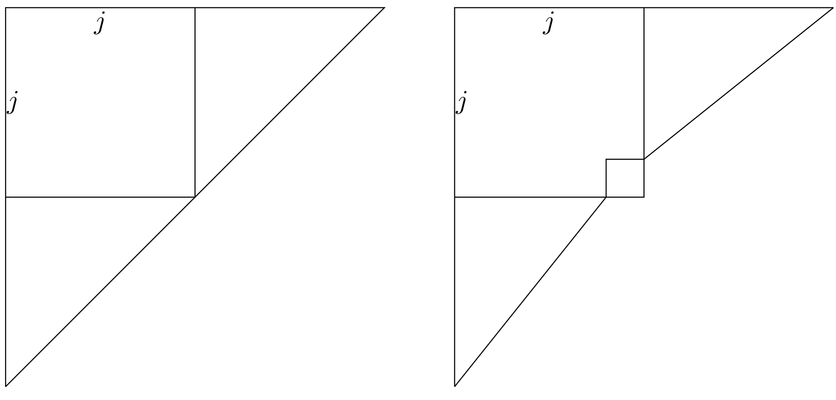
Proposition 3.3. *For any non-negative integer n, we have*

$$\overline{\begin{bmatrix} 2n \\ n \end{bmatrix}} = \sum_{j=0}^{n} q^{j^2} \left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2 \right).$$

Proof. It is clear that overpartitions fitting inside an $n \times n$ square are generated by (cf. Eqn (1.4))

$$\overline{\begin{bmatrix} n+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} 2n \\ n \end{bmatrix}}.$$

Now assume the overpartitions have Durfee squares of length j. Such overpartitions may be represented by either of the following diagrams depending on whether the last part of the Durfee square is not overlined or overlined, respectively.



In either diagram the Durfee square is generated by q^{j^2}, $j \geq 0$.

In the second diagram the subdiagram attached to the right side of the Durfee square represents an overpartition fitting inside an $(n-j) \times (j-1)$ rectangle, and the subdiagram attached below the Durfee square represents an overpartition fitting inside a $(j-1)\times(n-j)$ rectangle. So both subdiagrams are generated by

$$\begin{aligned}&\overline{\begin{bmatrix}(n-j)+(j-1)\\ j-1\end{bmatrix}}\times\overline{\begin{bmatrix}(j-1)+(n-j)\\ n-j\end{bmatrix}}\\ &=\overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}\times\overline{\begin{bmatrix}n-1\\ n-j\end{bmatrix}}\\ &=\overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}\times\overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}\quad\text{(by symmetry, (3.1))}\\ &=\overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}^2.\end{aligned}$$

Similarly for the first diagram, the subdiagram on the right side of the Durfee square represents an overpartition fitting inside an $(n-j)\times j$ rectangle, and the subdiagram below the Durfee square is an overpartition fitting inside a $j\times(n-j)$ rectangle. Thus they are generated by

$$\overline{\begin{bmatrix}(n-j)+j\\ j\end{bmatrix}}\times\overline{\begin{bmatrix}j+(n-j)\\ n-j\end{bmatrix}}=\overline{\begin{bmatrix}n\\ j\end{bmatrix}}\times\overline{\begin{bmatrix}n\\ n-j\end{bmatrix}}=\overline{\begin{bmatrix}n\\ j\end{bmatrix}}\times\overline{\begin{bmatrix}n\\ j\end{bmatrix}}=\overline{\begin{bmatrix}n\\ j\end{bmatrix}}^2.$$

Hence the generating function for the number of partitions into at most n parts with part-sizes $\leq n$, and Durfee square of length j, is given by

$$q^{j^2}\left(\overline{\begin{bmatrix}n\\ j\end{bmatrix}}^2+\overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}^2\right).$$

Lastly, we sum over $0\leq j\leq n$ to obtain the stated identity. □

Proposition 3.3 may be regarded as a 'finite' version of the following identity which provides another generating function for $\overline{p}(n)$ (cf. Eqn (1.1)).

Corollary 3.4. *We have*

$$\frac{(-q;q)_\infty}{(q;q)_\infty}=\sum_{j=0}^{\infty}2q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_j}\right)^2(1+q^{2j}).$$

Proof. Let $n\to\infty$ in Proposition 3.3. Then

$$\begin{aligned}\lim_{n\to\infty}\overline{\begin{bmatrix}2n\\ n\end{bmatrix}}&=\lim_{n\to\infty}\sum_{k=0}^{n}q^{\frac{k(k+1)}{2}}\frac{(q;q)_{2n-k}}{(q;q)_k(q;q)_{n-k}(q;q)_{n-k}}\\ &=\sum_{k=0}^{\infty}\frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k}\cdot\frac{1}{(q;q)_\infty}\\ &=\frac{(-q;q)_\infty}{(q;q)_\infty}.\end{aligned}$$

Proceeding to the limit we also have

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=0}^{n} q^{j^2}\left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2\right) &= \sum_{j=0}^{\infty} q^{j^2}\left(\left(\frac{(-q;q)_j}{(q;q)_j}\right)^2 + \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2\right) \\ &= \sum_{j=0}^{\infty} q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2\left(\left(\frac{1+q^j}{1-q^j}\right)^2+1\right) \\ &= \sum_{j=0}^{\infty} q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2 \cdot \frac{2(1+q^{2j})}{(1-q^j)^2} \\ &= \sum_{j=0}^{\infty} 2q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_j}\right)^2 (1+q^{2j}).\end{aligned}$$

Hence the result. □

Remark 3.5. Note that Corollary 3.4 may also be proved by pure combinatorial reasoning by splitting the set of overpartitions into two classes, in the spirit of the proof of Theorem 1.2.

Direct Combinatorial proof of Corollary 3.4. It is clear that

$$\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{1}{(q;q)_k}q^{\frac{k(k+1)}{2}} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

Now suppose we have an overpartition λ of N with Durfee square of side j. Separate λ into two classes as in the proof of Theorem 1.2 (see Section 2). In both cases, the Durfee square is obviously generated by q^{j^2}. However, in the first case the top-right and bottom-left subdiagrams are generated by $\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}$. Such overpartitions are generated by

$$q^{j^2}\frac{(-q;q)_{j-1}^2}{(q;q)_{j-1}^2}.$$

Similarly, in the second case, the top-right and bottom-left subdiagrams are generated by $\frac{(-q;q)_j}{(q;q)_j}$. Hence this class is generated by

$$q^{j^2}\frac{(-q;q)_j^2}{(q;q)_j^2}.$$

Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}(n)q^n &= \sum_{j=0}^{\infty} q^{j^2}\frac{(-q;q)_{j-1}^2}{(q;q)_{j-1}^2} + \sum_{j=0}^{\infty} q^{j^2}\frac{(-q;q)_j^2}{(q;q)_j^2} \\ &= \sum_{j=0}^{\infty} q^{j^2}\left(\left(\frac{(-q;q)_j}{(q;q)_j}\right)^2 + \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2\right)\end{aligned}$$

$$= \sum_{j=0}^{\infty} 2q^{j^2} \left(\frac{(-q;q)_{j-1}}{(q;q)_j} \right)^2 (1+q^{2j}). \qquad \square$$

4. Remarks on Conjecture 1.3

The combinatorial proof of the following lemma is given in [5]. For completeness we provide an algebraic proof below.

Lemma 4.1 (Dousse-Kim [5]). *For every positive integer n, we have*

$$\frac{(-zq;q)_n}{(zq;q)_n} = 1 + \sum_{k\geq 1} z^k q^k \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right).$$

Proof. We have that the right-hand side

$$\begin{aligned}
&= 1 + \sum_{k\geq 1} z^k q^k \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \sum_{k\geq 1} z^k q^k \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \\
&= \sum_{k\geq 0} z^k q^k \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \sum_{k\geq 0} z^{k+1} q^{k+1} \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} \\
&= \frac{(-zq^2;q)_{n-1}}{(zq;q)_n} + zq\frac{(-zq^2;q)_{n-1}}{(zq;q)_n} \qquad \text{(by Theorem 3.2)} \\
&= \frac{(-zq^2;q)_{n-1}}{(zq;q)_n}(1+zq) = \frac{(-zq;q)_n}{(zq;q)_n}. \qquad \square
\end{aligned}$$

This result enables the translation of the coefficients of the conjectured generating function into the coefficients of generating functions of overpartitions.

Set $z = q^{n-1}$ in Lemma 4.1 to get

$$\begin{aligned}
\frac{(-q^n;q)_n}{(q^n;q)_n} &= 1 + \sum_{k\geq 1} q^{k(n-1)} q^k \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right) \\
&= 1 + \sum_{k\geq 1} q^{kn} \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1-q)\frac{(-q^n;q)_n}{(q^n;q)_n} + q &= 1 + \sum_{k\geq 1} q^{kn}(1-q) \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right), \qquad (4.1) \\
&= 1 + \sum_{k\geq 1} q^{kn} \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} - q\overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right)
\end{aligned}$$

$$+\sum_{k\geq 1} q^{kn}\overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} - \sum_{k\geq 1} q^{kn+1}\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}}. \tag{4.2}$$

It is clear that the right-hand-sides of (4.1) and (4.2) enumerate overpartitions. Hence one approach to proving Conjecture 1.3 combinatorially relies on interpreting the right-hand-side of the equation as the generating function of a non-vacuous union of certain sets of restricted overpartitions.

It is hoped that this will enhance the discovery of a purely combinatorial proof of the conjecture.

Lastly, we lend credence to the conjecture by providing the results of a computational study of the actual coefficients of the associated generating function.

Let $[q^N]f(q)$ denote that coefficient of q^N in the Maclaurin series expansion of $f(q)$.

For all $n > 2$, the terms of the number sequences

$S(N,n) = [q^N]\Big((1-q)\frac{(-q^n;q)_n}{(q^n;q)_n} + q\Big)$, $N > 0$, are mostly positive, assuming zero values for few initial values of N. The following properties of the sequences were discovered using the computer algebra system Maple [9].

1. $S(0,n) = 1$ for all $n > 0$,
2. $S(1,1) = 2,\ S(N,1) = 0$ for $N > 1$,
3. $S(N,2) \in \{0,2\}$ for all $N > 0$,
4. $S(N,3) \in \{0,2\}$, $1 \leq N \leq 11$,
   $S(12,3) = 4,\ S(13,3) = S(14,3) = 0$,
   and $S(N,3) > 1$ for $N \geq 15$,
5. $S(N,n) \in \{0,2\}$, $1 \leq n \leq m$, and $S(N,n) \geq 2$, $n \geq m$, where $m = 3n + 2$, $n \geq 4$.

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Ruled like surfaces in three dimensional Euclidean space

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Abstract. In this paper, we introduce ruled like surfaces in three-dimensional Euclidean space, E^3. To form a ruled like surface in E^3, we consider a base curve $\gamma(s)$ and a director curve $X(s)$. Let parameter s be the angle between the tangent of $\gamma(s)$ and $X(s)$ when $X(s)$ lie on rectifying plane or in the osculating plane. Whereas, if $X(s)$ is in the normal plane, then parameter s will be the angle between the normal of $\gamma(s)$ and position vector of $X(s)$ at the corresponding point in E^3. Then we investigate some characterizations of such types of surfaces (say $\mathbb{S}(s,v)$). Moreover, we find the condition for the existence of Bertrand mate of $\gamma(s)$ in $\mathbb{S}(s,v)$. Finally, as examples, we construct the surfaces $\mathbb{S}(s,v)$ by using a straight line, circle and helix in E^3.

Keywords: Bertrand curve, Frenet frame, rectifying plane, osculating plane, normal plane, ruled surfaces

AMS Subject Classification: 53A05, 53A04

1. Introduction

Ruled surfaces are one of the basic and useful types of surfaces in differential geometry. Ruled surfaces are in the class of those surfaces which are broadly used in CAD systems. Ruled surfaces were introduced by G. Monge as a solution of a partial differential equation. Different properties depending upon geodesic curvature and the second fundamental form of ruled surfaces in E^3 were studied in [1]. Whereas the ruled surfaces generated by some special curves like circular

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helices, circular slant helices and Salkowski curves were considered in [15]. In [18], authors derived the isogeodesic surface pencil so that the geodesic curve is a directrix of the ruled surface.

The notion of pitch function for ruled surfaces was introduced by H. R. Müller in 1951. The pitch function and angle function of the pitch for non-developable ruled surfaces in E^3 and E^3_1 were further generalized in [9, 10]. For any non-developable ruled surface, if the base curve is a striction line and the directrix is a spherical curve, then the spherical Frenet frame can be obtained by using directrix. This spherical Frenet frame brings out three functions along the base curve on E^3, known as structural functions. In [19], authors studied the properties of non-developable ruled surfaces using structure functions. Ruled surfaces were also studied in Minkowski space [7, 17] and in three-dimensional Lie groups [16].

The idea of the Bertrand curve was given by Saint Venant in 1845 by the question “for any surface generated by a curve $\gamma(s)$, does there exist any other curve whose normal coincides with the normal of the initial curve”. Bertrand answered this question in 1850 [4] by the condition, “a curve $\gamma(s)$ on E^3 is a Bertrand curve if and only if there exists a linear relationship with constant coefficients between the curvature and torsion of the original curve”. In [3, 5, 11], authors studied the Bertrand curve in Minkowski space and three-dimensional sphere.

We organize our article as follows: Section 2, discusses some basic results of curves and surfaces in E^3. Ruled like surfaces, which are the core of our research article, are also defined in the same section. In Section 3, we talk about various characterizations of our surfaces, normal of the surface, Gaussian curvature, mean curvature etc. In Section 4, the conditions are obtained for the Bertrand mate of the curve $\gamma(s)$, which lie in the normal ruled like surface formed by $\gamma(s)$. In the final section, as examples, the surfaces are constructed using a straight line, plane curve circle and space curve helix.

2. Preliminaries and some results

Let $\gamma(s)$ be a unit speed space curve in R^3 with Frenet frame $\{T, N, B\}$ along $\gamma(s)$. Then, we know that

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N,$$

where κ is a curvature and τ is a torsion of $\gamma(s)$.

Definition 2.1 ([6])**.** Let $\gamma(s)$ be a smooth curve on E^3. Then $\gamma(s)$ is said to be a Bertrand curve if there exists another curve $\beta(\bar{s} = \phi(s))$ in E^3 such that the normals of $\gamma(s)$ and $\beta(\bar{s} = \phi(s))$ are linearly dependent to each other at corresponding points. Here ϕ is a bijection from $\gamma(s)$ to $\beta(\bar{s})$ and $\beta(\bar{s})$ is the Bertrand mate of $\gamma(s)$.

Definition 2.2 ([8])**.** The parametric representation of a ruled surface $\mathbb{S}(s, v)$ in E^3 is $\mathbb{S}(s, v) = \gamma(s) + v\delta(s)$, where $\gamma(s)$ is a space curve, $\delta\colon I \to \mathbb{R}^3 - \{0\}$ is a smooth map and I is an open interval or a unit circle. The curves $\gamma(s)$ and $\delta(s)$ are known as the base and director curves, respectively. The map $v \to \gamma(s) + v\delta(s)$ is known as a ruling of $\mathbb{S}(s, v)$.

Let $\mathbb{S}(s, v)$ be a ruled surface in E^3, then the various quantities associated with the surface are defined as follows:

(A). Unit surface normal: $\hat{N} = \frac{\mathbb{S}_s \times \mathbb{S}_v}{\|\mathbb{S}_s \times \mathbb{S}_v\|}$, where $\mathbb{S}_s = \frac{\partial \mathbb{S}}{\partial s}$ and $\mathbb{S}_v = \frac{\partial \mathbb{S}}{\partial v}$.

(B). First fundamental form: $I = \mathbb{E}ds^2 + 2\mathbb{F}dsdv + \mathbb{G}dv^2$, where $\mathbb{E} = \langle \mathbb{S}_s, \mathbb{S}_s \rangle$, $\mathbb{F} = \langle \mathbb{S}_s, \mathbb{S}_v \rangle$ and $\mathbb{G} = \langle \mathbb{S}_v, \mathbb{S}_v \rangle$.

(C). Second fundamental form: $II = \mathbb{L}ds^2 + 2\mathbb{M}dsdv + \mathbb{N}dv^2$, where $\mathbb{L} = \langle \mathbb{S}_{ss}, \hat{N} \rangle$, $\mathbb{M} = \langle \mathbb{S}_{sv}, \hat{N} \rangle$ and $\mathbb{N} = \langle \mathbb{S}_{vv}, \hat{N} \rangle$.

If K is a Gaussian curvature, H is a mean curvature and λ is a distribution parameter of $\mathbb{S}(s, v)$, then from [13]

(D). $K = \frac{\mathbb{L}\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\mathbb{G} - \mathbb{F}^2}$, $H = \frac{\mathbb{E}\mathbb{N} + \mathbb{G}\mathbb{L} - 2\mathbb{F}\mathbb{M}}{2(\mathbb{E}\mathbb{G} - \mathbb{F}^2)}$ and $\lambda = \frac{det(\gamma'(s), \delta(s), \delta'(s))}{\|\delta'(s)\|}$.

The second Gaussian curvature K_{II} of $\mathbb{S}(s, v)$ in E^3 is defined by replacing the components of the first fundamental form $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ by the components of the second fundamental form $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{N}$ in Brioschi's formulae respectively. In [2], the second Gaussian curvature of a surface is defined as

$$K_{II} = \frac{1}{(\mathbb{L}\mathbb{N} - \mathbb{M}^2)^2} \left(\begin{vmatrix} -\frac{1}{2}\mathbb{L}_{vv} + \mathbb{M}_{sv} - \frac{1}{2}\mathbb{N}_{ss} & \frac{1}{2}\mathbb{L}_s & \mathbb{M}_s - \frac{1}{2}\mathbb{L}_v \\ \mathbb{M}_v - \frac{1}{2}\mathbb{N}_s & \mathbb{L} & \mathbb{M} \\ \frac{1}{2}\mathbb{N}_v & \mathbb{M} & \mathbb{N} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\mathbb{L}_v & \mathbb{N}_s \\ \frac{1}{2}\mathbb{L}_v & \mathbb{L} & \mathbb{M} \\ \frac{1}{2}\mathbb{N}_s & \mathbb{M} & \mathbb{N}) \end{vmatrix} \right).$$

Let $\beta(s)$ be a curve in $\mathbb{S}(s, v)$, then the normal curvature κ_n, geodesic curvature κ_g and geodesic torsion τ_g of $\beta(s)$ [1] are given by

$$\kappa_n = \langle \hat{N}, T' \rangle, \quad \kappa_g = \langle \hat{N} \times T, T' \rangle, \quad \text{and} \quad \tau_g = \langle \hat{N} \times \hat{N}', T' \rangle.$$

The curve $\gamma(s)$ in $\mathbb{S}(s, v)$ can be characterized on the basis of the values of κ_g, κ_n and τ_g. That is
(1) $\gamma(s)$ will be a geodesic if and only if $\kappa_g = 0$.
(2) $\gamma(s)$ will be a asymptotic line if and only if $\kappa_n = 0$.
(3) $\gamma(s)$ will be a principal line if and only if $\tau_g = 0$.

In case of ruled surface $\mathbb{S}(s, v)$, the position vector of unit director curve $\delta(s)$ can be written as [1]

$$\delta(s) = f_1 T + f_2 N + f_3 B, \tag{2.1}$$

where $\{T, N, B\}$ is a Frenet frame along $\gamma(s)$ and f_i, $i \in \{1, 2, 3\}$, are fixed components, i.e., $f_1^2 + f_2^2 + f_3^2 = 1$.

In equation (2.1), it is clear that the components f_i of the director curve are fixed. Now, consider $\delta(s)$ lie on the normal plane of $\gamma(s)$, such that the angle between $\delta(s)$ and N is arc length parameter s at the corresponding point. Then the parametrization of $\mathbb{S}(s,v)$ is

$$\mathbb{S}(s,v) = \mathbb{S}^n(s,v) = \gamma(s) + v(\cos(s)N + \sin(s)B). \tag{2.2}$$

Obviously, the parametrized surface formed in (2.2), is not a ruled surface. Because the components $f_1 = 0$, $f_2 = \cos(s)$ and $f_3 = \sin(s)$ are not fixed. Similarly, we can construct the surfaces

$$\mathbb{S}(s,v) = \mathbb{S}^o(s,v) = \gamma(s) + v(\cos(s)T + \sin(s)N), \tag{2.3}$$

and

$$\mathbb{S}(s,v) = \mathbb{S}^r(s,v) = \gamma(s) + v(\cos(s)T + \sin(s)B), \tag{2.4}$$

by taking $\delta(s)$ in osculating plane $\{T, N\}$, and rectifying plane $\{T, B\}$ respectively, such that the angle between $\delta(s)$ and T is s at corresponding point. Here, we define the definition of a ruled like surface.

Definition 2.3. A surface $\mathbb{S}(s,v)$ with parametrization given by any one of the equations (2.2), (2.3) and (2.4) is said to be a ruled like surface generated by a curve $\gamma(s)$ on E^3. The surface $\mathbb{S}^n(s,v)$ is said to be a normal ruled like surface of $\gamma(s)$. Similarly, $\mathbb{S}^o(s,v)$ and $\mathbb{S}^r(s,v)$ are named as osculating ruled like surface and rectifying ruled like surface of $\gamma(s)$ on E^3.

3. Some characterization of ruled like surfaces

For any surface in E^3, unit surface normal, Gaussian curvature and Mean curvature are some basic properties that help to understand the surface. In this section, all these mentioned properties of ruled like surfaces generated by a space curve and a plane curve in E^3 are studied.

3.1. Normal ruled like surfaces

Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by space curve $\gamma(s)$ on E^3. Then the partial derivative of (2.2), gives us

$$\begin{cases} \mathbb{S}^n_s(s,v) = (1 - v\kappa\cos(s))T - v(1+\tau)\sin(s)N + v(1+\tau)\cos(s)B, \\ \mathbb{S}^n_v(s,v) = \cos(s)N + \sin(s)B, \end{cases}$$

as a natural frame $\{\mathbb{S}^n_s(s,v), \mathbb{S}^n_v(s,v)\}$ of tangent space on $\mathbb{S}^n(s,v)$. Also,

$$\| \mathbb{S}^n_s(s,v) \times \mathbb{S}^n_v(s,v) \|^2 = v^2(1+\tau)^2 + (1 - v\kappa\cos(s))^2 = 0,$$

if and only if $\tau = -1$ and $v = \frac{1}{\kappa\cos(s)}$, for all $s \in \mathbb{R} - \{(2n-1)\frac{\pi}{2}\}, n$ is an integer. Thus the singularity of $\mathbb{S}^n(s,v)$ can be removed by considering either $\tau \neq -1$ or $v \neq \frac{1}{\kappa\cos(s)}$, for all $s \in \mathbb{R} - \{(2n-1)\frac{\pi}{2}\}, n$ is an integer.

From now on, we will take only those ruled like surfaces that are generated by curves with $\tau(s) \neq -1$. The unit surface normal $\hat{N}^n$ of $\mathbb{S}^n(s,v)$ generated by a curve $\gamma(s)$ with $\tau(s) \neq -1$ is obtained as follows:

$$\hat{N}^n = \frac{-v(1+\tau)T - \sin(s)(1 - v\kappa\cos(s))N + \cos(s)(1 - v\kappa\cos(s))B}{\sqrt{v^2(1+\tau)^2 + (1 - v\kappa\cos(s))^2}}. \tag{3.1}$$

The coefficients of first and second fundamental forms of surface $\mathbb{S}^n(s,v)$ are

$$\begin{cases} \mathbb{E} = v^2(1+\tau)^2 + (1 - v\kappa\cos(s))^2, \\ \mathbb{F} = 0, \\ \mathbb{G} = 1, \end{cases}$$

and,

$$\begin{cases} \mathbb{L} = \frac{1}{\sqrt{\mathbb{E}}}\{v^2(1+\tau)(\kappa'\cos(s) - \kappa(2+\tau)\sin(s)) \\ \qquad -(1 - v\kappa\cos(s))(\kappa\sin(s)(1 - v\kappa\cos(s)) - v\tau')\}, \\ \mathbb{M} = \frac{1+\tau}{\sqrt{\mathbb{E}}}, \\ \mathbb{N} = 0, \end{cases}$$

respectively. Therefore the Gaussian curvature K and mean curvature H of the surface are given by

$$\begin{cases} K = -\frac{(1+\tau)^2}{\mathbb{E}^2}, \\ H = \frac{1}{2\mathbb{E}^{\frac{3}{2}}}\{v^2(1+\tau)(\kappa'\cos(s) - \kappa(2+\tau)\sin(s)) \\ \qquad -(1 - v\kappa\cos(s))(\kappa\sin(s)(1 - v\kappa\cos(s)) - v\tau')\}. \end{cases} \tag{3.2}$$

If $\gamma(s)$ is a plane curve, then for a normal ruled like surface of $\gamma(s)$ the unit surface normal $\hat{N}^n$, the Gaussian and the mean curvatures can be obtained simply by substituting $\tau = 0$, in equations (3.1) and (3.2), respectively. Here we discuss only the second Gaussian curvature K_{II} of $\mathbb{S}^n(s,v)$ generated by a plane curve. The second Gaussian curvature of $\mathbb{S}^n(s,v)$ is computed as:

$$\begin{aligned} K_{II} = &-\frac{\mathbb{L}_v\mathbb{E}_v}{4} + \frac{\mathbb{L}}{4\mathbb{E}}\left(\frac{\mathbb{E}_v^2}{2} - \mathbb{E}\mathbb{E}_{vv}\right) + \frac{\mathbb{E}_{vs}}{2\sqrt{\mathbb{E}}} - \frac{\mathbb{E}_s\mathbb{E}_v}{2\mathbb{E}^{\frac{3}{2}}} \\ &+ \frac{1}{\sqrt{\mathbb{E}}}\{\kappa'\cos(s) + \kappa\sin(s)\big(1 - \kappa^2\cos^2(s)\big)\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}_v &= 2\{v - \kappa\cos(s)(1 - v\kappa\cos(s))\}, \\ \mathbb{E}_s &= 2v(\kappa\sin(s) - \kappa'\cos(s))(1 - v\kappa\cos(s)), \\ \mathbb{E}_{vv} &= 2(1 + \kappa^2\cos^2(s)), \\ \mathbb{L} &= \frac{1}{\sqrt{\mathbb{E}}}\Big\{v^2(2\kappa\sin(s) + \kappa'\cos(s)) - \kappa\sin(s)(1 - v\kappa\cos(s))^2\Big\}, \end{aligned}$$

$$\mathbb{L}_v = \frac{1}{\mathbb{E}}\left[2\sqrt{\mathbb{E}}\{v(2\kappa\sin(s) + \kappa'\cos(s)) + \kappa^2\sin(s)\cos(s)(1 - v\kappa\cos(s))\} - \frac{1}{2}\mathbb{L}\mathbb{E}_v\right].$$

From all the above discussions, we obtain the following theorems and corollary.

Theorem 3.1. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a space curve $\gamma(s)$, $s \in I \subset \mathbb{R}$. Then the surface is singular if and only if $\tau(s) = -1$, where $\tau(s)$ is a torsion of $\gamma(s)$.*

Theorem 3.2. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a space curve $\gamma(s)$ with $\tau(s) \neq -1$. Then $\mathbb{S}^n(s,v)$ is neither a part of a sphere nor a plane.*

Corollary 3.3. *The Gaussian curvature and the mean curvature of a normal ruled like surface are related by $aH + bK = 0$, where $a = 2(1+\tau)^2$ and $b = \mathbb{E}\mathbb{L} = \sqrt{\mathbb{E}}\Big\{v^2(2\kappa\sin(s) + \kappa'\cos(s)) - \kappa\sin(s)(1 - v\kappa\cos(s))^2\Big\}$.*

Theorem 3.4. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by $\gamma(s)$ with $\tau(s) \neq -1$. Then $\mathbb{S}^n(s,v)$ is a minimal surface if and only if $\gamma(s)$ is a straight line.*

Proof. Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then from second part of equation (3.2), we have

$$\begin{aligned} v^2(1+\tau)(\kappa'\cos(s) - 2\kappa\sin(s)) &= (1 - v\kappa\cos(s))(\kappa\sin(s)(1 - v\kappa\cos(s)) - v\tau') \\ \implies v^2\{(1+\tau)(\kappa'\cos(s) - 2\kappa\sin(s)) &- \kappa^3\sin(s)\cos^2(s) - \kappa\tau'\cos(s)\} \\ &+ v\big(\tau' + 2\kappa^2\sin(s)\cos(s)\big) + \kappa\sin(s) = 0. \end{aligned}$$

Now, comparing the coefficients of v on both sides, we get

$$\begin{cases} (1+\tau)(\kappa'\cos(s) - 2\kappa\sin(s)) - \kappa^3\sin(s)\cos^2(s) - \kappa\tau'\cos(s) = 0, \\ \tau' + 2\kappa^2\sin(s)\cos(s) = 0, \\ \kappa\sin(s) = 0. \end{cases} \tag{3.3}$$

Because $s \in I \subset \mathbb{R}$, therefore $\sin(s) \neq 0 \ \forall \ s$. Thus, from the last part of (3.3), $\kappa = 0$. Hence $\gamma(s)$ is a straight line.

Conversely, assume that $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a straight line. Then taking $\kappa = 0$ and $\tau = 0$ in second part of equation (3.2), we have $H = 0$. Hence $\mathbb{S}^n(s,v)$ is a minimal surface. □

3.2. Osculating and rectifying ruled like surfaces

In this section, the coefficients of the first and the second fundamental forms, the Gaussian and the mean curvatures of osculating and rectifying ruled like surfaces are studied.

Let $\gamma(s)$ be a space curve in E^3 and $\mathbb{S}^o(s,v)$, $\mathbb{S}^r(s,v)$ are osculating and rectifying ruled like surfaces, respectively. Then natural frame $\{\mathbb{S}^o_s(s,v), \mathbb{S}^o_v(s,v)\}$ of

$\mathbb{S}^o(s,v)$, and $\{\mathbb{S}^r_s(s,v), \mathbb{S}^r_v(s,v)\}$ of $\mathbb{S}^r(s,v)$ are

$$\begin{cases}\mathbb{S}^o_s(s,v) = (1 - v(1+\kappa)\sin(s))T + v(1+\kappa)\cos(s)N + v\tau\sin(s)B,\\ \mathbb{S}^0_v(s,v) = \cos(s)T + \sin(s)N,\end{cases}$$

and,

$$\begin{cases}\mathbb{S}^r_s(s,v) = (1 - v\sin(s))T + v(\kappa\cos(s) - \tau\sin(s))N + v\cos(s)B,\\ \mathbb{S}^r_v(s,v) = \cos(s)T + \sin(s)B,\end{cases}$$

respectively. First, we will discuss various properties of $\mathbb{S}^o(s,v)$ in E^3. The unit surface normal for $\mathbb{S}^o(s,v)$ is obtained by using the relation $\hat{N} = \frac{\mathbb{S}^o_s \times \mathbb{S}^o_v}{\|\mathbb{S}^o_s \times \mathbb{S}^o_v\|}$, where

$$\mathbb{S}^o_s \times \mathbb{S}^o_v = -\tau v\sin^2(s)T + \tau v\sin(s)\cos(s)N + (\sin(s) - v(1+\kappa))B,$$
$$\| \mathbb{S}^o_s \times \mathbb{S}^o_v \|^2 = v^2\tau^2\sin^2(s) + (\sin(s) - v(1+\kappa))^2.$$

Now, $\| \mathbb{S}^o_s \times \mathbb{S}^o_v \|^2 = 0$ if and only if any one of the following conditions holds:
(1) $v = 0$ and $s = n\pi$, where n is an integer,
(2) $\tau = 0$ and $v = \frac{\sin(s)}{1+\kappa}$.

Therefore, if $\gamma(s)$ is neither a plane curve nor a straight line, then $\mathbb{S}^o(s,v)$, $s, v \in I$(open interval) $\subset \mathbb{R}$, have singularity only at $v = 0$ and $s = n\pi$, where n is an integer. The parametrization for $\mathbb{S}^o(s,v)$ can be further modified by removing $v = 0$.

But just for convenience we are considering the surface $\mathbb{S}^o(s,v)$ with parameters $s, v \in I$(open interval) $\subset \mathbb{R}$ such that $v > 1$ i.e., $v = (1, |a|)$, where $1 < |a| \in \mathbb{R}$. Thus the surface $\mathbb{S}^o(s,v)$ is now a regular surface for all $s \in I$, and $v = (1, |a|)$. The unit surface normal $\hat{N}^o$ of $\mathbb{S}^o(s,v)$, is obtained as

$$\hat{N}^o = \frac{-\tau v\sin^2(s)T + \tau v\sin(s)\cos(s)N + (\sin(s) - v(1+\kappa))B}{\sqrt{\tau^2v^2\sin^2(s) + (\sin(s) - v(1+\kappa))^2}}. \tag{3.4}$$

The components of the first and second fundamental forms, the Gaussian and mean curvatures of $\mathbb{S}^o(s,v)$ are

$$\mathbb{E} = \cos^2(s) + (\sin(s) - v(1+\kappa))^2 + \tau^2v^2\sin^2(s),\ \mathbb{F} = \cos(s),\ \mathbb{G} = 1.$$

$$\begin{cases}\mathbb{L} = \frac{1}{\sqrt{\mathbb{EG}-\mathbb{F}^2}}\{\tau v\sin(s)\big[v\kappa' + \cos(s)\big(\kappa - v\tau^2\sin(s)\big)\big]\\ \qquad + (\sin(s) - v(1+\kappa))(v\tau(2+\kappa)\cos(s) + v\tau'\sin(s))\},\\ \mathbb{M} = \frac{\tau\sin^2(s)}{\sqrt{\mathbb{EG}-\mathbb{F}^2}}, \quad \mathbb{N} = 0.\end{cases} \tag{3.5}$$

$$K^o = -\frac{\tau^2\sin^4(s)}{(\mathbb{EG} - \mathbb{F}^2)^2}, \quad H^o = \frac{\mathbb{L}}{2(\mathbb{EG} - \mathbb{F}^2)} - \cos(s)\sqrt{-K^o},$$

respectively. Similarly, for surface $\mathbb{S}^r(s,v)$, $\| \mathbb{S}^r_s \times \mathbb{S}^r_v \|^2 = v^2(\kappa\cos(s) - \tau\sin(s))^2 + (v - \sin(s))^2 = 0$ if and only if it satisfies any one of the following conditions:

(1) $v = 0$ and $s = n\pi$, where n is an integer,
(2) $\kappa \cos(s) - \tau \sin(s) = 0$ and $v = \sin(s)$.

Because $-1 \leq \sin(s) = v \leq 1$, therefore the surface $\mathbb{S}^r(s,v)$ is regular $\forall\ s \in I(\textit{open interval}) \subset \mathbb{R}$ and $v = (1, |a|)$, where $|a|$ is some real number greater then one. The unit surface normal, the Gaussian curvature and the mean curvature of $\mathbb{S}^r(s,v)$ are given by the following relations

$$\hat{N}^r = \frac{v\sin(s)(\kappa\cos(s)-\tau\sin(s))T+(v-\sin(s))N+v\cos(s)(\tau\sin(s)-\kappa\cos(s))B}{\sqrt{(v-\sin(s))^2+v^2(\kappa\cos(s)-\tau\sin(s))^2}}, \tag{3.6}$$

$$K^r = -\frac{\sin^2(s)(\kappa\cos(s)-\tau\sin(s))^2}{(\mathbb{E}\mathbb{G}-\mathbb{F}^2)^2}, \quad H^r = \frac{\mathbb{L}}{2(\mathbb{E}\mathbb{G}-\mathbb{F}^2)} - \cos(s)\sqrt{-K^r}. \tag{3.7}$$

where,

$$\mathbb{E}\mathbb{G} - \mathbb{F}^2 = (v - \sin(s))^2 + v^2(\kappa\cos(s) - \tau\sin(s))^2,$$

and,

$$\begin{aligned}\mathbb{L} = \tfrac{1}{\sqrt{\mathbb{E}\mathbb{G}-\mathbb{F}^2}}\{&\left[-v^2(\kappa\cos(s) - \tau\sin(s))^2(\kappa\sin(s) + \tau\cos(s))\right] \\ &+ (v - \sin(s))(v(\kappa\cos(s) - \tau\sin(s))' + \kappa(1 - v\sin(s)) - \tau v\cos(s))\}.\end{aligned}$$

Thus, we have the following theorems:

Theorem 3.5. *Let $\gamma(s)$ be a space curve with $\tau \neq 0$ and surfaces $\mathbb{S}^o(s,v)$, $\mathbb{S}^r(s,v)$, $s \in I(\textit{open interval}) \subset \mathbb{R}$, $1 < v \in J(\textit{open interval}) \subset \mathbb{R}$ are generated by $\gamma(s)$. Then at points $s = n\pi$, the surfaces are flat.*

Theorem 3.6. *Let $\gamma(s)$ be a plane curve and $\mathbb{S}^o(s,v)$, $s \in I(\textit{open interval}) \subset \mathbb{R}$, $1 < v \in J(\textit{open interval}) \subset \mathbb{R}$ is an osculating surface. Then $\mathbb{S}^o(s,v)$ is flat and minimal in E^3.*

Theorem 3.7. *Let $\mathbb{S}^r(s,v)$, $s \in I(\textit{open interval}) \subset \mathbb{R}$, $1 < v \in J(\textit{open interval}) \subset \mathbb{R}$ be a rectifying ruled like surfaces generated by $\gamma(s)$; $s \in I$. Then $\mathbb{S}^r(s,v)$ is a flat and minimal surface if and only if it is generated by a straight line.*

4. Characterizations of curves in normal ruled like surface $\mathbb{S}^n(s,v)$

Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then the different properties of $\gamma(s)$ in $\mathbb{S}^n(s,v)$ like, whether $\gamma(s)$ is a geodesic or not and asymptotic curve of $\mathbb{S}^n(s,v)$ or not are studied. Also, we find the condition for Bertrand mate of $\gamma(s)$ to lie on $\mathbb{S}^n(s,v)$.

Theorem 4.1. *Let $(\gamma(s), \beta(\bar{s}))$ be a Bertrand couple in E^3 and $\mathbb{S}^n(s,v)$ be a normal ruled like surface of unit speed space curve $\gamma(s)$ with $\tau(s) \neq 0$. Then unit speed* *curve $\beta(\bar{s})$ with $\bar{\kappa}(\bar{s}) \neq 0$ lies on $\mathbb{S}^n(s(\bar{s}), v(\bar{s}))$ if and only if the parameters $s(\bar{s})$ and $v(\bar{s})$ satisfies the following conditions*

$$\begin{cases} \sin(s)\ddot{v} + 2b\cos(s)(1+\tau)\dot{v} \\ \qquad + b^2\Big(\cos(s)(\eta ab(1+\tau)+\tau') - \sin(s)(1+\tau)^2\Big)v = 0, \\ -2\kappa\cos(s)\dot{v} + (\kappa\sin(s)(2+\tau) - \cos(s)(\kappa' + ab\eta\kappa))v + \eta ab^2 = 0, \\ \frac{ds}{d\bar{s}} = \frac{1}{\sqrt{(1-\eta\kappa)^2+\eta^2\tau^2}} = b \quad and \quad \eta(\kappa^2-\tau^2)' = 2(\kappa' - \frac{a}{b}), \end{cases} \tag{4.1}$$

where $\epsilon = \pm 1$, $\eta \neq 0$ is an arbitrary constant and $a = \epsilon\sqrt{(\kappa'^2 + \tau'^2)}$.

Proof. Let $(\gamma(s), \beta(\bar{s}))$ be a Bertrand couple in E^3 and $\gamma(s)$ be a space curve with $\tau(s) \neq 0$. Then

$$\beta(\bar{s}) = \gamma(s) + \eta(s)N, \tag{4.2}$$

where η is a smooth function on E^3 and N is a normal vector field of Frenet frame $\{T, N, B\}$ along $\gamma(s)$ on E^3. The derivative of equation (4.2), with respect to $\bar{s}$, gives the relation

$$\bar{T}(\bar{s}) = ((1-\eta(s)\kappa)T + \eta' N + \eta\tau B)\frac{ds}{d\bar{s}}, \tag{4.3}$$

where $\bar{T}$ is a tangent vector field of $\beta(\bar{s})$ in E^3. The scalar product of equation (4.3) with N, implies that $\eta(s) = constant \neq 0$. Now differentiating the equation (4.3) with respect to $\bar{s}$, and then taking the scalar product of differential equation with $\bar{T}$, $\bar{B}$, we have

$$(1-\eta\kappa)\frac{d^2s}{d\bar{s}^2} = \frac{\eta\kappa'}{((1-\eta\kappa)^2+\eta^2\tau^2)} \quad \text{and} \quad \eta\tau\frac{d^2s}{d\bar{s}^2} = -\frac{\eta\tau'}{((1-\eta\kappa)^2+\eta^2\tau^2)}.$$
$$\implies \frac{d^2s}{d\bar{s}^2} = \frac{\epsilon\eta\sqrt{(\kappa'^2+\tau'^2)}}{((1-\eta\kappa)^2+\eta^2\tau^2)^{\frac{3}{2}}}, \tag{4.4}$$

where $\epsilon = \pm 1$. Also

$$\frac{ds}{d\bar{s}} = \frac{1}{\sqrt{((1-\eta\kappa)^2+\eta^2\tau^2)}} \implies \frac{d^2s}{d\bar{s}^2} = \frac{\eta\kappa' - \eta^2(\kappa\kappa' + \tau\tau')}{((1-\eta\kappa)^2+\eta^2\tau^2)^2}. \tag{4.5}$$

Thus from (4.4) and (4.5), we get $\eta(\kappa^2-\tau^2)' = 2(\kappa' - \frac{a}{b})$, where $a = \epsilon\sqrt{(\kappa'^2+\tau'^2)}$ and $b = \frac{1}{\sqrt{((1-\eta\kappa)^2+\eta^2\tau^2)}}$.

Let $\beta(\bar{s})$ be a curve on surface $\mathbb{S}^n(s(\bar{s}), v(\bar{s}))$. Then $\beta(\bar{s})$ is given by

$$\beta(\bar{s}) = \mathbb{S}^n(s(\bar{s}), v(\bar{s})); \quad \bar{s} \mapsto (s(\bar{s}), v(\bar{s})). \tag{4.6}$$

Differentiating (4.6), two times with respect to $\bar{s}$, we have

$$\bar{\kappa}(\bar{s})\bar{N}(\bar{s}) = \mathbb{S}^n_v(s(\bar{s}), v(\bar{s}))\ddot{v} + 2\mathbb{S}^n_{sv}(s(\bar{s}), v(\bar{s}))\dot{s}\dot{v} + \mathbb{S}^n_{vv}(s(\bar{s}), v(\bar{s}))\dot{v}^2$$

$$+\,\mathbb{S}^n_{ss}(s(\bar{s}),v(\bar{s}))\dot{s}^2+\mathbb{S}^n_{s}(s(\bar{s}),v(\bar{s}))\ddot{s}, \tag{4.7}$$

where $\ddot{v}=\frac{d^2v}{d\bar{s}^2}$, $\dot{v}=\frac{dv}{d\bar{s}}$, $\ddot{s}=\frac{d^2s}{d\bar{s}^2}$ and $\dot{s}=\frac{ds}{d\bar{s}}$. The partial derivatives of $\mathbb{S}^n(s(\bar{s}),v(\bar{s}))$ with respect to s and v, are

$$\mathbb{S}^n_{v}(s(\bar{s}),v(\bar{s}))=(\cos(s)N+\sin(s)B), \tag{4.8}$$
$$\mathbb{S}^n_{sv}(s(\bar{s}),v(\bar{s}))=-\kappa\cos(s)T-\sin(s)(1+\tau)N+\cos(s)(1+\tau)B, \tag{4.9}$$
$$\mathbb{S}^n_{s}(s(\bar{s}),v(\bar{s}))=(1-v\kappa\cos(s))T-v\sin(s)(1+\tau)N+v\cos(s)(1+\tau)B, \tag{4.10}$$
$$\begin{aligned}\mathbb{S}^n_{ss}(s(\bar{s}),v(\bar{s}))&=v(\kappa(2+\tau)\sin(s)-\kappa'\cos(s))T\\&\quad+v\big(\tau'\cos(s)-(1+\tau)^2\sin(s)\big)B,\\&\quad+\big(\kappa(1-v\kappa\cos(s))-v\tau'\sin(s)-v(1+\tau)^2\cos(s)\big)N.\end{aligned} \tag{4.11}$$

Now, using the equations (4.8)–(4.11), in equation (4.7), and the fact that $\bar{N}$ and N are collinear, we get

$$\begin{cases}\sin(s)\ddot{v}+2\cos(s)(1+\tau)\dot{s}\dot{v}+\cos(s)(1+\tau)v\ddot{s}\\ \qquad+v\big(\tau'\cos(s)-(1+\tau)^2\sin(s)\big)\dot{s}^2=0,\\ -2\kappa\cos(s)\dot{v}\dot{s}+(1-v\kappa\cos(s))\ddot{s}+v(\kappa(2+\tau)\sin(s)-\kappa'\cos(s))\dot{s}^2=0.\end{cases} \tag{4.12}$$

Substituting $\dot{s}$ and $\ddot{s}$ from (4.3) and (4.4), in equation (4.12), we obtained the required conditions.

Conversely, Let $\beta(\bar{s})$ is a curve on surface $\mathbb{S}^n(s(\bar{s}),v(\bar{s}))$ such that the map $\bar{s}\mapsto(s(\bar{s}),v(\bar{s}))$, satisfies the equation (4.1). Then, on substituting (4.8)–(4.11), in equation (4.7), we obtain

$$\begin{aligned}\bar{\kappa}(\bar{s})\bar{N}(\bar{s})&=\{\sin(s)\ddot{v}+2\cos(s)(1+\tau)\dot{s}\dot{v}+\cos(s)(1+\tau)v\ddot{s}\\&\quad+v\big(\tau'\cos(s)-(1+\tau)^2\sin(s)\big)\dot{s}^2\}B+\{\cos(s)\ddot{v}-2\sin(s)(1+\tau)\dot{s}\dot{v}\\&\quad-\sin(s)(1+\tau)v\ddot{s}+\big(\kappa(1-v\kappa\cos(s))-v\tau'\sin(s)-v(1+\tau)^2\cos(s)\big)\dot{s}^2\}N\\&\quad\{-2\kappa\cos(s)\dot{v}\dot{s}+(1-v\kappa\cos(s))\ddot{s}+v(\kappa(2+\tau)\sin(s)-\kappa'\cos(s))\dot{s}^2\}T.\end{aligned}$$

As $\langle\bar{N},T\rangle=0$ and $\langle\bar{N},B\rangle=0$, hence $\bar{N}$ and N are collinear. Therefore, $\beta(\bar{s})$ is a Bertrand mate of $\gamma(s)$. □

Theorem 4.2. *Let $(\gamma(s),\beta(\bar{s}))$ be a Bertrand couple in E^3 and $\beta(\bar{s})$ is lying on normal ruled like surface $\mathbb{S}^n(s,v)$ of $\gamma(s)$ with $\tau(s)\neq 0$. Then the map $\bar{s}\mapsto v(\bar{s})$ satisfies the relation*

$$v=\begin{cases}\frac{\kappa\sin(s)\cos(s)b\big(\kappa-(\kappa(1-\eta\kappa)-\eta\tau^2)b^3\big)-\eta(1+\tau)ab}{\kappa\sin(s)(\kappa^2\cos^2(s)+(1+\tau)(2+\tau))+\cos(s)(\kappa\tau'-\kappa'(1+\tau))} & \text{if } \sin(s)\neq 0 \text{ and } \cos(s)\neq 0,\\ -\lambda\frac{\eta ab}{\kappa(2+\tau)} & \text{if } \sin(s)=\pm1=\lambda \text{ and } \cos(s)=0,\\ -\lambda\frac{\eta ab(1+\tau)}{\tau'\kappa-\kappa'(1+\tau)} & \text{if } \cos(s)=\pm1=\lambda \text{ and } \sin(s)=0,\end{cases}$$

where $\epsilon=\pm1$, $\eta\neq 0$ is an arbitrary constant, $a=\epsilon\sqrt{(\kappa'^2+\tau'^2)}$ and $b=\frac{1}{\sqrt{(1-\eta\kappa)^2+\eta^2\tau^2}}$.

Proof. Let $(\gamma(s), \beta(\bar{s}))$ be a Bertrand couple in E^3 and $\beta(\bar{s})$, lying on normal ruled like surface $\mathbb{S}^n(s,v)$ of $\gamma(s)$ with $\tau(s) \neq 0$. Then, substituting (4.8)–(4.11), in (4.7), and taking the scalar product with T, N and B, we have

$$\begin{cases} -2\kappa\cos(s)\dot{v}\dot{s} + (1 - v\kappa\cos(s))\ddot{s} + v(\kappa(2+\tau)\sin(s) - \kappa'\cos(s))\dot{s}^2 = 0, \\ \cos(s)\ddot{v} - 2\sin(s)(1+\tau)\dot{s}\dot{v} - \sin(s)(1+\tau)v\ddot{s} \\ \qquad +\big(\kappa(1 - v\kappa\cos(s)) - v\tau'\sin(s) - v(1+\tau)^2\cos(s)\big)\dot{s}^2 = \bar{\kappa}\langle N, \bar{N}\rangle, \\ \sin(s)\ddot{v} + 2\cos(s)(1+\tau)\dot{s}\dot{v} + \cos(s)(1+\tau)v\ddot{s} \\ \qquad + v\big(\tau'\cos(s) - (1+\tau)^2\sin(s)\big)\dot{s}^2 = 0. \end{cases} \tag{4.13}$$

Now, if both $\cos(s) \neq 0$ and $\sin(s) \neq 0$, then from second and third part of (4.13), we get

$$\begin{aligned} &2(1+\tau)\dot{v}\dot{s} + (1-\tau)v\ddot{s} + (-\kappa\sin(s)(1 - v\kappa\cos(s)) + v\tau')\dot{s}^2 \\ &= -\bar{\kappa}\sin(s)\langle N, \bar{N}\rangle, \end{aligned} \tag{4.14}$$

Using equations (4.3), (4.4) and (4.14) in the first part of (4.13), we obtain

$$v = \frac{\kappa\sin(s)\cos(s)b\big(\kappa - \bar{\kappa}\langle N, \bar{N}\rangle b\big) - \eta(1+\tau)ab}{\kappa\sin(s)(\kappa^2\cos^2(s) + (1+\tau)(2+\tau) + \cos(s)(\kappa\tau' - \kappa'(1+\tau)))}, \tag{4.15}$$

where $a = \epsilon\sqrt{(\kappa'^2 + \tau'^2)}$ and $b = \frac{1}{\sqrt{((1-\eta\kappa)^2 + \eta^2\tau^2)}}$. Also, if we differentiate (4.3) with respect to $\bar{s}$, and take the scaler product with the normal, then

$$\bar{\kappa}\langle N, \bar{N}\rangle = \frac{\kappa(1-\eta\kappa) - \eta\tau^2}{(1-\eta\kappa)^2 + \eta^2\tau^2} = b^2\big(\kappa(1-\eta\kappa) - \eta\tau^2\big). \tag{4.16}$$

Hence, equations (4.15) and (4.16) together prove the first part of the theorem. To prove the other two parts consider $\cos(s) = 0$, $\sin(s) = \pm 1 = \lambda$ and $\sin(s) = 0$, $\cos(s) = \pm 1 = \lambda$ in equation (4.13), we get

$$\begin{cases} \ddot{s} + \lambda\kappa(2+\tau)v\dot{s}^2 = 0, \\ -2\lambda(1+\tau)\dot{s}\dot{v} - \lambda(1+\tau)v\ddot{s} + (\kappa - v\tau'\lambda)\dot{s}^2 = \bar{\kappa}\langle N, \bar{N}\rangle, \\ \lambda\ddot{v} - v\lambda(1+\tau)^2\dot{s}^2 = 0, \end{cases} \tag{4.17}$$

and,

$$\begin{cases} -2\kappa\lambda\dot{v}\dot{s} + (1 - v\kappa\lambda)\ddot{s} - \kappa'\lambda v\dot{s}^2 = 0, \\ \lambda\ddot{v} + \big(\kappa(1 - v\lambda\kappa) - v(1+\tau)^2\lambda\big)\dot{s}^2 = \bar{\kappa}\langle N, \bar{N}\rangle, \\ 2\lambda(1+\tau)\dot{s}\dot{v} + \lambda(1+\tau)v\ddot{s} + \tau'\lambda v\dot{s}^2 = 0. \end{cases} \tag{4.18}$$

The second part of the theorem is proved by the first part of (4.17), (4.3) and (4.4). Whereas to prove the third part of the theorem, solve the first and third parts of (4.18) by replacing the values of $\dot{v}\dot{s}$, and then use equations (4.3) and (4.4) to get the required result. □

Theorem 4.3. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then $\gamma(s)$ is neither an asymptotic curve nor a geodesic of $\mathbb{S}^n(s,v)$.*

Proof. Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then the unit surface normal of $\mathbb{S}^n(s,v)$ is given by the equation (3.1). Now from [14, p. 166], we have

$$\kappa_g = \kappa\langle N, \hat{N}\times T\rangle \ \ \textit{and} \ \ \kappa_n = \kappa\langle N, \hat{N}\rangle. \tag{4.19}$$

Thus the unit surface normal $\hat{N}$ and $\hat{N}\times T$ along $\gamma(s)$, from (3.1) we have

$$\begin{cases} \hat{N}(s,0) = -\sin(s)N + \cos(s)B, \\ \hat{N}(s,0)\times T = \cos(s)N + \sin(s)B. \end{cases} \tag{4.20}$$

Therefore, from (4.19) and (4.20), $\kappa_g = \kappa\cos(s) \neq 0$ and $\kappa_n = -\kappa\sin(s) \neq 0$ for all s. Hence $\gamma(s)$ is neither an asymptotic curve nor a geodesic of $\mathbb{S}^n(s,v)$. □

Corollary 4.4. *The geodesic torsion of the curve $\gamma(s)$ on normal ruled like surface $\mathbb{S}^n(s,v)$ is given by $\tau_g = \kappa\cos(s)\sin(s)$.*

Proof. From relation $\tau^g_\Upsilon = \langle \hat{N}(s,0)\times \hat{N}_s(s,0), \kappa N\rangle$, we get the solution of this corollary by direct calculation. □

As we know $\mathbb{S}^n(s,v) = \gamma(s) + v(\cos(s)N + \sin(s)B)$, where $X(s) = \cos(s)N + \sin(s)B)$; $\langle X(s), X(s)\rangle = 1$ and $\langle T, X\rangle = 0$. Therefore, we can make another frame $\{T, X(s), T\times X = Y\}$ in $\mathbb{S}^n(s,v)$, such that the derivative of T, X and Y satisfies the equations

$$\begin{vmatrix} T' \\ X' \\ Y' \end{vmatrix} = \begin{vmatrix} 0 & \kappa\cos(s) & -\kappa\sin(s) \\ -\kappa\cos(s) & 0 & (1+\tau) \\ \kappa\sin(s) & -(1+\tau) & 0 \end{vmatrix}\begin{vmatrix} T \\ X \\ Y \end{vmatrix}, \tag{4.21}$$

and this frame coincides with the Darboux frame along $\gamma(s)$ in $\mathbb{S}^n(s,v)$.

Theorem 4.5. *Let $\mathbb{S}^n(s,v) = \gamma(s) + vX(s)$, where $X(s) = \cos(s)N + \sin(s)B$. Then orthogonal trajectory of $X(s)$ lies in $\mathbb{S}^n(s,v)$ if and only if $v = \frac{\kappa\cos(s)}{\kappa^2\cos^2(s) + (1+\tau)^2}$.*

Proof. Let $\delta(s)$ be an orthogonal trajectory of $X(s)$ lying on $\mathbb{S}^n(s,v)$. Then

$$\delta(s) = \gamma(s) + v(s)X(s) \ \ \textit{and} \ \ \langle \delta'(s), X'(s)\rangle = 0$$

Also, from 4.21, we get

$$\begin{aligned} 0 =& \langle \delta'(s), X'(s)\rangle = \langle T, X'(s)\rangle + v\langle X'(s), X'(s)\rangle, \\ &\implies v = \frac{\kappa\cos(s)}{\kappa^2\cos^2(s) + (1+\tau)^2}. \end{aligned} \tag{4.22}$$

Equation (4.22) proves the first part of the theorem.

Now to prove converse part, let $\mathbb{S}^n(s,v) = \gamma(s) + vX(s)$, with $X(s) = \cos(s)N + \sin(s)B$ and $v = \frac{\kappa\cos(s)}{\kappa^2\cos^2(s)+(1+\tau)^2}$. Then by taking

$$\delta(s) = \gamma(s) + \frac{\kappa\cos(s)}{\kappa^2\cos^2(s)+(1+\tau)^2}(\cos(s)N + \sin(s)B),$$

it is easy to prove that $\langle \delta'(s), X'(s)\rangle = 0$(use Frenet frame of $\gamma(s)$). Hence $\delta(s)$ is an orthogonal trajectory of $\gamma(s)$ in $\mathbb{S}^n(s,v)$. □

Note. Similar way, we can also study the characterizations of curves lying on osculating and rectifying ruled like surfaces.

5. Examples for ruled like surfaces

In this section, we form the normal, osculating and rectifying ruled like surfaces generated from a straight line, circle and helix. Also, we plot the orthogonal trajectory of $X(s) = \cos(s)N + \sin(s)B$ in a normal ruled like surface.

Example 5.1. Let $\gamma(s) = (s,0,0)$ be a straight line in E^3. Then Frenet frame along $\gamma(s)$ can be taken as follows

$$T(s) = (1,0,0), \quad N(s) = (0,1,0), \quad B(s) = (0,0,1).$$

Then, the parametrization for normal, osculating and rectifying ruled like surfaces for a straight line are given by

$$\begin{cases} \mathbb{S}^n(s,v) = \Big(s, v\cos(s), v\sin(s)\Big), \ \forall\ s \in I, v \in J \text{ and } I, J \subset \mathbb{R}, \\ \mathbb{S}^o(s,v) = \Big(s + v\cos(s), v\sin(s), 1\Big), \ \forall\ s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}, \\ \mathbb{S}^n(s,v) = \Big(s + v\cos(s), 1, v\sin(s)\Big)\ \forall\ s \in I \subset \mathbb{R}, v \in (1,b); \text{ and } 1 < b \in \mathbb{R}. \end{cases}$$

Now, we will discuss these surfaces one by one.

Case 1. Consider the surface $\mathbb{S}^n(s,v) = (s, v\cos(s), v\sin(s))$, $\forall\ s \in I$ and $v \in J$; $I, J \subset \mathbb{R}$. Then the natural frame $\{\mathbb{S}^n_s(s,v), \mathbb{S}^n_v(s,v)\}$ on $\mathbb{S}^n(s,v)$ are

$$\mathbb{S}^n_s(s,v) = (1, -v\sin(s), v\cos(s)), \ \textit{and} \ \mathbb{S}^n_v(s,v) = (0, \cos(s), \sin(s)).$$

Therefore the unit surface normal of $\mathbb{S}^n(s,v)$ is $\hat{N}^n = \frac{1}{\sqrt{1+v^2}}(-v, \sin(s), \cos(s))$. The coefficients of first fundamental form are $E = (1+v^2)$, $F = 0$ and $G = 1$. Whereas coefficients of the second fundamental form are $\mathbb{L} = 0$, $\mathbb{M} = \frac{1}{\sqrt{1+v^2}}$ and $\mathbb{N} = 0$.

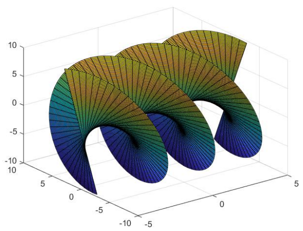
Thus the surface $\mathbb{S}^n(s,v)$ is minimal and a surface of negative Gaussian curvature in E^3.

Case 2. Let $\mathbb{S}^o(s,v) = (s + v\cos(s), v\sin(s), 1)$, $\forall\ s \in I, v \in (1,b), I \subset \mathbb{R}$ and $1 < b \in \mathbb{R}$. Then $\{\mathbb{S}^o_s(s,v), \mathbb{S}^0_v(s,v)\}$ is a natural frame of $\mathbb{S}^o(s,v)$ and $\mathbb{S}^o_s(s,v)$, $\mathbb{S}^0_v(s,v)$ are obtained as follows

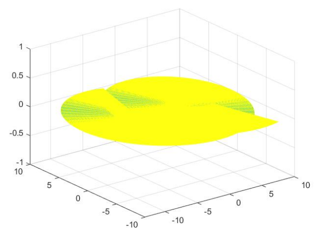
$$\mathbb{S}^o_s(s,v) = (1 - v\sin(s), v\cos(s), 0),\ \textit{and}\ \ \mathbb{S}^0_v(s,v) = (\cos(s), \sin(s), 0).$$

The unit surface normal $\hat{N}^o$ of $\mathbb{S}^o(s,v)$ is $\hat{N}^o = (0,0,1)$. Thus the first I and the second II fundamental forms of $\mathbb{S}^o(s,v)$ are $I = \big((s + v\cos(s)^2 + v^2\sin^2(s))\big)ds^2 + 2Fdsdv + dv^2$ and $II = 0$, respectively. Hence the surfaces of type $\mathbb{S}^o(s,v)$ generated by the straight line in E^3 are minimal and flat.

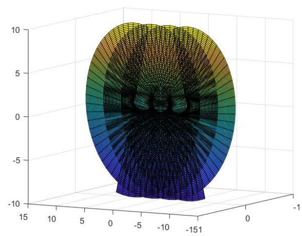
The nature of rectifying surface of a straight line is not much different as compared to the osculating surface. Because the rectifying and osculating ruled like surfaces of straight-line look the same. Therefore we give figures only for regular osculating surfaces and irregular rectifying surfaces in E^3.



(a) Normal ruled like surface of the straight line for $-5 < s < 5$ and $-10 < v < 10$.



(b) Osculating ruled like surface of the straight line for $-5 < s < 5$ and $1 < v < 10$.



(c) Rectifying ruled like surface of the straight line(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 1. Ruled like surfaces of a straight line.

Example 5.2. Let $\gamma(s) = (\cos(s), \sin(s), 0)$ be a circle in E^3. Then Frenet frame of $\gamma(s)$ on E^3 are

$$T(s) = (-\sin(s), \cos(s), 0),\ \ N(s) = (-\cos(s), -\sin(s), 0),\ \ B(s) = (0,0,-1).$$

Therefore the ruled like surfaces of the circle are given by

$$\begin{cases} \mathbb{S}^n(s,v) = \Big(\cos(s) - v\cos^2(s), \sin(s) - v\sin(s)\cos(s), -v\sin(s)\Big), \\ \qquad\qquad\qquad\qquad \forall\, s \in I, v \in J \text{ and } I, J \subset \mathbb{R}, \\ \mathbb{S}^o(s,v) = \Big(\cos(s) - 2v\sin(s)\cos(s), \sin(s) + v(\cos^2(s) - \sin^2(s)), 0\Big), \\ \qquad\qquad\qquad\qquad \forall\, s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}, \\ \mathbb{S}^r(s,v) = \Big(\cos(s) - v\sin(s)\cos(s), \sin(s) + v\cos^2(s), -v\sin(s)\Big) \\ \qquad\qquad\qquad\qquad \forall\, s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}. \end{cases}$$

Thus, the unit surface normal of the surfaces from equations (3.1), (3.4) and (3.6) are

$$\begin{cases} N^n(s,v) = \frac{1}{\sqrt{v^2+(1-v\cos(s))^2}} \Big\{ v\sin(s) + \sin(s)\cos(s)\big(1 - v\cos(s)\big), \\ \qquad\qquad -v\cos(s) + \sin^2(s)(1 - v\cos(s)), \cos(s)\big(1 - v\cos(s)\big) \Big\}, \\ N^o(s,v) = (0,0,-1), \\ N^r(s,v) = \frac{1}{\sqrt{(v-\sin(s))^2+v^2\cos(s)^2}} \Big\{ v\sin^2(s)\cos(s) - \cos(s)(v - \sin(s)), \\ \qquad\qquad v\sin(s)\cos^2(s) - \sin(s)(v - \sin(s)), v\cos(s)\cos(s) \Big\}. \end{cases}$$

Similarly, the Gaussian and the mean curvatures for the surfaces can be obtained from (3.2), (3.5) and (3.7). Also, the orthogonal trajectory of $X(s) = \cos(s)N + \sin(s)B = (-\cos^2(s), -\sin(s)\cos(s), -\sin(s))$ from Theorem 4.5 is (see the Figure 2)

$$\delta(s) = \left(\frac{\cos(s)}{1+\cos^2(s)}, \frac{\sin(s)}{1+\cos^2(s)}, \frac{-\sin(s)\cos(s)}{1+\cos^2(s)} \right).$$

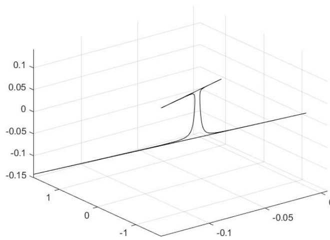
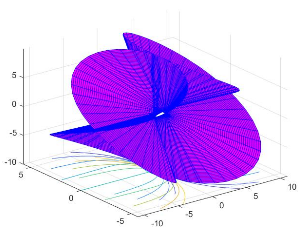


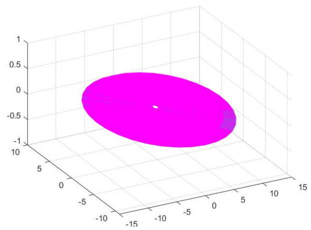
Figure 2. Orthogonal trajectory of $X(s)$ for $-5 < s < 5$ in Figure 3a.

Example 5.3. Let $\gamma(s) = \frac{1}{\sqrt{2}}(\cos(s), \sin(s), s)$ be a circular helix in E^3. Then Frenet frame along $\gamma(s)$ are

$$\begin{cases} T(s) = \frac{1}{\sqrt{2}}(-\sin(s), \cos(s), 1), \\ N(s) = (-\cos(s), -\sin(s), 0), \\ B(s) = \frac{1}{\sqrt{2}}(\sin(s), -\cos(s), 1). \end{cases}$$

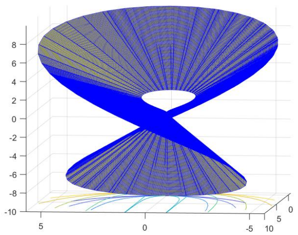


(a) Normal ruled like surface of the circle for $-5 < s < 5$ and $-10 < v < 10$.

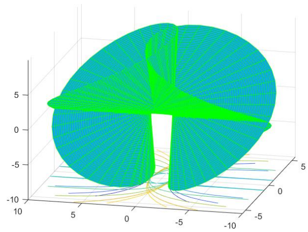


(b) Osculating ruled like surface of a circle for $-5 < s < 5$ and $1 < v < 10$.

Figure 3. Normal and osculating ruled like surfaces of the circle.



(a) Rectifying ruled like surface of the circle(regular) for $-5 < s < 5$ and $1 < v < 10$.



(b) Rectifying ruled like surface of the circle(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 4. Rectifying ruled like surfaces of the circle.

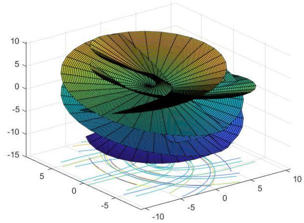
Thus, the ruled like surfaces of the circular helix are given by the following equations:

$$\begin{cases} \mathbb{S}^n(s,v) = \Big(\frac{\cos(s)}{\sqrt{2}} + v(-\cos^2(s) + \frac{\sin^2(s)}{\sqrt{2}}), \frac{\sin(s)}{\sqrt{2}} - v(\cos(s)\sin(s))(\frac{1+\sqrt{2}}{\sqrt{2}}), \\ \qquad\qquad \frac{s}{\sqrt{2}} + \frac{v\sin(s)}{\sqrt{2}}\Big), \forall\ s \in I, v \in J \text{ and } I, J \subset \mathbb{R}, \\ \mathbb{S}^o(s,v) = \Big(\frac{\cos(s)}{\sqrt{2}} - v(\frac{\sin(s)\cos(s)}{\sqrt{2}} + \sin(s)\cos(s)), \frac{\sin(s)}{\sqrt{2}} + v(\frac{\cos^2(s)}{\sqrt{2}} - \sin^2(s)), \\ \qquad\qquad \frac{s}{\sqrt{2}} + \frac{v\cos(s)}{\sqrt{2}}\Big),\ \forall\ s \in I \subset \mathbb{R}, v \in (1,b)\ \text{ and } 1 < b \in \mathbb{R}, \\ \mathbb{S}^r(s,v) = \frac{1}{\sqrt{2}}\Big(\cos(s) + v\sin(s)(\sin(s) - \cos(s)), \sin(s) + v\cos(s)\big(\cos(s) \\ \qquad - \sin(s)\big), s + v(\sin(s) + \cos(s))\Big),\ \forall\ s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}. \end{cases}$$

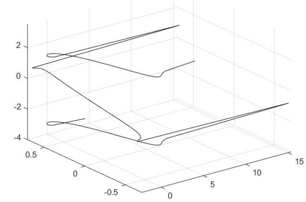
The unit surface normal for these surfaces can be obtained by using equations (3.1), (3.4) and (3.6), respectively. Also, the orthogonal trajectory of $X(s) = \cos(s)N + \sin(s)B = (-\cos^2(s), -\sin(s)\cos(s), -\sin(s))$ from Theorem 4.5 is (see Figure 5b)

$$\delta(s) = \Bigg(\frac{1}{\sqrt{2}}\cos(s) + \frac{\sqrt{2}\cos(s)}{\cos^2(s) + (1+\sqrt{2})^2}\Bigg(\frac{\sin^2(s)}{\sqrt{2}} - \cos^2(s)\Bigg), \frac{1}{\sqrt{2}}\sin(s)$$

$$- \frac{(1+\sqrt{2})\sin(s)\cos^2(s)}{\cos^2(s)+(1+\sqrt{2})^2}, \frac{s}{\sqrt{2}} + \frac{\sin(s)\cos(s)}{\cos^2(s)+(1+\sqrt{2})^2}\Big).$$

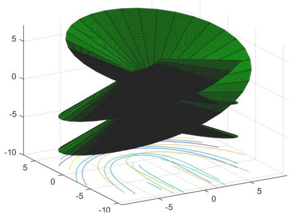


(a) Normal ruled like surface of the helix for $-5 < s < 5$ and $-10 < v < 10$.

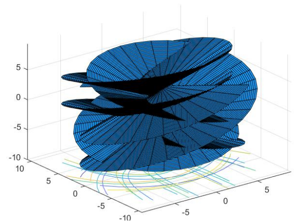


(b) Orthogonal trajectory of $X(s)$ for $-5 < s < 5$ in Figure 5a.

Figure 5. Normal ruled like surface of the helix and Orthogonal trajectory of $X(s)$.

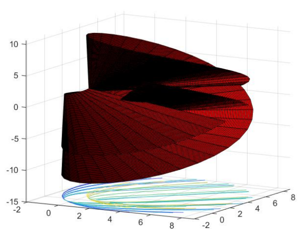


(a) Osculating ruled like surface of the helix for $-5 < s < 5$ and $1 < v < 10$.

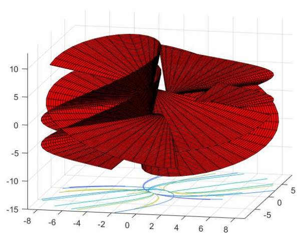


(b) Osculating ruled like surface of the helix(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 6. Osculating ruled like surfaces of the helix.



(a) Rectifying ruled like surface of the helix for $-5 < s < 5$ and $1 < v < 10$.



(b) Rectifying ruled like surface of the helix(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 7. Rectifying ruled like surfaces of the helix.

6. Conclusion

For normal ruled like surfaces, we consider only those surfaces which are generated by curves with $\tau(s) \neq -1$, therefore in the case of Salkowski curves [12] heaving $\tau(s) = \tan(s)$ regular normal ruled like surfaces are not possible with the same parametrization. Whereas in the case of rectifying ruled like surfaces generated by a curve, we got a case for some curve whose ratio of curvature and torsion holds the equation

$$\frac{\kappa}{\tau} = \begin{cases} \tan(s), & \text{if } s \neq (2n+1)\frac{\pi}{2}, \\ 0, & \text{if } s = (2n+1)\frac{\pi}{2}. \end{cases}$$

Thus exploring more details about this curve may give some new results. Furthermore, we believe that using this way of parametrization, one can find different surfaces in Minkowski space as well.

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On the generalized Fibonacci like sequences and matrices

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Abstract. In this paper, we study the generalized Fibonacci like sequences $\{t_{k,n}\}_{k\in\{2,3\},n\in\mathbb{N}}$ with arbitrary initial seed and give some new and well-known identities like Binet's formula, trace sequence, Catalan's identity, generating function, etc. Further, we study various properties of these generalized sequences, establish a recursive matrix and relationships with Fibonacci and Lucas numbers and sequence of Fibonacci traces. In this study, we examine the nature of identities and recursive matrices for arbitrary initial values.

Keywords: Binet's formula, Fibonacci like sequences, generating function, recursive matrix, trace sequence

AMS Subject Classification: 11B37, 11B39, 11B83, 65Q30

1. Introduction

In recent years, several papers [1, 2, 4, 18] published involving new identities and results based on Fibonacci-like sequences and their generalizations which have many interesting properties. One can refer to the book [8] of T. Koshy for more such sequences, generalizations, and rich applications.

In spite of many articles, books, and literature reviews on Fibonacci-like sequences and their generalizations [3–10, 13, 17], investigating new identities, results and their applications are interesting areas among researchers. Ongoing through the available literature review on generalizations of Fibonacci sequences, it can be noted that mainly the work may be generalized in two directions. Either the re-

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cursive formula can be generalized and extended or the formula is preserved with arbitrary initial assumptions. Kalman et al. [6] discussed some well-known results of classical Fibonacci-like sequences and demonstrated that many of the properties of these sequences can be established for much more general classes.

The recursive matrices corresponding to recursive sequences always attract researchers to investigate new identities and establish some well-known results such as Binet's formula, determinants, permanents, etc. For instance, Kumari et al. [9] have proposed some new families of identities of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials. Tianxiao et al. [16] presented a recursive matrix for recursive sequences of order three $a_{k+3} = pa_k + qa_{k+1} + ra_{k+2}$ with arbitrary initial conditions, and discussed some special third order recurrences such as Padavon and Perrin numbers. Saba et al. [14] introduced the concept of bivariate Mersenne Lucas polynomials then established Binet's formula and obtained many well-known identities using Binet's formula. Özkan et al. [11] obtained the elements of the Lucas polynomials by using two matrices and extended the study to the n-step Lucas polynomials, whereas Testan et al. [15] given some families of generalized Fibonacci and Lucas polynomials and developed some properties of these families and established interrelationships.

1.1. Fibonacci and Lucas matrices

The well-known integer sequences, Fibonacci $\{f_{2,n}\}$ and Lucas $\{u_{2,n}\}$ sequence are defined as

$$f_{2,n+2} = f_{2,n} + f_{2,n+1} \quad \text{and} \quad u_{2,n+2} = u_{2,n} + u_{2,n+1}; \quad n \geq 0, \tag{1.1}$$

with $f_{2,0} = 0,\ f_{2,1} = 1$ for $\{f_{2,n}\}$ and $u_{2,0} = 2,\ u_{2,1} = 1$ for $\{u_{2,n}\}$. These sequences are also extendable in the negative direction which can be achieved by rearranging Eqn. (1.1). It is also noted that $f_{2,-n} = (-1)^{n+1} f_{2,n}$ and $u_{2,-n} = (-1)^n u_{2,n}$ for $n \in \mathbb{N} \cup \{0\}$.

A matrix sequence [8] corresponding to above integer sequences are given as

$$Q_2^n = \begin{bmatrix} f_{2,n+1} & f_{2,n} \\ f_{2,n} & f_{2,n-1} \end{bmatrix} \quad \text{and} \quad L_2^{(n)} = \begin{bmatrix} u_{2,n+1} & u_{2,n} \\ u_{2,n} & u_{2,n-1} \end{bmatrix}. \tag{1.2}$$

Further in [12], Prasad et al. have obtained some interesting properties of generalized Fibonacci matrices (Q_k^n) given in the following theorem. We use these identities to establish some new identities and results in this paper.

Theorem 1.1 ([12])**.** *Let $n, l \in \mathbb{Z},\ k(\geq 2) \in \mathbb{N}$ and Q_k^n be a generalized Fibonacci matrix of order k, then we have*

1. $(Q_k^1)^n = Q_k^n$,
2. $Q_k^0 = I_k$, *where I_k is identity matrix of order k,*
3. $Q_k^n Q_k^l = Q_k^{n+l}$,

4. $\det(Q_k^n) = (-1)^{(k-1)n}$.

Note. Throughout the paper, we adopt the notation $t_{k,n}$ to denote the nth term of the sequence $\{t_{k,n}\}$ of order k with arbitrary initial values.

2. The $\{t_{2,n}\}$ sequence and some properties

Consider the second order linear difference equation given by

$$t_{2,n+2} = t_{2,n+1} + t_{2,n}, \quad n \geq 0 \quad \text{with} \quad t_{2,0} = a \quad \text{and} \quad t_{2,1} = b. \tag{2.1}$$

Similar to the Fibonacci sequence, the sequence $\{t_{2,n}\}$ can also be extended in the negative direction by rearranging Eqn. (2.1) as $t_{2,-n} = t_{2,-n+2} - t_{2,-n+1}$; $n \in \mathbb{N}$ with the same initial values.

Thus, the first few terms of the sequence are as follows:

n	...	-3	-2	-1	**0**	**1**	2	3	4	5	6	...
$t_{2,n}$	...	-3a+2b	2a-b	-a+b	**a**	**b**	a+b	a+2b	2a+3b	3a+5b	5a+8b	...
$f_{2,n}$	...	2	-1	1	**0**	**1**	1	2	3	5	8	...
$l_{2,n}$	...	-4	3	-1	**2**	**1**	3	4	7	11	18	...

Remark 2.1. For a sequence $\{t_{2,n}\}_{n\geq 0}$ satisfying Eqn. (2.1), we have

$$t_{2,n} = af_{2,n-1} + bf_{2,n}, \quad \text{where} \quad f_{2,0} = 0 \quad \text{and} \quad f_{2,1} = 1. \tag{2.2}$$

2.1. Matrix formation

The matrix sequence $\{T_2^{(n)}\}_{n\geq 0}$ associated with the integer sequence $\{t_{2,n}\}$ is defined as

$$T_2^{(n)} = \begin{bmatrix} t_{2,n+1} & t_{2,n} \\ t_{2,n} & t_{2,n-1} \end{bmatrix} \quad \text{with} \quad T_2^{(0)} = \begin{bmatrix} b & a \\ a & b-a \end{bmatrix}, \tag{2.3}$$

where $\det(T_2^{(0)}) = b(-a+b) - a^2 = b^2 - ab - a^2 = K(\text{say})$.

In next theorems and results, we present some interesting recursive and explicit formulas for the matrix sequence $T_2^{(n)}$ associated with the Fibonacci matrices.

Theorem 2.2. *The determinant of matrix $T_2^{(n)}$ is given by*

$$\det(T_2^{(n)}) = (a^2 + ab - b^2)(-1)^{n-1} = K(-1)^n.$$

Proof. To prove it, we use the following result of Fibonacci numbers

$$f_{2,n+1}f_{2,n-2} - f_{2,n}f_{2,n-1} = (-1)^{n-1}. \tag{2.4}$$

Therefore,

$$\det(T_2^{(n)}) = t_{2,n+1}t_{2,n-1} - t_{2,n}^2$$

$$\begin{aligned}
&= (af_{2,n} + bf_{2,n+1})(af_{2,n-2} + bf_{2,n-1}) - (af_{2,n-1} + bf_{2,n})^2 \\
&= a^2(f_{2,n}f_{2,n-2} - f_{2,n-1}^2) + b^2(f_{2,n+1}f_{2,n-1} - f_{2,n}^2) \\
&\quad + ab(f_{2,n}f_{2,n-1} + f_{2,n+1}f_{2,n-2} - 2f_{2,n}f_{2,n-1}) \\
&= a^2[(-1)^{n-1}] + b^2[(-1)^n] + ab(f_{2,n+1}f_{2,n-2} - f_{2,n}f_{2,n-1}) \\
&= a^2[(-1)^{n-1}] + b^2[(-1)^n] + ab[(-1)^{n-1}] \quad \text{(using Eqn. (2.4))} \\
&= (a^2 - b^2 + ab)(-1)^{n-1} = -K(-1)^{n-1} = K(-1)^n
\end{aligned}$$

as required. □

Corollary 2.3. $\det(T_2^{(n+1)}) = (-1)\det(T^{(n)})$.

Example 2.4 (Fibonacci matrix)**.** For $a = 0,\ b = 1$, we have $\det(T_2^{(n)}) = (-1)^n$.

Example 2.5 (Lucas matrix)**.** For $a = 2,\ b = 1$, we have $\det(T_2^{(n)}) = (-1)^n 5$.

Theorem 2.6. *Let $T_2^{(n)}$ be a matrix as defined in (2.3) and Q_2^n is the Fibonacci matrix, then we write*

$$T_2^{(n)} = Q_2^n T_2^{(0)} = T_2^{(0)} Q_2^n, \quad \forall n \in \mathbb{Z}.$$

Proof. We have

$$\begin{aligned}
Q_2^n T_2^{(0)} &= \begin{bmatrix} f_{2,n+1} & f_{2,n} \\ f_{2,n} & f_{2,n-1} \end{bmatrix} \begin{bmatrix} b & a \\ a & b-a \end{bmatrix} = \begin{bmatrix} bf_{2,n+1} + af_{2,n} & af_{2,n+1} + (b-a)f_{2,n} \\ bf_{2,n} + af_{2,n-1} & af_{2,n} + (b-a)f_{2,n-1} \end{bmatrix} \\
&= \begin{bmatrix} af_{2,n} + bf_{2,n+1} & bf_{2,n} + af_{2,n-1} \\ af_{2,n-1} + bf_{2,n} & bf_{2,n-1} + af_{2,n-2} \end{bmatrix} \quad \text{(using relation (1.1))} \\
&= \begin{bmatrix} t_{2,n+1} & t_{2,n} \\ t_{2,n} & t_{2,n-1} \end{bmatrix} \quad \text{(using relation (2.2))} \\
&= T_2^{(n)}.
\end{aligned}$$

By a similar argument, we have $T_2^{(0)} Q_2^n = T_2^{(n)}$. □

Corollary 2.7. *If $a = 0,\ b = 1$ then $T_2^{(0)} = I_2$ and $T_2^{(n)} = Q_2^n$, where I_2 is an identity matrix of order 2.*

Corollary 2.8. *For $n \in \mathbb{N}$, we have $T_2^{(n)} = Q_2 T_2^{(n-1)} = Q_2^{-1} T_2^{(n+1)}$.*

Theorem 2.9. *Let $T_2^{(n)}$ be a matrix as defined in (2.3), then we write*

$$T_2^{(n)} T_2^{(-n)} = (T_2^{(0)})^2.$$

Proof. By definition of $T_2^{(n)}$, we have

$$T_2^{(n)} T_2^{(-n)} = Q_2^{(n)} T_2^{(0)} Q_2^{(-n)} T_2^{(0)}$$

$$= T_2^{(0)} Q_2^{(n)} Q_2^{(-n)} T_2^{(0)}$$
$$= T_2^{(0)} I T_2^{(0)} = T_2^{(0)} T_2^{(0)} = (T_2^{(0)})^2$$

as required. □

From Theorem 2.2, it is clear that the matrix $T_2^{(n)}$ is invertible if and only if $T_2^{(0)}$ is invertible i.e $\det(T_2^{(0)}) = K \neq 0$. Thus from Theorem 2.9, we have the inverse of $T_2^{(n)}$ given by

$$\mathrm{Inv}(T_2^{(n)}) = T_2^{(-n)} H^{-1}, \quad \text{where} \quad H = (T_2^{(0)})^2 \text{ and } a, b \text{ are such that } K \neq 0.$$

2.2. The trace sequence

Let us define another sequence $\{l_{2,n}\}$ of order two for the given sequence $\{t_{2,n}\}$ as follows

$$l_{2,n} = \mathrm{trace}(T_2^{(n)}) = t_{2,n+1} + t_{2,n-1}, \tag{2.5}$$

whose initial values in terms of a and b are obtained as

$$l_{2,0} = t_{2,1} + t_{2,-1} = b + (b-a) = -a + 2b,$$
$$l_{2,1} = t_{2,2} + t_{2,0} = (a+b) + a = 2a + b.$$

Thus, Eqn. (2.5) can be re-stated free from $t_{2,n}$, recursively as

$$l_{2,n+2} = l_{2,n+1} + l_{2,n} \quad \text{with} \quad l_{2,0} = -a + 2b,\ l_{2,1} = 2a + b. \tag{2.6}$$

In particular, for $a = 0,\ b = 1$, $\{t_{2,n}\}$ becomes $\{f_{2,n}\}$ and its corresponding sequence of traces coincides with the standard Lucas sequence $\{u_{2,n}\}$.

Moreover, the matrix $M_2^{(n)}$ corresponding to trace sequence $\{l_{2,n}\}$ is given by

$$M_2^{(n)} = \begin{bmatrix} l_{2,n+1} & l_{2,n} \\ l_{2,n} & l_{2,n-1} \end{bmatrix} \quad \text{with} \quad M_2^{(0)} = \begin{bmatrix} l_{2,1} & l_{2,0} \\ l_{2,0} & l_{2,-1} \end{bmatrix} = \begin{bmatrix} 2a+b & 2b-a \\ 2b-a & 3a-b \end{bmatrix}. \tag{2.7}$$

Theorem 2.10. *The determinant of matrix $M_2^{(n)}$ is given by*

$$\det(M_2^{(n)}) = 5K(-1)^{n+1} \quad \forall n \in \mathbb{Z}.$$

Proof. From Eqn. (2.7), we have

$$M_2^{(n)} = \begin{bmatrix} l_{2,n+1} & l_{2,n} \\ l_{2,n} & l_{2,n-1} \end{bmatrix} = \begin{bmatrix} t_{2,n+2} + t_{2,n} & t_{2,n+1} + t_{2,n-1} \\ t_{2,n+1} + t_{2,n-1} & t_{2,n} + t_{2,n-2} \end{bmatrix}$$
$$= \begin{bmatrix} t_{2,n+1} & t_{2,n} \\ t_{2,n} & t_{2,n-1} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = T_2^{(n)} L_2^{(0)} \quad \text{(from Eqn. (2.1) and Eqn. (1.2))}.$$

Thus, $\det(M_2^{(n)}) = |T_2^{(n)} L_2^{(0)}| = |Q_2^n T_2^{(0)} L_2^{(0)}| = |Q_2^n||T_2^{(0)}||L_2^{(0)}| = 5K(-1)^{n+1}$. □

In particular for $n = 0$, we have $\det(M_2^{(0)}) = 5a^2 + 5ab - 5b^2 = -5K$. The first few terms of the trace sequence $\{l_{2,n}\}_{n\in\mathbb{Z}}$ are as follows:

n	...	-3	-2	-1	**0**	**1**	2	3	4	...
$l_{2,n}$	...	7a-4b	-4a+3b	3a-b	**-a+2b**	**2a+b**	a+3b	3a+4b	4a+7b	...

Remark 2.11. If $l_{2,n} = k_1a + k_2b$ for $n > 0$, then we have

$$l_{2,-n+1} = (k_2a - k_1b)(-1)^n.$$

2.3. Binet's formula, identities and generating function

The characteristics equation for the second order linear difference equation (2.1) is given by

$$x^2 = x + 1. \tag{2.8}$$

Equation (2.8) has two real roots, $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$, which satisfy

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1 - \alpha_2 = \sqrt{5}, \quad \alpha_1\alpha_2 = -1 \quad \text{and} \quad \frac{\alpha_1}{\alpha_2} = \frac{3+\sqrt{5}}{-2}. \tag{2.9}$$

And from the theory of difference equation we know that the general term of the Eqn. (2.1) can be expressed as:

$$t_{2,n} = c_1\alpha_1^n + c_2\alpha_2^n, \tag{2.10}$$

where c_1 and c_2 are arbitrary constants (to be evaluated) and α_1 and α_2 are characteristics roots.

Theorem 2.12 (Binet's formula)**.** *For $n \geq 0$, we have*

$$t_{2,n} = \frac{-A\alpha_1^n + B\alpha_2^n}{\sqrt{5}}, \tag{2.11}$$

where $A = a\alpha_2 - b$ and $B = a\alpha_1 - b$.

Proof. To establish the result, we eliminate arbitrary constants c_1 and c_2 from Eqn. (2.10). Now, putting the values of α_1 and α_2 in Eqn. (2.10), we get

$$t_{2,n} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2.12}$$

To determine the values of c_1 and c_2, we set $t_{2,0} = a$ and $t_{2,1} = b$ in Eqn. (2.12). Therefore,

$$t_{2,0} = a = c_1 + c_2 \quad \text{and} \quad t_{2,1} = b = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\implies b = \frac{1}{2}[a + \sqrt{5}(c_1 - c_2)],$$

which gives $c_1 + c_2 = a$ and $c_1 - c_2 = (2b-a)/\sqrt{5}$ and on solving we get

$$c_1 = \frac{a\sqrt{5}-(a-2b)}{2\sqrt{5}} \quad \text{and} \quad c_2 = \frac{a\sqrt{5}+(a-2b)}{2\sqrt{5}}.$$

Thus, from Eqn. (2.12), we have

$$\begin{aligned} t_{2,n} &= \frac{1}{2\sqrt{5}}\left[(a\sqrt{5}-(a-2b))\left(\frac{1+\sqrt{5}}{2}\right)^n + (a\sqrt{5}+(a-2b))\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ &= \frac{1}{\sqrt{5}}[-A\alpha_1^n + B\alpha_2^n] \end{aligned}$$

as required. □

Theorem 2.13. *For $n \in \mathbb{N}$, we have*

$$t_{2,-n} = (-1)^n \frac{-A\alpha_2^n + B\alpha_1^n}{\sqrt{5}}.$$

Proof. Replacing n by $-n$ in the Binet's formula (2.11), we get

$$\begin{aligned} t_{2,-n} &= \frac{-A\alpha_1^{-n} + B\alpha_2^{-n}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\frac{-A}{\alpha_1^n} + \frac{B}{\alpha_2^n}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{-A\alpha_2^n + B\alpha_1^n}{\alpha_1^n\alpha_2^n}\right) \\ &= \frac{-A\alpha_2^n + B\alpha_1^n}{\sqrt{5}(-1)^n} = (-1)^n\frac{-A\alpha_2^n + B\alpha_1^n}{\sqrt{5}} \quad (\text{using } \alpha_1\alpha_2 = -1) \end{aligned}$$

as required. □

Theorem 2.14 (Catalan's identity)**.** *For the sequence $\{t_{2,n}\}$, we have*

$$t_{2,n-r}t_{2,n+r} - t_{2,n}^2 = \frac{(-1)^n(b^2-a^2-ab)}{2^r.5}[2^{r+1} - (\sqrt{5}-3)^r - (-\sqrt{5}-3)^r].$$

Proof. Using the Binet's formula (2.11), we write

$$\begin{aligned} &t_{2,n-r}t_{2,n+r} - t_{2,n}^2 \\ &= \left(\frac{-A\alpha_1^{n-r} + B\alpha_2^{n-r}}{\sqrt{5}}\right)\left(\frac{-A\alpha_1^{n+r} + B\alpha_2^{n+r}}{\sqrt{5}}\right) - \left(\frac{-A\alpha_1^n + B\alpha_2^n}{\sqrt{5}}\right)^2 \\ &= \frac{1}{5}\left[AB(2\alpha_1^n\alpha_2^n - \alpha_1^{n-r}\alpha_2^{n+r} - \alpha_1^{n+r}\alpha_2^{n-r})\right] \\ &= \frac{1}{5}AB\alpha_1^n\alpha_2^n\left[(2 - \alpha_1^{-r}\alpha_2^r - \alpha_1^r\alpha_2^{-r})\right] \\ &= \frac{AB\alpha_1^n\alpha_2^n}{5}\left[2 - \left(\frac{3-\sqrt{5}}{-2}\right)^r - \left(\frac{3+\sqrt{5}}{-2}\right)^r\right] \quad (\text{using } (2.9)) \end{aligned}$$

$$= \frac{(-1)^n(b^2-a^2-ab)}{2^r.5}[2^{r+1}-(\sqrt{5}-3)^r-(-\sqrt{5}-3)^r]$$

as required. □

Corollary 2.15 (Cassini's identity). *For the sequence $\{t_{2,n}\}_{n\in\mathbb{N}}$, we have*

$$t_{2,n-1}t_{2,n+1}-t_{2,n}^2=(-1)^n(b^2-a^2-ab).$$

Theorem 2.16 (d'Ocagne's identity). *For positive integers r and n, we have*

$$t_{2,n}t_{2,r+1}-t_{2,n+1}t_{2,r}=\frac{(b^2-a^2-ab)}{\sqrt{5}}[\alpha_1^n\alpha_2^r-\alpha_1^r\alpha_2^n].$$

Proof. Using the Binet's formula (2.11), we write

$$\begin{aligned}
&t_{2,n}t_{2,r+1}-t_{2,n+1}t_{2,r}\\
&=\left(\frac{-A\alpha_1^n+B\alpha_2^n}{\sqrt{5}}\right)\left(\frac{-A\alpha_1^{r+1}+B\alpha_2^{r+1}}{\sqrt{5}}\right)\\
&\quad-\left(\frac{-A\alpha_1^{n+1}+B\alpha_2^{n+1}}{\sqrt{5}}\right)\left(\frac{-A\alpha_1^r+B\alpha_2^r}{\sqrt{5}}\right)\\
&=\frac{AB}{5}(\alpha_1^{n+1}\alpha_2^r+\alpha_2^{n+1}\alpha_1^r-\alpha_1^n\alpha_2^{r+1}-\alpha_1^{r+1}\alpha_2^n)\\
&=\frac{AB}{5}[\alpha_1^n\alpha_2^r(\alpha_1-\alpha_2)-\alpha_1^r\alpha_2^n(\alpha_1-\alpha_2)]\\
&=\frac{AB}{5}[(\alpha_1^n\alpha_2^r-\alpha_1^r\alpha_2^n)(\alpha_1-\alpha_2)] \quad \text{(substituting the value of A and B)}\\
&=\frac{(b^2-a^2-ab)}{\sqrt{5}}[\alpha_1^n\alpha_2^r-\alpha_1^r\alpha_2^n] \quad (\text{using } \alpha_1-\alpha_2=\sqrt{5})
\end{aligned}$$

as required. □

Now, we aim to give the generating function for $\{t_{2,n}\}$ and $\{l_{2,n}\}$ sequences in terms of a and b.

Generating function

Let $g(x)=\sum_{n=0}^{\infty}t_{2,n}x^n$ be a generating function for the sequence $\{t_{2,n}\}$. Now, multiplying Eqn. (2.1) by x^{n+2} and then taking summation over 0 to ∞, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}x^{n+2}t_{n+2}-\sum_{n=0}^{\infty}x^{n+2}t_{n+1}-\sum_{n=0}^{\infty}x^{n+2}t_n=0\\
&\Longrightarrow (g(x)-t_0-t_1x)-(g(x)-t_0)x-g(x)x^2=0\\
&\Longrightarrow g(x)(1-x-x^2)-(t_0+t_1x-t_0x)=0\\
&\Longrightarrow g(x)=\frac{a+(b-a)x}{(1-x-x^2)}. \qquad (2.13)
\end{aligned}$$

Theorem 2.17. *Let $q(x)$ be the generating function for trace sequence $\{l_{2,n}\}$ (2.6), then we have*

$$q(x) = -g(x) + 2\left(\frac{g(x) - a}{x}\right).$$

Proof. Lat $A = -a + 2b$ and $B = 2a + b$ (initial value of trace sequence), then in Eqn. (2.13) replace a by A and b by B, we get

$$\begin{aligned} q(x) &= \frac{A + (B - A)x}{(1 - x - x^2)} = \frac{(-a + 2b) + (2a + b - (-a + 2b))x}{(1 - x - x^2)} \\ &= \frac{(-a + 2b) + (3a - b)x}{(1 - x - x^2)} \\ &= \frac{-a - (b - a)x}{(1 - x - x^2)} + 2\frac{[b + (a + b - b)x]}{(1 - x - x^2)} \\ &= -g(x) + 2\left(\frac{g(x) - a}{x}\right) \end{aligned}$$

as required. □

For $a = 0,\ b = 1$ and $a = 2,\ b = 1$, Eqn. (2.13) gives the generating function for Fibonacci and Lucas sequence, respectively.

3. The $\{t_{3,n}\}$ sequence and some properties

Let us consider the sequence $\{t_{3,n}\}_{n\geq 0}$ given by a third order linear difference equation as follows

$$t_{3,n+3} = t_{3,n+2} + t_{3,n+1} + t_{3,n} \quad \text{with} \quad t_{3,0} = a,\ t_{3,1} = b,\ t_{3,2} = c. \tag{3.1}$$

The recurrence relation (3.1) can also be extended in negative direction and it can be achieved by rearranging the relation as $t_{3,n} = t_{3,n+3} - t_{3,n+2} - t_{3,n+1},\quad n \leq 0$.

In particular for $a = b = 0,\ c = 1$, Eqn. (3.1) gives tribonacci sequence while for $a = 3,\ b = 1,\ \ c = 3$, same is known as trucas (Tribonacci-Lucas) sequence [8].

The first few terms of sequence $\{t_{3,n}\}$ are given in the following table:

Index (n)	$t_{3,n}$	Value	Index ($-n$)	$t_{3,-n}$	Value
0	$t_{3,0}$	a	0	$t_{3,0}$	a
1	$t_{3,1}$	b	-1	$t_{3,-1}$	$c - a - b$
2	$t_{3,2}$	c	-2	$t_{3,-2}$	$2b - c$
3	$t_{3,3}$	$a + b + c$	-3	$t_{3,-3}$	$2a - b$
4	$t_{3,4}$	$a + 2b + 2c$	-4	$t_{3,-4}$	$2c - 3a - 2b$
5	$t_{3,5}$	$2a + 3b + 4c$	-5	$t_{3,-5}$	$5b - 3c + a$
6	$t_{3,6}$	$4a + 6b + 7c$	-6	$t_{3,-6}$	$4a - 4b + c$

The matrix representation corresponding to Eqn. (3.1) is given by a square matrix $T_3^{(n)}$ of order 3 defined as

$$T_3^{(n)} = \begin{bmatrix} t_{3,n+2} & t_{3,n+1}+t_{3,n} & t_{3,n+1} \\ t_{3,n+1} & t_{3,n}+t_{3,n-1} & t_{3,n} \\ t_{3,n} & t_{3,n-1}+t_{3,n-2} & t_{3,n-1} \end{bmatrix} \text{with } T_3^{(0)} = \begin{bmatrix} c & a+b & b \\ b & c-b & a \\ a & b-a & c-a-b \end{bmatrix} \quad (3.2)$$

and the determinant of $T_3^{(0)}$ is given as

$$\det(T_3^{(0)}) = a^3 + 2a^2b + a^2c + 2ab^2 - 2\,a\,b\,c - a\,c^2 + 2\,b^3 - 2\,b\,c^2 + c^3 \ (= K, \text{say}).$$

Theorem 3.1. *Let $\{f_{3,k}\}_{n\geq 0}$ be tribonacci sequence [A000073] with initial values $0, 0, 1,$ then*

$$t_{3,n} = b(f_{3,n+1} - f_{3,n}) + af_{3,n-1} + cf_{3,n}, \quad \forall n \in \mathbb{Z}.$$

Proof. We prove it using mathematical induction on n. For $n = 0$, the result obviously holds. For $n = 1$, we have

$$t_{3,1} = b(f_{3,2} - f_{3,1}) + af_{3,0} + cf_{3,1} = b + a0 + c0 = b.$$

Now assuming the result is true for $n = k$. For $n = k + 1$, we write

$$\begin{aligned} t_{k+1} &= t_k + t_{k-1} + t_{k-2} \\ &= [b(f_{k+1} - f_k) + af_{k-1} + cf_k] + [b(f_k - f_{k-1}) + af_{k-2} + cf_{k-1}] \\ &\quad + [b(f_{k-1} - f_{k-2}) + af_{k-3} + cf_{k-2}] \\ &= b(f_{k+1} - f_{k-2}) + a(f_{k-1} + f_{k-2} + f_{k-3}) + c(f_k + f_{k-1} + f_{k-2}) \\ &= b(f_{k+2} - f_{k+1}) + af_k + cf_{k+1} \qquad \text{(using tribonacci sequence)} \end{aligned}$$

as required. □

Theorem 3.2. *Let $T_3^{(0)}$ be the initial matrix defined in Eqn. (3.2) and Q_3^n be tribonacci matrix, then we have $T_3^{(n)} = Q_3^n T_3^{(0)}$, $\forall n \in \mathbb{Z}$.*

Proof. It can be easily proved using mathematical induction on n and Theorem 3.1. □

Corollary 3.3. *For $n \in \mathbb{N}$, we have, $T_3^{(n)} = Q_3 T_3^{(n-1)} = Q_3^{-1} T_3^{(n+1)}$.*

Remark 3.4. Matrices Q_3^n and $T_3^{(0)}$ commutes i.e. $Q_3^n T_3^{(0)} = T_3^{(0)} Q_3^n, \quad \forall n \in \mathbb{Z}$.

Theorem 3.5. *For recursive matrix $T_3^{(n)}$, we write*

$$T_3^{(n)} T_3^{(-n)} = (T_3^{(0)})^2, \quad \forall n \in \mathbb{Z}.$$

Proof. Using definition of $T_3^{(n)}$, we have

$$\begin{aligned} T_3^{(n)}T_3^{(-n)} &= Q_3^n T_3^{(0)} Q_3^{-n} T_3^{(0)} \\ &= Q_3^n Q_3^{-n} T_3^{(0)} T_3^{(0)} = I T_3^{(0)} T_3^{(0)} = (T_3^{(0)})^2 \end{aligned}$$

as required. □

Remark 3.6. Determinant of $T_3^{(n)}$ is invariant of n, i.e. $\det(T_3^{(n)}) = \det(T_3^{(0)}) = K$. Since by the properties of determinant, we write

$$\begin{aligned} \det(T_3^{(n)}) &= \det(Q_3^n T_3^{(0)}) = \det(Q_3^n)\det(T_3^{(0)}) \\ &= (-1)^{2n}\det(T_3^{(0)}) = \det(T_3^{(0)}) = K. \end{aligned}$$

Thus, $T_3^{(n)}$ is invertible if and only if $T_3^{(0)}$ is invertible, so for the existence of inverse of $T_3^{(n)}$, we consider only those values of a, b, c such that $\det(T_3^{(0)}) \neq 0$.

Example 3.7 (Tribonacci)**.** Let $a = b = 0$ and $c =1$ then $\det(T_3^{(n)}) = 1$.

Example 3.8 (Trucas)**.** Let $a = 3, b = 1$ and $c = 3$ then $\det(T_3^{(n)}) = 44$.

Remark 3.9. $\mathrm{Inv}(T_3^{(n)}) = T_3^{(-n)} H^{-1}$ provided $\det(T_3^{(0)}) \neq 0$, where $H = (T_3^{(0)})^2$.

3.1. Matrix representation for sequence of traces

The Lucas sequence of order 3 (also known as trucas, ref. A001644, A007486) is given by following recurrence relation

$$l_{3,n+3} = l_{3,n+2} + l_{3,n+1} + l_{3,n}, \quad \text{with} \quad l_{3,0} = 3,\ l_{3,1} = 1,\ l_{3,2} = 3. \tag{3.3}$$

In terms of tribonacci sequence, trucas is given by $l_{3,n} = trace(Q_3^n) = f_{3,n+2} + f_{3,n} + 2f_{3,n-1}$. Now, redefining the trucas (3.3) for $\{t_{3,n}\}$ sequence with the relation

$$l_{3,n} = \mathrm{trace}(T_3^{(n)}).$$

Since $\mathrm{trace}(T_3^{(n)}) = t_{3,n+2} + t_{3,n} + 2t_{3,n-1}$, so from Theorem 3.1, we have

$$\begin{aligned} \mathrm{trace}(T_3^{(n)}) &= [b(f_{n+3} - f_{n+2}) + af_{n+1} + cf_{n+2}] + [b(f_{n+1} - f_n) + af_{n-1} + cf_n] \\ &\quad + 2[b(f_n - f_{n-1}) + af_{n-2} + cf_{n-1}] \\ &= b(f_{3,n+3} + f_{3,n+1} + f_{3,n} - f_{3,n+2} - 2f_{3,n-1}) \\ &\quad + a(f_{3,n+1} + f_{3,n-1} + 2f_{3,n-2}) + c(f_{3,n+2} + f_{3,n} + 2f_{3,n-1}) \\ &= 2b(f_{3,n+3} - f_{3,n+2} - f_{3,n-1}) + al_{3,n-1} + cl_{3,n}. \end{aligned} \tag{3.4}$$

Remark 3.10. For $a = b = 0,\ c = 1$, Eqn. (3.4) gives the standard trucas sequence.

The corresponding matrix sequence $\{M_3^{(n)}\}$ for the sequence $\{l_{3,n}\}$ is given by

$$M_3^{(n)} = \begin{bmatrix} l_{3,n+2} & l_{3,n+1}+l_{3,n} & l_{3,n+1} \\ l_{3,n+1} & l_{3,n}+l_{3,n-1} & l_{3,n} \\ l_{3,n} & l_{3,n-1}+l_{3,n-2} & l_{3,n-1} \end{bmatrix}.$$

Theorem 3.11. *Let $L_3^{(0)}$ be the initial trucas matrix (it can be obtained by putting $a=3,\ b=1,\ c=3$ in $T_3^{(0)}$ in Eqn. (3.2)), then we have*

$$M_3^{(n)} = T_3^{(n)} L_3^{(0)}. \tag{3.5}$$

Proof. It can be easily proved with mathematical induction on n. $\square$

Theorem 3.12. *If K is determinant of $T_3^{(0)}$, then $\det(M_3^{(n)}) = 44K$.*

Proof. Using properties of the determinant and Eqn. (3.5), we have

$$\begin{aligned} \det(M_3^{(n)}) &= |T_3^{(n)} L_3^{(0)}| = |T_3^{(n)}||L_3^{(0)}| = |Q_3^n||T_3^{(0)}||L_3^{(0)}| \\ &= (-1)^{2n} K 44 = 44K \end{aligned}$$

as required. $\square$

Thus, it is concluded that if $T_3^{(n)}$ is invertible implies inverse for $M_3^{(n)}$ exists for all $n \in \mathbb{Z}$, i.e. $M_3^{(n)}$ is invertible if and only if $T_3^{(0)}$ is invertible.

Generating function

Let $g(x) = \sum_{n=0}^{\infty} t_{3,n} x^n$ be a generating function for $\{t_{3,n}\}$ sequence. On multiplying each term of Eqn. (3.1) with x^{n+3} and then taking summation over $n=0$ to ∞, we get

$$\sum_{n=0}^{\infty} x^{n+3} t_{n+3} - \sum_{n=0}^{\infty} x^{n+3} t_{n+2} - \sum_{n=0}^{\infty} x^{n+3} t_{n+1} - \sum_{n=0}^{\infty} x^{n+3} t_n = 0.$$

Thus, we have

$$\begin{aligned} &(g(x) - t_0 - t_1 x - t_2 x^2) - (g(x) - t_0 - t_1 x)x - (g(x) - t_0)x^2 - g(x)x^3 = 0 \\ &\implies g(x)(1 - x - x^2 - x^3) - t_0(1 - x - x^2) - t_1(x - x^2) - t_2 x^2 = 0 \\ &\implies g(x) = \frac{a(1-x-x^2) + b(x-x^2) + cx^2}{(1-x-x^2-x^3)} \\ &\implies g(x) = \frac{a + (b-a)x + (c-b-a)x^2}{(1-x-x^2-x^3)}. \end{aligned} \tag{3.6}$$

In particular, setting $a=b=0,\ c=1$ and $a=3,\ b=1,\ c=3$ in Eqn. (3.6) give the generating functions for tribonacci and trucas sequence, respectively.

3.2. Binet's formula

To establish any identity involving nth term of the sequence, the Binet's formula plays an important role. Here, we derive an explicit formula for generalized third order sequences $\{t_{3,n}\}$.

Let us assume that the three characteristic roots of difference Eqn. (3.1) are r_1, r_2 and r_3. Clearly, r_1, r_2 and r_3 satisfy the relations

$$r_1 + r_2 + r_3 = 1, \quad r_1r_2 + r_2r_3 + r_3r_1 = -1 \quad \text{and} \quad r_1r_2r_3 = 1. \tag{3.7}$$

Theorem 3.13 (Binet's formula)**.** *For $n \geq 0$, we have*

$$t_{3,n} = \frac{P{r_1}^n + Q{r_2}^n}{r_1 - r_2} + R{r_3}^n, \tag{3.8}$$

where $P = (r_2 - r_3)R - ar_2 + b$, $Q = (r_3 - r_1)R + ar_1 - b$, $R = \frac{c-(r_1+r_2)b+r_1r_2a}{r_3^2-(r_1+r_2)r_3+r_1r_2}$.

Proof. Using the relation between roots and the coefficients of a polynomial, rewriting Eqn. (3.1) as

$$t_{k,n+3} = (r_1 + r_2 + r_3)t_{k,n+2} - (r_1r_2 + r_2r_3 + r_3r_1)t_{k,n+1} + r_1r_2r_3t_{k,n}.$$

It can also be written as,

$$\begin{aligned} &t_{k,n+3} - (r_1 + r_2)t_{k,n+2} + (r_1r_2)t_{k,n+1} \\ &= r_3t_{k,n+2} - r_3(r_1 + r_2)t_{k,n+1} + r_1r_2r_3t_{k,n} \\ &= r_3[t_{k,n+2} - (r_1 + r_2)t_{k,n+1} + r_1r_2t_{k,n}]. \end{aligned} \tag{3.9}$$

Similarly, we have

$$t_{k,n+2} - (r_1 + r_2)t_{k,n+1} + r_1r_2t_{k,n} = r_3[t_{k,n+1} - (r_1 + r_2)t_{k,n} + r_1r_2t_{k,n-1}]. \tag{3.10}$$

Substitute Eqn. (3.10) in Eqn. (3.9), we get

$$t_{k,n+3} - (r_1 + r_2)t_{k,n+2} + (r_1r_2)t_{k,n+1} = r_3^2[t_{k,n+1} - (r_1 + r_2)t_{k,n} + r_1r_2t_{k,n-1}].$$

Continuing this substitution process, we obtain a recursive relation

$$t_{k,n+3} - (r_1 + r_2)t_{k,n+2} + (r_1r_2)t_{k,n+1} = r_3^{n+1}[t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0}].$$

Now, divide both side of the above equation by r_3^{n+3}, we get

$$\frac{t_{k,n+3}}{r_3^{n+3}} - \frac{(r_1 + r_2)}{r_3^{n+3}}t_{k,n+2} + \frac{(r_1r_2)}{r_3^{n+3}}t_{k,n+1} = \frac{1}{r_3^2}[t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0}]. \tag{3.11}$$

For simplicity, consider $t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0} = K$ and $\frac{t_{k,n+3}}{r_3^{n+3}} = H_{k,n+3}$ in Eqn. (3.11), we write

$$H_{k,n+3} - \frac{(r_1 + r_2)}{r_3}H_{k,n+2} + \frac{(r_1r_2)}{r_3^2}H_{k,n+1} = \frac{1}{r_3^2}K, \tag{3.12}$$

which is a second order non-homogeneous linear difference equation and its solution is given by $H_{k,n} = H(C)+H(P)$, where $H(C)$ represents the solution corresponding homogeneous part and $H(P)$ is particular solution.

Since, roots of the characteristic equation for homogeneous part of Eqn. (3.12) are $\alpha_1 = \frac{r_1}{r_3}$ and $\alpha_2 = \frac{r_2}{r_3}$. So, the solution for homogeneous part is given by

$$H(C) = A\left(\frac{r_1}{r_3}\right)^n + B\left(\frac{r_2}{r_3}\right)^n, \text{ where A and B are arbitrary constants.}$$

Furthermore, the non-homogeneous part of Eqn. (3.12) is a constant, so particular solution is also a constant and it is given by $H(P) = \frac{K}{r_3^2-(r_1+r_2)r_3+r_1r_2}$. Thus, general solution of Eqn. (3.12) is

$$H_{k,n} = H(C) + H(P) = A\left(\frac{r_1}{r_3}\right)^n + B\left(\frac{r_2}{r_3}\right)^n + \frac{K}{r_3^2 - (r_1 + r_2)r_3 + r_1r_2}.$$

Replacing $H_{k,n}$ by $\dfrac{t_{k,n}}{r_3^n}$ and K by $t_{k,2}-(r_1+r_2)t_{k,1}+r_1r_2t_{k,0}$ in the above equation, we get

$$t_{k,n} = Ar_1{}^n + Br_2{}^n + r_3{}^n R, \quad \text{where } R = \left[\frac{t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0}}{r_3^2 - (r_1 + r_2)r_3 + r_1r_2}\right]. \tag{3.13}$$

Hence, using initial values from Eqn. (3.1) in Eqn. (3.13), we have

$$A = \frac{(r_2 - r_3)R - ar_2 + b}{r_1 - r_2} \quad \text{and} \quad B = \frac{(r_3 - r_1)R + ar_1 - b}{r_1 - r_2},$$

where $R = \frac{c-(r_1+r_2)b+r_1r_2a}{r_3^2-(r_1+r_2)r_3+r_1r_2}$, as required. □

Remark 3.14. Setting $a = b = 0$ and $c = 1$ in Eqn. (3.8) gives the Binet's formula for the standard tribonacci sequence (the Fibonacci sequence of order three).

Remark 3.15. Setting $a = 3,\ b = 1$ and $c = 3$ in Eqn. (3.8) gives the Binet's formula for the Tribonacci-Lucas sequence.

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Units of the semisimple group algebras $\mathcal{F}_q SL(2,8)$ and $\mathcal{F}_q SL(2,9)$

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Abstract. In this paper, we characterize the structure of the unit group of the semisimple group algebras $\mathcal{F}_q SL(2,8)$ and $\mathcal{F}_q SL(2,9)$ of the special linear groups of 2×2 matrices with determinant 1 over the finite fields of order 8 and 9, respectively.

Keywords: Unit group, Group algebra, Special linear group, Wedderburn Decomposition

AMS Subject Classification: 16U60, 20C05

1. Introduction

The group algebra of the finite group G over the finite field $\mathcal{F}_q$ is denoted by $\mathcal{F}_q G$. Let q and p be the order and characteristics of the finite field $\mathcal{F}_q$, respectively and $q = p^k$. Let $\mathcal{U}(\mathcal{F}_q G)$ be the group of units of the group algebra $\mathcal{F}_q G$. Group theory frequently runs into the issues with unit group of the group algebra. The characterization of the unit groups is crucial for a number of applications, including the study of the isomorphism problem [14], one of the most significant research problems in the theory of group algebras, the development of convolutional codes in group algebra (see [5, 9]) and other applications. The structure of the unit

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group of the semisimple group algebra $\mathcal{F}_qG$ has been extensively studied (see [3, 4, 10, 12–18, 20, 22]). The study by Bakshi et al. [4], in which the unit groups of the semisimple group algebras of all metabelian groups are studied, is one of the most significant ones in this field. As a result, the majority of research in this field focuses on understanding the unit group of non-metabelian group algebras. The unit groups of the group algebras of non-metabelian groups up to order 72 were described by Mittal et al. in [16]. Further, Sharma et al. determined the unit group of the semisimple group algebra (SGA) of the respective groups $SL(2,3)$ (special linear group over the finite field of 3 elements) and $SL(2,5)$ (See [11] and [19]). In continuation, Sivaranjani et al. [21] determined the unit group of the SGA of the group $SL(2,7)$. In addition, Arvind et al. [1] investigated the unit group of the SGA of the group $SL(3,\mathbb{Z}_2)$. The main objective of this paper is to derive the unit group of the SGA of the groups $SL(2,8)$ and $SL(2,9)$, respectively. We notice that the difficulty of exactly identifying the unit group of the SGA increases as the size of the group increases. One may refer [15, 17] for some of the recent works in this area. Our first goal in determining the unit group is to infer the Wedderburn decomposition (WD) of $\mathcal{F}_qSL(2,8)$ and $\mathcal{F}_qSL(2,9)$, respectively. Further, it is easy to derive the unit group from the WD. The rest of this paper is structured as follows. The prerequisites for the article are covered in Section 2. In sections 3 and 4, we deduce the unit group of the group algebras $\mathcal{F}_qSL(2,8)$ and $\mathcal{F}_qSL(2,9)$ in the form of theorems 3.1 and 4.1, respectively. Section 5 concludes the paper.

2. Preliminaries

Throughout this paper, $SL(n,r)$ denotes the special linear group of $n \times n$ matrices with determinant 1 over the finite field of order r. The order of $SL(n,r)$ is given by

$$(r^n-1)(r^n-r)\cdots(r^n-r^{n-1})(r-1)^{-1}.$$

Next, we discuss some notations and results from [7]. Let $J(\mathcal{F}_qG)$ denote the Jacobson radical of $\mathcal{F}_qG$. Let s be the least common multiple of the orders of p-regular elements of group G and let η be the primitive s^{th} root of unity over a finite field $\mathcal{F}$. Let $T_{G,\mathcal{F}} = \{t : \eta \to \eta^t \text{ is an automorphism of } \mathcal{F}(\eta) \text{ over } \mathcal{F}\}$. Since the Galois group $\mathrm{Gal}(\mathcal{F}(\eta):\mathcal{F})$ is cyclic, for any $\sigma \in \mathrm{Gal}(\mathcal{F}(\eta):\mathcal{F})$, there exists a positive integer s such that $\sigma(\eta) = \eta^s$. For any p-regular element $g \in G$ (i.e., p does not divide order of g), we define $\gamma_g = \sum h$, where h runs over all the elements in the conjugacy class C_g of g. The cyclotomic $\mathcal{F}$-class of γ_g is defined as $S\mathcal{F}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G,\mathcal{F}}\}$. The following theorem characterizes the set $T_{G,\mathcal{F}}$.

Theorem 2.1 ([16, Theorem 2.3]). *Let $\mathcal{F}$ be a finite field with prime power order d such that $\gcd(d,s)=1$ and $e = \mathrm{order}_s(d)$ is the multiplicative order of d modulo s, then $T_{G,\mathcal{F}} = \{1, d, \dots, d^{e-1}\} \mod s$.*

To uniquely identify the Wedderburn decomposition (WD) of the group algebra, the following six results will play an important role.

Proposition 2.2 ([7, Proposition 1.2])**.** *The number of non isomorphic simple components of $\mathcal{F}G/J(\mathcal{F}G)$ is equal to the number of cyclotomic $\mathcal{F}$-classes in G.*

Theorem 2.3 ([7, Theorem 1.3])**.** *Assume that G has t cyclotomic $\mathcal{F}$-classes and* $\mathrm{Gal}(\mathcal{F}(\eta):\mathcal{F}))$ *is a cyclic group, then $|S_i| = [\mathcal{F}_i : \mathcal{F}]$ with appropriate index ordering if $S_1, S_2, \cdots, S_t$ are the cyclotomic $\mathcal{F}$-classes of G and $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$ are the simple components of $Z(\mathcal{F}G/J(\mathcal{F}G))$.*

Proposition 2.4 ([14, Proposition 3.6.11])**.** *Let G' be the commutator subgroup of G and let $\mathcal{F}G$ be a semisimple group algebra, then*

$$\mathcal{F}G \simeq \mathcal{F}(G/G') \oplus \triangle(G, G'),$$

where $\mathcal{F}(G/G')$ is the sum of all commutative simple components of $\mathcal{F}G$ and $\triangle(G, G')$ is the sum of all others.

Proposition 2.5 ([14, Proposition 3.6.7])**.** *Let N be a normal subgroup of G and let $\mathcal{F}G$ be a semisimple group algebra (SGA), then*

$$\mathcal{F}G \simeq \mathcal{F}(G/N) \oplus \triangle(G, N),$$

where $\triangle(G, N)$ is an ideal of $\mathcal{F}G$ generated by the set $\{n - 1 : n \in N\}$.

Proposition 2.6 ([6, Proposition 1])**.** *Let $\mathcal{F}G$ be a finite SGA, where characteristics of $\mathcal{F}$ is p. Let $\mathcal{F}G \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i)$, where $\mathcal{F}_i$ are finite extensions of $\mathcal{F}$ and r is a positive integer. Then p does not divide any of the n_i.*

Lemma 2.7 ([24])**.** *Let p_1 and p_2 be two primes. Let $\mathcal{F}_{q_1}$ be a field with $q_1 = p_1^{k_1}$ elements and let $\mathcal{F}_{q_2}$ be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $\mathcal{F}_{q_1}G, \mathcal{F}_{q_2}G$ be semisimple. Suppose that*

$$\mathcal{F}_{q_1}G \cong \oplus_{i=1}^{t}\mathbf{M}(n_i, \mathcal{F}_{q_1}),\ n_i \geq 1$$

and $M\big(n, \mathcal{F}_{q_2^r}\big)$ is a Wedderburn component of the group algebra $\mathcal{F}_{q_2}G$ for some $r \geq 1$ and any positive integer n, i.e.,

$$\mathcal{F}_{q_2}G \cong \oplus_{i=1}^{s-1}\mathbf{M}(m_i, \mathcal{F}_{q_{2,i}}) \oplus \mathbf{M}(n, \mathcal{F}_{q_2^r}),\ m_i \geq 1.$$

Here $\mathcal{F}_{q_{2,i}}$ is a field extension of $\mathcal{F}_{q_2}$. Then $M(n, \mathcal{F}_{q_1})$ must be a Wedderburn component of the group algebra $\mathcal{F}_{q_1}G$ and it appears atleast r times in the WD of $\mathcal{F}_{q_1}G$.

Proposition 2.8 ([1, Corollary 3.8])**.** *Let $\mathcal{F}G$ be a finite SGA, where characteristics of $\mathcal{F}$ is p. If there exists an irreducible representations of degree n over $\mathcal{F}$, then one of the Wedderburn component of $\mathcal{F}G$ is $\mathbf{M}(n, \mathcal{F})$.*

3. Unit group of $\mathcal{F}_q SL(2,8)$

Let $G_1 := SL(2,8)$. Clearly, the order of G_1 is 504. The group algebra $\mathcal{F}_qG_1$ is semi simple for $p \neq 2,3,7$ by Maschke's theorem [14]. Also, One can note that the degrees of irreducible representations of G_1 are $1,7,8$ and 9 whenever $|S\mathcal{F}_q(\gamma_g)| = 1, \forall g \in G_1$ (see [23]). The group G_1 has 9 conjugacy classes (let the representative of these classes be denoted by g_i for $i = 1,\dots,9$). The representatives (R) of the conjugacy classes, sizes (S) and the orders (O) of representatives are tabulated below.

R	I_2	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & x^2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}$	$\begin{bmatrix} x & 0 \\ 0 & \beta \end{bmatrix}$	$\begin{bmatrix} x^2 & 0 \\ 0 & \alpha+1 \end{bmatrix}$	$\begin{bmatrix} x+1 & 0 \\ 0 & \alpha \end{bmatrix}$
S	1	63	56	56	56	56	72	72	72
O	1	2	3	9	9	9	7	7	7

Here I_2 is 2×2 identity matrix, x is the generator of multiplicative group of finite field of order 8, $\alpha = x^2 + x$ and $\beta = x^2 + 1$. Also, G_1 can be generated by two elements a and b, where

$$a = \begin{bmatrix} x & 0 \\ 0 & x^2+1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.1}$$

The exponent of G_1 is 126. In this section, we characterize the unit group of the group algebra $\mathcal{F}_qG_1$ for $p \neq 2,3,7$ such that the group algebra $\mathcal{F}_qG_1$ is semisimple and $q = p^k$. In the following theorems, $\mathcal{F}_i$ denotes the finite extensions of $\mathcal{F}_q$ and n_i, r are positive integers.

Theorem 3.1. *The unit group of $\mathcal{F}_qG_1$, where $q = p^k$ and $p \neq 2,3,7$ is given as follows:*
(1) *for* $p^k \equiv \{1,55,71,125\} \mod 126$, *we have*

$$U(\mathcal{F}_qG_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q)^4 \oplus GL(8,\mathcal{F}_q) \oplus GL(9,\mathcal{F}_q)^3.$$

(2) *for* $p^k \equiv \{13,29,41,43,83,85,97,113\} \mod 126$, *we have*

$$U(\mathcal{F}_qG_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q) \oplus GL(8,\mathcal{F}_q) \oplus GL(9,\mathcal{F}_q)^3 \oplus GL(7,\mathcal{F}_{q^3}).$$

(3) $p^k \equiv \{17,19,37,53,73,89,107,109\} \mod 126$, *we have*

$$U(\mathcal{F}_qG_1) \simeq \mathcal{F}_q^* \oplus GL(7,\mathcal{F}_q)^4 \oplus GL(8,\mathcal{F}_q) \oplus GL(9,\mathcal{F}_{q^3}).$$

(4) $p^k \equiv \{5, 11, 23, 25, 31, 47, 59, 61, 65, 67, 79, 95, 101, 103, 115, 121\} \mod 126$, *we have*

$$U(\mathcal{F}_qG_1) \simeq \mathcal{F}_q^* \oplus GL(7, \mathcal{F}_q) \oplus GL(8, \mathcal{F}_q) \oplus GL(7, \mathcal{F}_{q^3}) \oplus GL(9, \mathcal{F}_{q^3}).$$

Proof. The group algebra $\mathcal{F}_qG_1$ is semi simple, it follows from the Wedderburn decomposition theorem [14] that $\mathcal{F}_qG_1 \simeq \oplus_{i=1}^r \mathbf{M}(n_i, \mathcal{F}_i)$. The derived subgroup G_1' of G_1 is G_1 itself (i.e., G_1 is a perfect one). This accompanying with Proposition 2.2 gives

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{r-1} \mathbf{M}(n_i, \mathcal{F}_i), \quad n_i \geq 2. \tag{3.2}$$

Using Theorem 2.1, we construct the set $T_{G,\mathcal{F}}$ of group G_1 and divide the proof into the following 4 cases.
Case 1: $p^k \equiv \{1, 55, 71, 125\} \mod 126$. In this case, we note that the cardinality of cyclotomic $\mathcal{F}_q$-class of γ_g is 1, for all g in G_1. By employing this with Proposition 2.2 and Theorem 2.3, we further rewrite (3.2) as

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{8} \mathbf{M}(n_i, \mathcal{F}_q) \implies 503 = \sum_{i=1}^{8} n_i^2, \; n_i \geq 2. \tag{3.3}$$

We have discussed earlier in this section that the degrees of irreducible representations of G_1 are $1, 7, 8$ and 9, whenever $| S\mathcal{F}_q(\gamma_g) |= 1, \forall g \in G_1$. We note that there are 158 choices of n_i's fulfilling (4.3). The only choice that only contains $7, 8$ and 9 is $(7^4, 8, 9^3)$. Hence, the Wedderburn decomposition (WD) is

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q)^4 \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(9, \mathcal{F}_q)^3.$$

Case 2: $p^k \equiv \{13, 29, 41, 43, 83, 85, 97, 113\} \mod 126$. In this case, the cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } i = 1, 2, 3, 7, 8, 9, \quad S\mathcal{F}_q(\gamma_{g_4}) = \{\gamma_{g_4}, \gamma_{g_5}, \gamma_{g_6}\}.$$

By incorporating Proposition 2.2, we derive from (3.2) that

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{5} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_6, \mathcal{F}_{q^3}) \implies 503 = \sum_{i=1}^{5} n_i^2 + 3n_6^2, \; n_i \geq 2. \tag{3.4}$$

Due to Lemma 2.7, it follows from (3.4) that $n_i \geq 7$. Consequently, the possible choices of n_i's fulfilling (3.4) are $(7^4, 8, 9)$, $(7^3, 8, 10, 8)$, $(7, 8, 9^3, 7)$ and $(8^4, 10, 7)$. Again, Lemma 2.7 implies that $\mathbf{M}(10, \mathcal{F}_q)$ can not be a Wedderburn component. Therefore, we are only remaining with two choices of n_i's given by $(7^4, 8, 9)$ and $(7, 8, 9^3, 7)$. Next, to uniquely identify the correct choice, we show that $\mathbf{M}(9, \mathcal{F}_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 13$

and consider the following mapping from G_1 to $GL(9, \mathcal{F}_{13})$:

$$a \to \begin{bmatrix} 5 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 4 & 8 & 3 & 0 & 0 & 0 & 0 \\ 6 & 12 & 11 & 2 & 12 & 11 & 0 & 0 & 0 \\ 0 & 6 & 2 & 11 & 9 & 4 & 2 & 4 & 0 \\ 10 & 5 & 10 & 7 & 7 & 6 & 12 & 7 & 11 \\ 3 & 2 & 2 & 4 & 9 & 10 & 12 & 9 & 8 \\ 11 & 1 & 3 & 8 & 3 & 5 & 11 & 8 & 9 \\ 4 & 5 & 3 & 12 & 6 & 10 & 11 & 6 & 1 \end{bmatrix},$$

$$b \to \begin{bmatrix} 8 & 9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 12 & 2 & 4 & 3 & 1 & 12 & 0 & 0 \\ 3 & 10 & 1 & 4 & 6 & 0 & 8 & 1 & 11 \\ 2 & 5 & 3 & 8 & 1 & 10 & 0 & 10 & 2 \\ 9 & 8 & 5 & 6 & 2 & 8 & 1 & 5 & 11 \\ 6 & 1 & 5 & 0 & 9 & 2 & 8 & 8 & 0 \\ 12 & 4 & 0 & 10 & 8 & 12 & 6 & 7 & 10 \end{bmatrix}.$$

This mapping is a homomorphism from G_1 to $GL(9, \mathcal{F}_{13})$ (as a and b given in (3.1) generates G_1). It should be noted that this map is an irreducible representation of G_1 over $\mathcal{F}_{13}$. Therefore, according to Proposition 2.8, $\mathbf{M}(9, \mathcal{F}_{13})$ will always be a Wedderburn component of $\mathcal{F}_q G_1$. Hence, it follows that $(7, 8, 9^3, 7)$ is the only possible value for n_i. Hence, the WD is

$$\mathcal{F}_q G \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(9, \mathcal{F}_q)^3 \oplus \mathbf{M}(7, \mathcal{F}_{q^3}).$$

Case 3: $p^k \equiv \{17, 19, 37, 53, 73, 89, 107, 109\} \bmod 124$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } 1,2,3,4,5,6, \quad S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_8}, \gamma_{g_9}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (3.2) that

$$\mathcal{F}_q G_1 \simeq \mathcal{F}_q \bigoplus_{i=1}^{5} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_6, \mathcal{F}_{q^3}) \implies 503 = \sum_{i=1}^{5} n_i^2 + 3n_6^2, \ n_i \geq 2. \quad (3.5)$$

By proceeding on the similar lines of case 2, one can show that we need to deduce the unique choices among the 2 choices $(7^4, 8, 9)$ and $(7, 8, 9^3, 7)$. For this, we take

$p = 17$ and show that there are two distinct homomorphisms from G_1 to $GL(7, \mathcal{F}_{17})$. We consider the following mappings:

$$a \to \begin{bmatrix} 13 & 10 & 0 & 0 & 0 & 0 & 0 \\ 1 & 11 & 10 & 4 & 0 & 0 & 0 \\ 0 & 8 & 6 & 11 & 7 & 9 & 0 \\ 0 & 1 & 14 & 12 & 14 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 & 15 & 4 \\ 0 & 0 & 1 & 1 & 4 & 3 & 10 \\ 0 & 0 & 0 & 12 & 12 & 3 & 12 \end{bmatrix}, \quad b \to \begin{bmatrix} 8 & 14 & 16 & 0 & 0 & 0 & 0 \\ 9 & 2 & 6 & 13 & 4 & 0 & 0 \\ 0 & 5 & 14 & 1 & 6 & 15 & 4 \\ 0 & 2 & 3 & 11 & 1 & 13 & 15 \\ 0 & 1 & 12 & 14 & 4 & 3 & 12 \\ 0 & 0 & 12 & 8 & 10 & 14 & 10 \\ 0 & 0 & 1 & 9 & 1 & 11 & 16 \end{bmatrix},$$

$$a \to \begin{bmatrix} 14 & 14 & 0 & 0 & 0 & 0 & 0 \\ 13 & 16 & 0 & 4 & 0 & 0 & 0 \\ 16 & 12 & 12 & 15 & 0 & 0 & 0 \\ 5 & 8 & 9 & 0 & 15 & 7 & 2 \\ 10 & 9 & 13 & 13 & 9 & 1 & 12 \\ 9 & 1 & 6 & 3 & 13 & 15 & 0 \\ 1 & 5 & 11 & 8 & 2 & 14 & 2 \end{bmatrix}, \quad b \to \begin{bmatrix} 6 & 8 & 16 & 0 & 0 & 0 & 0 \\ 12 & 1 & 16 & 14 & 0 & 16 & 0 \\ 11 & 12 & 13 & 7 & 4 & 12 & 0 \\ 0 & 15 & 9 & 6 & 0 & 13 & 0 \\ 9 & 7 & 2 & 9 & 13 & 13 & 14 \\ 12 & 14 & 10 & 6 & 9 & 9 & 4 \\ 5 & 9 & 14 & 13 & 4 & 16 & 4 \end{bmatrix}.$$

These mappings are 2 irreducible representations of G_1 over $\mathcal{F}_{17}$. Therefore, Proposition 2.8 derives that $\mathbf{M}(7, \mathcal{F}_{17})^2$ is a summand of the group algebra $\mathcal{F}_{17}G_1$. Thus, the required choices of n_i's fulfilling (3.5) is $(7^4, 8, 9)$ Hence, the WD is

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q)^4 \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(9, \mathcal{F}_{q^3}).$$

Case 4: $p^k \equiv \{5, 11, 23, 25, 31, 47, 59, 61, 67, 79, 101, 103, 115, 121, 65, 95\} \bmod 124$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } 1, 2, 3, \ S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7}, \gamma_{g_8}, \gamma_{g_9}\}, S\mathcal{F}_q(\gamma_{g_4}) = \{\gamma_{g_4}, \gamma_{g_5}, \gamma_{g_6}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (3.2) that

$$\mathcal{F}_qG_1 \simeq (\mathcal{F}_q) \bigoplus_{i=1}^{2} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_3, \mathcal{F}_{q^3}) \oplus \mathbf{M}(n_4, \mathcal{F}_{q^3})$$

$$\implies 503 = \sum_{i=1}^{2} n_i^2 + 3(n_3^2 + n_4^2), \ n_i \geq 2. \tag{3.6}$$

By following the procedure as in case 1, we can show that the $n_i \geq 7$ in (3.6). Hence, the only possible choice of n_i's is $(7, 8, 7, 9)$, which means that

$$\mathcal{F}_qG_1 \simeq \mathcal{F}_q \oplus \mathbf{M}(7, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q) \oplus \mathbf{M}(7, \mathcal{F}_{q^3}) \oplus \mathbf{M}(9, \mathcal{F}_{q^3}).$$

This completes the proof. $\square$

4. Unit group of $\mathcal{F}_q SL(2,9)$

Let $G_2 := SL(2,9)$. The order of G_2 is 720. Since $p > 5$, it does not divide the order of G_2, the group algebra $\mathcal{F}_q G_2$ is semisimple. Also, from [8] one can note that G_2 has irreducible representations of degrees $1, 4, 5, 8, 9$ and 10 whenever $|S\mathcal{F}_q(\gamma_g)| = 1, \forall g \in G_2$. The group G_2 has 13 conjugacy classes and it is represented as $g_i's$. The representative of the conjugacy classes, size and the order of representatives are tabulated below:

R	S	O
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$	40	6
$\begin{bmatrix} 0 & 2x+2 \\ x+2 & 1 \end{bmatrix}$	40	6
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	1	2
$\begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$	40	3
$\begin{bmatrix} 0 & 2x+2 \\ x+2 & 2 \end{bmatrix}$	40	3
$\begin{bmatrix} 0 & 2 \\ 1 & 2x+2 \end{bmatrix}$	72	5
$\begin{bmatrix} 0 & 2 \\ 1 & x+2 \end{bmatrix}$	72	5
$\begin{bmatrix} 0 & 2 \\ 1 & x+1 \end{bmatrix}$	72	10
$\begin{bmatrix} 0 & 2 \\ 1 & 2x+1 \end{bmatrix}$	72	10
$\begin{bmatrix} 2x+2 & 0 \\ 0 & 2x+1 \end{bmatrix}$	90	8
$\begin{bmatrix} x & 0 \\ 0 & 2x \end{bmatrix}$	90	4
$\begin{bmatrix} x+2 & 0 \\ 0 & x+1 \end{bmatrix}$	90	8

Here x is the generator of multiplicative group of finite field of order 9. Also, G_2 can be generated by two elements a and b, where

$$a = \begin{bmatrix} x+1 & 0 \\ 0 & x+2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}. \tag{4.1}$$

In this section, we characterize the unit group of the group algebra $\mathcal{F}_q G_2$ for $p > 5$ such that the group algebra $\mathcal{F}_q G_2$ is semisimple and $q = p^k$. It is clear from the above table that the exponent of G_2 is 120.

Theorem 4.1. *The unit group of $\mathcal{F}_q G_2$ is as follows:*
(1) *for* $p^k \equiv \{1, 31, 41, 49, 71, 79, 89, 119\} \mod 120$*, we have*

$$U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(8,\mathcal{F}_q)^4 \oplus GL(9,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_q)^3.$$

(2) *for* $p^k \equiv \{7, 17, 23, 47, 73, 97, 103, 113\} \mod 120$*, we have*

$$U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(9,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_q)^3 \oplus GL(8,\mathcal{F}_{q^2})^2.$$

(3) $p^k \equiv \{11, 19, 29, 59, 61, 91, 101, 109\} \mod 120$*, we have*

$$U(\mathcal{F}_q G_2) \simeq \mathcal{F}_q^* \oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(9,\mathcal{F}_q) \oplus GL(8,\mathcal{F}_q)^4$$

$$\oplus\, GL(10,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_{q^2}).$$

(4) $p^k \equiv \{13,37,43,53,67,77,83,107\} \mod 120$ *, we have*

$$\begin{aligned}U(\mathcal{F}_qG_2) \simeq \mathcal{F}_q^* &\oplus GL(4,\mathcal{F}_q)^2 \oplus GL(5,\mathcal{F}_q)^2 \oplus GL(9,\mathcal{F}_q) \oplus GL(10,\mathcal{F}_q)\\ &\oplus GL(10,\mathcal{F}_{q^2}) \oplus GL(8,\mathcal{F}_{q^2})^2.\end{aligned}$$

Proof. It follows from the Wedderburn decomposition theorem that $\mathcal{F}_qG_2 \simeq \bigoplus_{i=1}^{r} \mathbf{M}(n_i,\mathcal{F}_i)$. Also, G_2 is a perfect group. This and Proposition 2.2 imply that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{r-1} \mathbf{M}(n_i,\mathcal{F}_i), \quad n_i \geq 2. \tag{4.2}$$

As in the previous theorem, we construct the set $T_{G,\mathcal{F}}$ of group G_2 and divide the proof into the following 4 cases.

Case 1: $p^k \equiv \{1,31,41,49,71,79,89,119\} \mod 120$. In this case, it can be verified that $|S\mathcal{F}_q(\gamma_g)| = 1, \forall g \in G_2$. By utilizing this along with Proposition 2.2, we further rewrite (4.2) as

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{12} \mathbf{M}(n_i,\mathcal{F}_q) \implies 719 = \sum_{i=1}^{12} n_i^2,\ n_i \geq 2. \tag{4.3}$$

Next, we consider the normal subgroup N of G_2 generated by $\left[\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right]$. One can observe that with $G_2/N \simeq A_6$. We recall from [2, Proposition 4.7] that

$$\mathcal{F}_qA_6 \simeq \mathcal{F}_q \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q) \oplus \mathbf{M}(8,\mathcal{F}_q)^2. \tag{4.4}$$

Utilizing (4.4) and Proposition 2.5 in (4.3) to derive that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q) \oplus \mathbf{M}(8,\mathcal{F}_q)^2 \bigoplus_{i=1}^{6} \mathbf{M}(n_i,\mathcal{F}_q) \tag{4.5}$$

with $360 = \sum_{i=1}^{6} n_i^2,\ n_i \geq 2$. We note that G_2 has irreducible representations of degrees $1,4,5,8,9$ and 10. This means n_i's in (4.5) are among the set $\{4,5,8,9,10\}$. Among all the possible choices of n_i's fulfilling $360 = \sum_{i=1}^{6} n_i^2$, the only choice that contains elements from the set $\{4,5,8,9,10\}$ is $(4^2,8^2,10^2)$. Hence, (4.5) implies that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4,\mathcal{F}_q)^2 \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(8,\mathcal{F}_q)^4 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q)^3.$$

Case 2: $p^k \equiv \{7,17,23,47,73,97,103,113\} \mod 120$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\},\ \text{for 1-6, 11-3},\ S\mathcal{F}_q(\gamma_{g_7}) = \{\gamma_{g_7},\gamma_{g_8}\},\ S\mathcal{F}_q(\gamma_{g_9}) = \{\gamma_{g_9},\gamma_{g_{10}}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$\begin{aligned} &\mathcal{F}_q G_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{8} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_9, \mathcal{F}_{q^2}) \oplus \mathbf{M}(n_{10}, \mathcal{F}_{q^2}), \\ &719 = \sum_{i=1}^{8} n_i^2 + 2(n_9^2 + n_{10}^2), \end{aligned} \tag{4.6}$$

where $n_i \geq 2$. We observe the WD of $\mathcal{F}_q A_6$ in this case is (see [2, Proposition 4.7])

$$\mathcal{F}_q A_6 \simeq \mathcal{F}_q \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_{q^2}). \tag{4.7}$$

Using (4.7) and Proposition 2.5, we further obtain from (4.3) that

$$\begin{aligned} \mathcal{F}_q G_2 \simeq\ & \mathcal{F}_q \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \\ & \oplus \mathbf{M}(8, \mathcal{F}_{q^2}) \bigoplus_{i=1}^{4} \mathbf{M}(n_i, \mathcal{F}_q) \oplus \mathbf{M}(n_5, \mathcal{F}_{q^2}), \end{aligned} \tag{4.8}$$

with

$$360 = \sum_{i=1}^{4} n_i^2 + 2n_5^2, \;\; n_i \geq 2. \tag{4.9}$$

According to Lemma 2.7 and Case 1, $4 \leq n_i \leq 10$. Moreover, Proposition 2.6 confirms that $n_i \neq 7$ in this case. Thus, we are remaining with the three choices of n_i's fulfilling (4.9) given by $(4^2, 8^2, 10)$, $(4^2, 10^2, 8)$ and $(8^2, 10^2, 4)$. Next, to uniquely identify the correct choice, we show that $\mathbf{M}(4, \mathcal{F}_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 7$ and consider the following mapping from G_2 to $GL(4, \mathcal{F}_7)$:

$$a \to \begin{bmatrix} 3 & 6 & 2 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 5 & 4 & 2 \\ 5 & 2 & 1 & 1 \end{bmatrix}, \quad b \to \begin{bmatrix} 4 & 2 & 5 & 2 \\ 4 & 5 & 4 & 0 \\ 4 & 6 & 3 & 5 \\ 0 & 3 & 4 & 0 \end{bmatrix}.$$

This mapping is a homomorphism from G_2 to $GL(4, \mathcal{F}_7)$ (as a and b given in (4.1) generates G_2). It should be noted that this map is an irreducible representation of G_2 over $\mathcal{F}_7$. Therefore, according to Proposition 2.8, $\mathbf{M}(4, \mathcal{F}_7)$ will always be a Wedderburn component of $\mathcal{F}_q G_2$. Consequently, we are left with two possible choices of n_i's given by $(4^2, 8^2, 10)$ and $(4^2, 10^2, 8)$. Finally, we show that $\mathbf{M}(10, \mathcal{F}_q)^2$

is a summand of $\mathcal{F}_qG_2$. For this, we define the following two maps:

$$a \to \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0\\ 6&2&3&4&0&0&0&0&0&0\\ 4&6&3&3&0&0&0&0&0&0\\ 2&5&0&2&0&0&0&0&0&0\\ 5&1&0&6&0&1&6&0&1&0\\ 3&3&0&5&1&1&3&3&4&0\\ 3&5&6&0&4&1&2&5&0&3\\ 6&0&4&4&2&3&6&6&1&5\\ 6&6&1&4&4&1&2&3&5&1\\ 3&2&5&1&1&0&4&1&4&4 \end{bmatrix}, \ b \to \begin{bmatrix} 0&4&4&0&0&0&0&0&0&0\\ 2&1&4&3&5&0&0&0&0&0\\ 6&3&4&3&3&4&0&0&0&0\\ 2&5&2&5&2&4&3&0&0&0\\ 6&0&3&3&1&4&6&0&0&0\\ 3&2&6&2&5&6&1&0&0&0\\ 4&3&0&3&2&2&5&0&0&0\\ 4&1&6&5&5&1&2&4&3&4\\ 2&3&6&6&0&0&6&0&1&3\\ 4&1&0&3&5&4&4&3&3&2 \end{bmatrix}$$

$$a \to \begin{bmatrix} 4&5&0&0&0&0&0&0&0&0\\ 0&4&3&5&0&0&0&0&0&0\\ 6&6&2&4&6&0&4&0&0&0\\ 4&1&4&5&1&3&1&6&0&0\\ 4&3&0&3&4&3&4&0&4&5\\ 3&6&2&2&5&0&4&4&6&5\\ 2&0&5&3&3&3&4&1&0&5\\ 3&2&3&1&0&3&0&0&0&6\\ 5&4&0&1&6&2&2&5&3&0\\ 5&1&0&4&1&6&3&1&0&5 \end{bmatrix}, \ b \to \begin{bmatrix} 1&0&4&0&0&0&0&0&0&0\\ 0&6&1&6&3&0&0&0&0&0\\ 0&4&0&5&3&5&6&0&0&0\\ 0&1&0&5&2&2&0&3&6&0\\ 0&3&3&0&6&5&4&2&1&2\\ 0&3&2&2&4&1&1&4&1&0\\ 0&1&3&1&0&0&2&6&3&6\\ 0&1&3&3&6&1&5&1&0&1\\ 0&6&4&2&2&6&5&4&1&6\\ 0&5&1&5&0&6&5&3&0&6 \end{bmatrix}$$

These mappings are 2 irreducible representations of G_2 over $\mathcal{F}_7$. Therefore, Proposition 2.8 derives that $\mathbf{M}(10,\mathcal{F}_7)^2$ is a summand of the group algebra $\mathcal{F}_7G_2$. Consequently, the required choice of n_i's is $(4^2, 10^2, 8)$. Hence, using (4.8), we get

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4,\mathcal{F}_q)^2 \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q)^3 \oplus \mathbf{M}(8,\mathcal{F}_{q^2})^2.$$

Case 3: $p^k \equiv \{11, 19, 29, 59, 61, 91, 101, 109\} \bmod 120$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for 1-10, 12}, \ S\mathcal{F}_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{13}}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$\mathcal{F}_qG_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{10} \mathbf{M}(n_i,\mathcal{F}_q) \oplus \mathbf{M}(n_{11},\mathcal{F}_{q^2}), \ 719 = \sum_{i=1}^{10} n_i^2 + 2n_{11}^2, \ n_i \geq 2. \tag{4.10}$$

We observe the WD of $\mathcal{F}_qA_6$ in this case same as in Case 1. Using this and Proposition 2.5, we further obtain from (4.10) that

$$\begin{aligned} \mathcal{F}_qG_2 \simeq{}& \mathcal{F}_q \oplus \mathbf{M}(5,\mathcal{F}_q)^2 \oplus \mathbf{M}(8,\mathcal{F}_q)^2\mathbf{M}(9,\mathcal{F}_q) \oplus \mathbf{M}(10,\mathcal{F}_q) \\ & \bigoplus_{i=1}^{4} \mathbf{M}(n_i,\mathcal{F}_q) \oplus \mathbf{M}(n_5,\mathcal{F}_{q^2}). \end{aligned} \tag{4.11}$$

By proceeding as in the previous case, we are remaining with the three choices of n_i's fulfilling (4.9) given by $(4^2, 8^2, 10), (4^2, 10^2, 8)$ and $(8^2, 10^2, 4)$. Further, on the similar lines of the previous case, we can show that the final choice of n_i's is $(4, 4, 8, 8, 10)$. Hence, it follows from (4.11) that the WD is

$$\begin{aligned}\mathcal{F}_q G_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4, \mathcal{F}_q)^2 \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_q)^4 \\ \oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_{q^2}).\end{aligned}$$

Case 4: $p^k \equiv \{13, 37, 43, 53, 67, 77, 83, 107\} \bmod 120$. The cyclotomic $\mathcal{F}_q$ classes of γ_g are

$$S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } i = 1\text{-}6, 12, S\mathcal{F}_q(\gamma_{g_i}) = \{\gamma_{g_i}, \gamma_{g_{i+1}}\} \text{ for } i = 7, 9,$$
$$S\mathcal{F}_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{13}}\}.$$

By incorporating Proposition 2.2 and Theorem 2.3, we derive from (4.2) that

$$\begin{aligned}\mathcal{F}_q G_2 \simeq \mathcal{F}_q \bigoplus_{i=1}^{6} \mathbf{M}(n_i, \mathcal{F}_q) \bigoplus_{i=7}^{9} \mathbf{M}(n_i, \mathcal{F}_{q^2}) \\ \implies 719 = \sum_{i=1}^{6} n_i^2 + 2\sum_{i=7}^{9} n_i^2, \ n_i \geq 2.\end{aligned} \tag{4.12}$$

In this case, the WD of $\mathcal{F}_q A_6$ is given by (4.7). Using this and proposition 2.5, we further obtain from (4.12) that

$$\begin{aligned}\mathcal{F}_q G_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q) \oplus \mathbf{M}(8, \mathcal{F}_{q^2}) \\ \bigoplus_{i=1}^{2} \mathbf{M}(n_i, \mathcal{F}_q) \bigoplus_{i=3}^{4} \mathbf{M}(n_i, \mathcal{F}_{q^2}),\end{aligned}$$

with $360 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2, \ n_i \geq 2$. According to lemma 2.7 and case 1, $4 \leq n_i \leq 10$ for each i. Furthermore, lemma 2.7 and case 1 implies that $n_i \neq 7$ for any i. This leaves us with three possible values of n_i's given by $(4, 4, 8, 10)$, $(8, 8, 4, 10)$ and $(10, 10, 4, 8)$. Next, to uniquely identify the correct choice, we show that $\mathbf{M}(4, \mathcal{F}_q)$ will always be a Wedderburn component in this case. In particular, we take $p = 13$ and consider the following mapping from G_2 to $GL(4, \mathcal{F}_{13})$:

$$a \to \begin{bmatrix} 0 & 12 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 9 \\ 0 & 1 & 2 & 8 \end{bmatrix}, \quad b \to \begin{bmatrix} 0 & 2 & 7 & 0 \\ 9 & 3 & 3 & 3 \\ 1 & 10 & 0 & 1 \\ 0 & 10 & 1 & 8 \end{bmatrix}.$$

This mapping is an irreducible representation of G_2 over $\mathcal{F}_{13}$. Therefore, according to proposition 2.8, $\mathbf{M}(4, \mathcal{F}_q)$ will always be a Wedderburn component of $\mathcal{F}_q G_2$. Hence, we get

$$\mathcal{F}_q G_2 \simeq \mathcal{F}_q \oplus \mathbf{M}(4, \mathcal{F}_q)^2 \oplus \mathbf{M}(5, \mathcal{F}_q)^2 \oplus \mathbf{M}(9, \mathcal{F}_q) \oplus \mathbf{M}(10, \mathcal{F}_q)$$

$$\oplus\, \mathbf{M}(10, \mathcal{F}_{q^2}) \oplus \mathbf{M}(8, \mathcal{F}_{q^2})^2.$$

This completes the proof. □

5. Conclusion

In this paper, we focused on deriving the unit groups of the semisimple group algebras of groups $SL(2,8)$ and $SL(2,9)$. In order to derive these, we computed the Wedderburn decomposition using the findings from the classical theory of group algebras. Having the wide range of possible Wedderburn components, it is evident that it becomes more and more challenging to characterize the Wedderburn decomposition with increasing group size. Finally, this paper further motivates to deduce the unit groups of special linear groups of higher order.

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Development opportunities for mobile and ICT learning in teacher education

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Abstract. Learning is nowadays a continuous intellectual readiness to be able to cope with the current needs of the world of work. Students' knowledge must be adapted accordingly and they must be capable of continuous development throughout their lives [8]. Changes in teaching-learning habits have already been observed at the beginning of the 21st century and those involved in education need to adapt to these changes with the appropriate transition to digital education what is more important today than ever before. We asked our teacher-candidate students in an online questionnaire at the beginning and at the end of the "Teaching Methods in Mathematics" course. Our non-representative survey provided valuable data for course development. The aim of the survey was to find out students' perceptions of the methods and tools they had learned in education. In the questionnaire at the beginning of the semester, we inquired our students about the types of work and methods they were familiar with, and then they gave their opinions about the different ICT tools, educational programmes and applications and how they were used. At the end of the semester, we interwieved them again whether they still held similar views on the subject or they had managed to change their views during the course.

Keywords: ICT, teacher training, Mathematics, educational environment

AMS Subject Classification: 97U10, 97U50, 97U60

1. Introduction

In this paper, we aim to show how the changing teaching-learning culture has influenced the structure of education of Mathematics and teaching methods in the teacher training programme in Széchenyi István University Győr. We have experienced the reduction in the number of lessons and the changes in the attitudes

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of students as a challenge. These are the problems that everyone has faced in education in Hungary. In Győr we have tried to do our best. In these changes, we have also taken into account the applications of new information and communication technologies (ICT) but we have not neglected to present well-established manipulative tools and their potential use in teaching. Our previous research and publication have presented ideas on how to strike a balance between traditional and ICT-enhanced visualisation [3, 13]. The aim of our methods is to increase students' motivation and change their attitudes towards learning in a positive way. After a theoretical introduction the results of our online questionnaire and the conclusions are presented.

2. Theoretical framework

2.1. The generation growing up in the 21st century and the way they acquire knowledge

Growing up in the digital age is characterised by the need to acquire knowledge quickly and in the age of the Internet students get their knowledge from the information space. Their social relationships and social interaction habits have changed and they find a sense of belonging to a community through social portals, blogs, networking games [17].

They have no problem with navigating in parallel, side-by-side, so-called multitask applications. They expect instant, fast access to programs, quick reinforcement and rewards in solving tasks. However, during training, we need to make sure that they have the right ICT competences and are not only familiar with information-sharing applications on social networking sites. We need to build on their existing ICT skills to create the most appropriate learning environment for them.

2.2. Learning process in the 21st century

Learning involves modelling the outside world. In the course of education, we can shape and change this representation, forming a world view. Throughout history there have been several educational paradigms. Some have emphasised the direct transmission of knowledge, others the demonstration, and still others the action. Different pedagogical trends have alternated and complemented each other and have given rise to new theories, such as behaviourism, cognitivism or even constructivism. The process did not stop there, as new arenas of knowledge flow and knowledge sharing emerged with the rise of digital culture and digital education. The latest network-based forms of learning such as connectivism, adapt to students' forms of knowledge acquisition and their community organisation [11]. The focus is not only on acquisition but also on knowledge creation and sharing, where learners participate in the creation of collaborative content. It requires cooperation between all those involved in education. Of course, this also requires that students receive as broad methodological training as possible on the opportunities of using ICT tools and teaching methods [6]. The attitude of students to Science as a subject cannot be described positive nowadays which is why the teaching of these subjects requires even more the implementation of a high level of demonstrations and developments.

2.3. Expectations and skills in the 21st century

In addition to learning methods, another important factor is the need to change the educational environment as soon as possible. These are the expectations in the workplace. The only way to prepare students for their future professions is to develop 21st century skills and competences. Several educationalists and researchers have attempted to study these skills and competences, and although there are differences on a few points, the main features are the same.

Taking the ITL (Innovative Teaching and Learning) research as a starting point, we have the following classification:

- *knowledge building*
- *problem solving and innovation*
- *communication skills*
- *collaboration*
- *self-regulation*
- *ICT use* [16]

The graph (Figure 1) shows the parts and conditions of knowledge applicability: self-expression, creativity, continuous readiness for self-development, flexibility to react to problems are definitely needed.

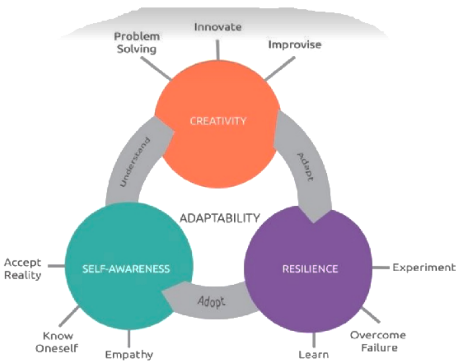

Figure 1. The metaskills we need to thrive in the 21st century [18].

Anyone who reads carefully the skills listed above will see that the traditional frontal classroom work, lecturing and explanation are no longer effective as methods. Changes in students' learning habits, the rise of digital tools and 21st century guidelines have made it necessary to change traditional teaching methods in higher education. Collaborative learning, problem-solving methods and the education in the use of ICT should be promoted. Our previous research, cited several times in this paper confirms the need for these changes [13, 14].

3. Methods used in the course

After describing the new generation's learning processes and expectations, we will now give some ideas on the methods we use in our teaching and which were also asked about in the above mentioned questionnaire. Overall, we found the methods described below to be appropriate for achieving the objectives we set ourselves in the development of the subject.

3.1. Flipped classroom

The flipped classroom model is a kind of inversion of traditional education. It is a learning management solution where students can watch at home the lecture prepared by the instructor and individually study the recommended teaching aids and online resources [7]. The flipped classroom is an engaging and student-interest based method where students' passivity is transformed into activity. In this learning management process, students are more independent and interested in the activity and know that there will be time for questions and discussion. In contrast to traditional lectures, the emphasis here is on practice rather than frontal teaching, according to the students' own needs. During or after the lecture, for example, students are asked to complete a test as feedback, and the results are known within a short time. Homework is done in the classroom, where the focus is usually on interactive activities and collaborative work. The flipped classroom is not an online course, the video is not a substitute for the teacher. The students not only work independently in front of the computer but also prepare themselves to work together with the teacher.

The advantages are that during the contact lessons, based on the students' prior preparation, we can answer their questions, organise group work, implement many activities that make the class interactive and allow the students to be active participants in the learning process. It can be effective in cases of lack of motivation or discipline [1, 15]. Students can take responsibility for their own learning and progress at their own pace, according to their own timetable. The content created can be archived and retrieved, so that no absences are missed. The downside, however, is that we need to be sure that students completed the tasks at home during the pre-learning stage, i.e. they watched the videos posted by the instructor. In the context of the flipped classroom method, the video is not just a resource, but the basis for all subsequent work. Before introducing this method, it is necessary to discuss with the students what they need to pay attention to, and what will make the collaborative activities work well [10].

3.2. Gamification

In today's education, face-to-face teaching is becoming less important, and computer-networked forms of training are now available, which can be used to effectively support the teaching-learning process and to acquire the necessary knowledge with the right methods of knowledge transfer and learning. New technologies can be used to encourage multichannel, collaborative learning. More interactive technologies can increase students' control and provide more opportunities for repetition in the learning process. This can be helped by gamification, an English term first used by Nick Pelling around 2002, so we can see that as a term it has been known for less than two decades. A relatively all-encompassing definition of gamification is: gamification is a strategy in which game elements are used in a non-game environment to move some behaviour in a positive direction. This definition can be used in higher education because gamification is a way of solving a problem if students have not done something before then we try to encourage them to do it so it can help with motivational problems. This is the role of gamification that we want to exploit in our courses [5].

In preparing teacher-candidates for their future profession, in line with the new learning theories, we aim to reduce the number of frontal lectures and develop new activities in which the whole educational process is guided from outside by means of assigned tasks. A kind of asynchronous teaching and learning is achieved, be it through ICT or through manual demonstration, action and experimentation. This type of learning provides the learner with a direct experience of success, which strengthens motivation to learn and thus encourages independent learning. When designing a course based on gamification, the learning material needs to be structured in modules and easily learnt. The student is an autonomous learner and therefore the learning material should be sufficiently motivating. However, this form of learning also requires tasks that can be completed together. In an online, group-editable submission, everyone can contribute to the creation of the product. Communication, working together, also develops human relationships. This model combines traditional classroom teaching with the opportunities offered by the internet and digital media [12]. Previous research has shown that students do not yet rank cooperative working as a top priority. We would definitely like to change this during their studies, as we have seen in the list of 21st century skills that communication and collaborative working will be needed in the future [14].

3.3. Project work

In the general approach, the aim of a project is always to create something new and socially important. The aim of a pedagogical project is: the learners want to produce the final result defined in the project. However, given that the project is subordinate to educational objectives, from the definition of the theme to the presentation of the final result, an important objective is also the result of the educational process that is the outcome of the activity, which can be of many kinds, such as posters, oral or written reports, blueprints, exhibitions.

Project-based education must meet a number of criteria. For example, the starting point should be the pupils' question and the design should be a collaborative process. The solution of the project should be achieved through activities, but there should also be opportunities for individual and group work, reinforcing communication and the development of cooperative skills. It covers a longer learning period, during which students will also be exposed to situations outside school, in a spirit of interdisciplinarity. Partners with different competences work together to achieve success and students are responsible for their own decisions, while the teacher is only a trainer or mentor in the learning process. It has the advantage that there are no major risks if it is not used carefully. The expected consequences for the students are increasing motivation, autonomy, self-awareness, self-awareness and self-esteem. Cooperation with peers improves and creativity develops. Of course, project work also has prerequisites without which it cannot work. These include the teacher's openness to working with students and the students' readiness for independent and cooperative learning, and whether the institutional framework is appropriate for this type of education [9].

The advantage of this method is that students have a responsibility to complete their work. Disadvantages include the difficulty of finding balance between teacher direction and student autonomy [4].

After presenting the main methods used in our courses and the questionnaire, we will now describe the course organisation, our research and its results.

4. Course organisation

In the first term of 2021/2022, two methods were used simultaneously in the teaching of the subject: Teaching Methods of Mathematics. One of these methods was the "mirrored classroom" method and the other was gamification. The mirrored classroom method was used to highlight the frontal teaching by means of pre-recorded videos. This served two purposes. One was saving time. This was emphasised due to the reduction in the number of lessons, and this enabled us to achieve the other objective, giving students the opportunity to practice and do more teamwork and projects. The other method that was introduced was gamification. The term was divided into blocks and within these blocks students had to solve tasks both collectively and individually. The tasks that could be completed during the lessons included some theoretical knowledge and a lot of practice through exercises, solving problems and their methodological analysis. The videos with the necessary theoretical material had to be previewed by the students before the lessons so that they could test their knowledge each time using the Kahoot! application. This was a feedback for the students and for teachers too on which topics were difficult for the students to complete. It also created a competitive environment for the students as they could compete with themselves and their classmates without any fear. Other work that was given to be done was "tool making": e.g. cubes, octaeders etc. It is very important that students must be aware of how to motivate children with less abilities and what tools can be used to help the pupils understand. For this reason, manipulative tools were also created. They had to prepare a project on a topic too. In this term, they had to work cooperatively to create a multi-curricular project for Animals' Day, combining Maths, Reading, Science, Environmental studies and other subjects. In this way, they practised cooperative work, document creation, editing and information retrieval using ICT tools. They also explored the possibilities of using ICT tools. The LearingApp, Genially and many other apps and web applications for textbooks were presented. Students learned to produce their own lesson plans using online interfaces. We tried to do all this with as little frontal teaching as possible. They could choose the tasks freely and students were given pre-defined prompts for each topic. The evaluation was continuous so students could check their progress lesson by lesson. In this way, they were also able to get their final grade, avoiding the so called campaign learning that leads to quick forgetting after finishing the exams. With this learning organisation and the 'mirrored classroom" method, students could do more work at home and allocate their time more easily. Because they were free to choose which tasks to do we tried to encourage them to learn. Those who did all the tasks in each section and got the maximum scores got a good mark. We were able to do this in this system because if the student completes all the assignments it means that they have studied continuously during the whole course.

An important question arises as this method puts a lot of work on the instructor. Of course, this method needs a lot more in preparation but we have to find the "golden mean" because if we don't change anything, the student's knowledge and attitude will not change either. Making videos and online surveys is a lot of work at first and later the lecturer always have to make corrections. We hoped that the positive change in the attitude of the students that we expected would take place and that they would be successfully prepared for their future profession.

5. Questionnaire

5.1. The context of the study

An online questionnaire was conducted at the beginning and at the end of the course in the subject of Teaching Methods in Mathematics. Out of an already small number of students (34) only 18 responded at the beginning and even fewer (13) at the end of the semester. Therefore we cannot call our survey representative, but we have extracted valuable data for future course development. The questionnaire included open-ended, expository questions and questions on the 5-point Likert scale. Our aim was to assess students' perceptions of the methods and ICT tools they had learned in education. In the first part of the questionnaire they were asked about the known forms of teaching, methods and their frequency of use, followed by their opinions about different ICT tools, educational programmes and applications and their use by the students. At the end of the semester, we wanted to know whether they still held similar views on the subject or whether they had managed to change their views.

5.2. Research questions, theses

At the beginning of the term, we assumed that the students would prefer traditional, frontal forms of teaching work as this was what they had mostly encountered. It was hoped that there would be a change in their minds and that their future careers would include forms of teaching that were appropriate to 21st century skills. We also assumed that they would learn how collaborative working and collaborative document editing could help the learning process through the opportunities they would learn during the course.

6. Outcomes

6.1. Results for the questions on working methods

The questionnaire was completed by 18 students at the beginning of the semester: 53 percent of the students. At the beginning of the questionnaire we asked if the students would like to use a mobile device for learning purposes in the course.

There were 15 yes and 3 no answers to this question. The positive answers were supported by the following comments:

- *I think that nowadays most children in primary school have a mobile phone and it is difficult for them to break away from it. I think it would be easier to teach with the cell phones using their advantages than to wean them off.*
- *A more practical, quicker, more informative outline could be made.*
- *I wonder how it could be used for a maths subject.*
- *I think in today's world it is necessary to get acquainted with such content, it can make the lessons we will have in the future more colourful.*
- *In my opinion, it would make the lessons more exciting and interesting.*
- *Because learning is much easier and more effective.*
- *If it makes the curriculum more understandable, then yes.*
- *Due to the fact that a mobile device can illustrate certain topics better than paper.*
- *This would make the course more varied.*
- *A test could be used to check the mastery of the course material. This would provide quick feedback. Textbooks could be opened in digital format.*

- *Because I want to learn how to use my mobile phone for learning purposes.*
- *If there is any useful information about mathematics, I would like to know it, it can only benefit learning.*
- *It is often useful in today's world to access information quickly, and as a student, this means searching for it in a matter of seconds instead of learning a lot of material.*

And those who do not, gave the following reasons for their opinions:

- *It distracts from the essential things while surfing*
- *I do not want to completely lose the school experience.*

It is natural that there are always people who are afraid or do not want to change certain old things or perhaps do not want to break away from previous habits. But constantly choosing between different methods and applying them to the right part of the curriculum tends to move the learning process forward. This is why teacher candidates need to learn as many ICT tools and methods as possible, so that they can apply them correctly in different situations. In our courses, we also try to show the positive side of the different opportunities to those who are doubtful and possibly reluctant.

At the end of the term, we asked students again about their views. 13 students responded to our questionnaire and 84.6 percent of them liked this type of education.

Some of the students' answers are the followings:

- *The competitive spirit of Kahoot! increases performance.*
- *The introduction of technology makes the class interactive. It is much more enjoyable and exciting to complete a test using a mobile phone and apps that instantly grade assignments. Interactive tasks can be very motivating for students.*
- *Because we learned a lot of new things and because the phone is always at hand, it was a good idea to incorporate them during the lesson.*
- *Today's generation is attached to the phone anyway, so at least we could put it to good use.*
- *It made the class more varied, I felt more active and I learned more than if I had just taken notes.*
- *The competitive experience of the playful tasks made learning and revision exciting.*
- *Because it was much easier.*

- *It was simpler that way.*
- *It made learning playful and 21st century. For example: the Kahoot tests did not have that typical test feel, more like a playful quiz. So I was not stressed and I could think with a cool head. I learned to give a quick and accurate answer, which I think is good.*
- *It made the lesson more interesting and exciting.*
- *It made the lesson more interactive and exciting.*
- *There was constant repetition and no paper to use and less stress.*

The downside of the method, noted by one student, was that the wifi connection was not always good. We tried to eliminate this by telling them that if they lost the connection, they should write down the answer quickly on a piece of paper. Unfortunately, technical problems can always arise, but as the above comments show, the new methods were generally well received.

In this section we also asked the teacher candidates which teaching methods they would like to use in their work and how often. At the beginning of the semester the traditional forms of work – lecture, explanation, individual work – were the most prominent. All the things that the students already have experienced of, having encountered them during their studies. Cooperative work, online tests and the project method, which can be used continuously in teaching.

Students are curious about working with ICT tools but they were unsure in using them according to the survey at the beginning of the semester. They have a desire to learn and they do not completely reject working with modern tools but they are not convinced that completely online education could be a successful and effective way of learning. They also consider essential to familiarise children with ICT tools in order to achieve educational goals more effectively.

The question asked at the beginning of the term was also asked at the end of the semester. We were curious to know how much they would use the methods and forms of work they had learned during the course in the future. We can only look at the changes in general, not individually because the questionnaire was anonym. The answers were then quantified and subjected to a statistical test. The graph below shows the use of methods at the beginning and end of the semester.

What can be seen very clearly is that the formerly usual frontal teaching was replaced by teamwork and online tests, which may even facilitate continuous accountability, and the mirrored classroom has become one of the more frequently used methods. These responses illustrate our view that students' attitudes towards different forms of work and methods could be changed and that we can move from frontal teaching to teamwork, that makes stronger the development of the ability to work together.

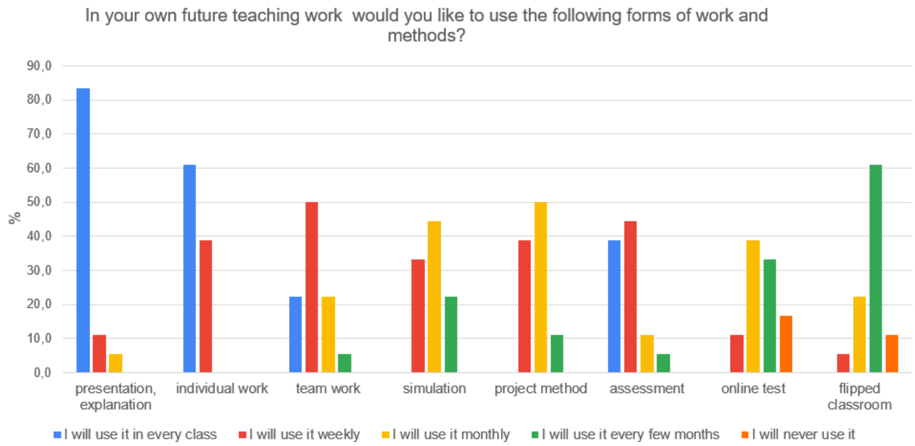

Figure 2. Methods at the beginning of the semester.

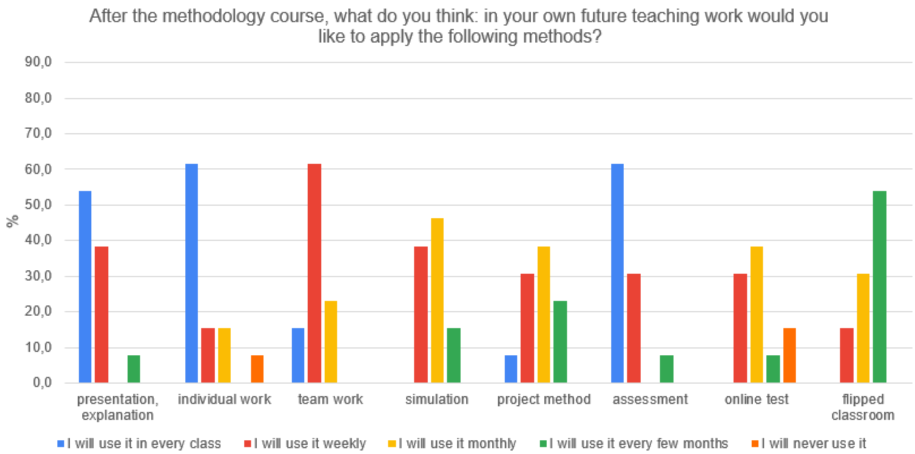

Figure 3. Methods at the end of the semester.

Using a hypothesis test, it was quantifiably shown that there was a change of students' opinion about the methods they had learned (Table 1). The responses were scored on the 5-point scale and then subjected to a t-test (level of significance 95 percent). In all cases, there was evidence of a change in students' attitudes towards the new methods. Students' attitudes toward the new ICT supported methods changed significantly in each case. The result of the t-test shows that there is no change in the individual work but the value of the t-test of the lesser-known methods exceeds the value in the table in absolute value. The results of our study show that students' attitudes moved towards collaborative methods during the term. They have learnt and hopefully will apply these educational innovations to start developing 21st century skills in future generations.

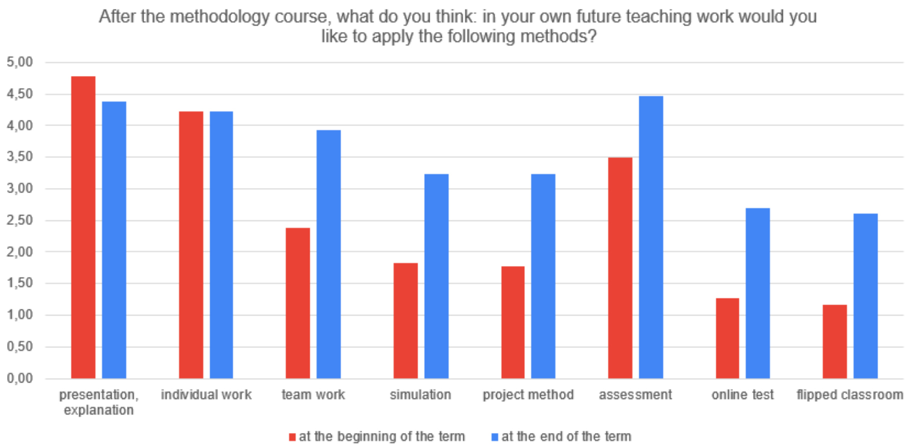

Figure 4. Comparison of methods at the beginning and end of the semester.

Table 1. Result of t test.

		presentation, explanation	individual work	team work	simulation	project method	assessment	online test	flipped classroom
at the beginning of the term (n=18)	average	4,78	4,22	2,39	1,83	1,78	3,50	1,28	1,17
	deviation	0,84	1,19	0,62	0,70	0,89	0,84	1,02	0,74
at the end of the term (n=13)	average	4,38	4,23	3,92	3,23	3,23	4,46	2,69	2,62
	deviation	0,53	0,92	1,34	1,01	1,13	0,90	0,65	0,50
F-test (F_{table}=2,6)		2,46	1,68	0,21	0,47	0,62	0,88	2,46	2,18
Independent-samples t-test (t_{table}=1,699)		1,60	-0,02	-3,85	-4,29	-3,84	-3,02	-4,71	-6,52

6.2. Testing the use of ICT tools

It was important for us to find out when our students had used ICT tools. This was the basis for designing the course and applying the different ICT tools and methods. In the graph below (Figure 5 and 6), you can see that the different tools are used more for monitoring social media, not really present in the learning-teaching process. It is important to make teacher candidates aware that there are lots of complex possibilities. For example, learning together with peers, editing different documents online and collaboratively, developing animations that can be used for teaching, working in a team on a project or doing tests, homework, creating reports.

By the end of the term, we had the following results:

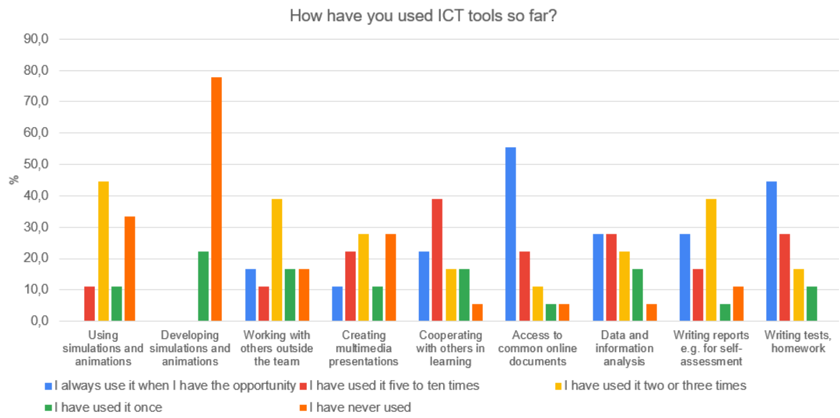

Figure 5. Use of ICT tools at the beginning of the semester.

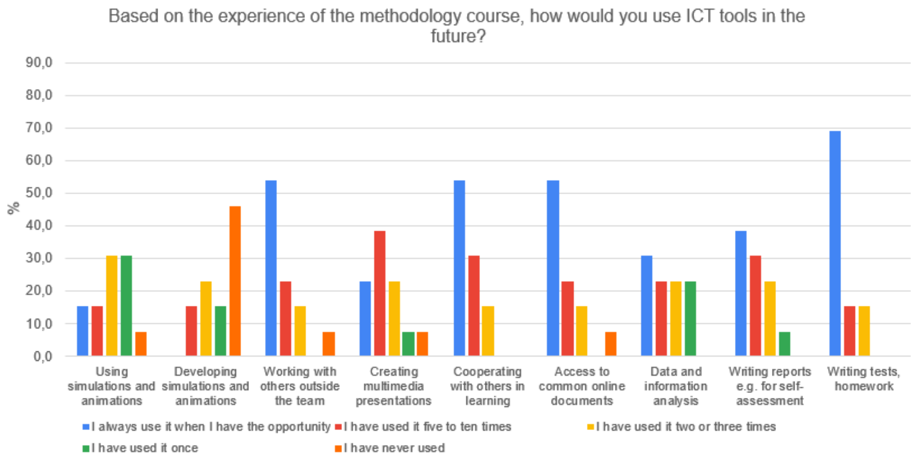

Figure 6. Use of ICT tools at the end of the semester.

7. Conclusion

The information technology revolution and changing learning habits have challenged teachers, lecturers and educational institutions as well. The growing body of knowledge and changing educational needs have made it necessary to find the new ways of teaching and to introduce new methods [2]. The information technology revolution has brought not only problems but also a range of possible solutions. By the 21st century some researchers have recognized that motivating methods in the recent development of gamification can be successfully applied in education too.

Our research shows that even at the beginning of their studies the teacher candidates are not excluded from using ICT but they are not using it consciously although social networking sites are always present in their daily lives. Later in their works they are going to use ICT mostly in traditional ways – for example PowerPoint for explanations. During their university studies they should be aware of the ICT supported appropriate methods and they should be shown that there are many other ways in which they can apply digital tools successfully: organising a diagnostic survey, online tests for competitions or creating a collaborative online product and document.

Gamification does not mean the use of games in teaching but the integration of game mechanisms into everyday practice of teaching work processes or into the preparing for lessons. For the "digital generation" which is socialising and growing up nowadays it is highly important being familiar with ICT.

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The role of creation in the didactical traditions in Hungary*

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Abstract. In the didactics of mathematics, a lot of such research appeared in the past twenty-thirty years that consider both mathematics and mathematics education as a cultural activity [20]. In this approach, didactical texts and social space are carriers of beliefs created about mathematics education. Similarly, the investigation of beliefs has strengthened in the past two decades [33]. In our paper, we examine what relationship appears between the beliefs of the Hungarian didactic tradition existing in written texts or personal contacts, and those of Hungarian mathematics teachers. To carry it out, we use means of cultural history and a questionnaire.

Keywords: Belief, culture, literature, mathematical creation

AMS Subject Classification: 97A99

1. Introduction

At the very beginning, we intend to clarify some notions:

1. *Culture*
   We examine mathematical activity as a cultural activity. In the descriptive or scientific determinations, mainly the common patterns referring to groups of people are connected to the notion of culture [20]. On the one hand, culture exists in the space of behaviour with the help of tools and symbols, but parallelly with this, it can also be examined in the environment of thoughts

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and values. This ambivalence resulted in a kind of “cultural turn”, and 10-15 years ago the number of research considering mathematics education as a cultural activity increased. [20] Our article joins this direction; insofar it examines the presence of a thought – more exactly the thought of creation – connected to mathematical activity in historical and cultural context. The culture of didactics of mathematics can also be investigated; we take this as a starting point.

2. *Beliefs*
   Beliefs about mathematics and about mathematics education form a part of the culture of mathematics and its didactics. The research of teachers' knowledge, beliefs and attitudes started intensively in the last two decades, see [30] and got an overview by the Springer book [33]. According to Philipps' definition, a belief is the following: “Psychologically held understandings, premises, or propositions about the world that are thought to be true” [30]

3. *Creation in mathematics and in mathematics education*
   In mathematics, *creation* is possible at two different levels.

   (a) One of them happens when something new is discovered in an existing system. An existing, but a not yet recognised connection is found out. It can be a new statement, the proof of a new theorem or an old conjecture by linking two fields in a creative way, or proof given to an old theorem in a more elegant way, etc.

   (b) At a higher level: when completely new mathematical worlds, new universes are created as a result of a mathematician's creation process, such creations that show the nature of mathematics itself. Later on, we will show two examples to this (Bolyai's geometric creation and the significance of Gödel's role). This level is strongly connected with artistic creation as an act of creation; insofar the artist also creates an absolute world as a result of the process of creation.

According to us, the equivalent of the mathematical creation's notion can also be found in didactics. An example to this can be mathematics education by discovering which has a long tradition not only in Hungary. Here in Hungary this tradition is connected to the methodology of Tamás Varga [14, 37, 38], and also to the discovery method of Lajos Pósa [18]. In case of the discovery method, the result of the creation process can be experienced, the teacher gives such space to the student who – by going through it – can create own, individual results. The experience of creation is more strongly present in radical constructivism [4], in case of which such space is given to students where they similarly create some parts of knowledge by themselves, but more independently from the teacher. Constructivism based on this, merely has a tradition in Hungary. The connection between artistic creation and mathematics also appears in STEAM movement, which is a modern educational trend [24].

2. Question

With the help of the approaches and notions above, we examine a part of the comprehensive question, what kind of relationship exists in Hungary between the written tradition and personal tradition in didactics and in mathematics education. In what form does a thought – detected in written tradition – appear among the beliefs of teachers?

We intend to give answer to these questions, on the one hand, with the help of presenting such an effect history curve, in which we prove connections between mathematical texts and texts of mathematical didactics. On the other hand, having interpreted the outcomes of a questionnaire's particular items, we present in what way the thought appears – which is examined in the texts – among the teachers' beliefs, and from what sources the teachers' picture of mathematics derives.

3. The written tradition

The written tradition is interpreted as texts having mutual effect, as a net of (at times only loosely connected) texts referring to or quoting each other, and we examine this with the aid of the phenomenon of intertextuality [21]. Hungarian didactics also exists as a cultural activity. From this huge environment, we focus on the texts first, and on the fact as well, in what way the texts are in a dialogue with each other. If we approach the culture of mathematics and mathematics education from the viewpoint of beliefs and intellectual history, then one of the notions in the Hungarian tradition can be creation. Creation does not exclusively appear in Hungarian didactics as being equal to mathematical activity, but it also has a remarkable tradition among the conceptualisations regarding school mathematics ranging from the first grade of elementary school to university mathematics education (see Varga's reform and Rózsa Péter's work in the context of the teacher training). It has yet a connection with the artistic creation and the world of a literature. The proof of this can be seen by examining the history of one of Bolyai's sentences.

3.1. The sources of a sentence

The concept of creation flows along the history of European culture starting from the biblical Genesis creation ("God said"), through the creator divine Word (Word = God) which is known from the Gospel of John. To the history of the artistic creation belongs the myth of Prometheus, or by Shakespeare and Rousseau (see below). According to the conceptualisation of Sturm und Drang, in the Genieperiode, the creator is a genius, the self-contained artist is legalised to create a whole world with his own artistic power through breaking the rules [1]. The same power appears in János Bolyai's famous line in connection with mathematical creation. János Bolyai, who was the 19th century's greatest Hungarian mathematician and one of the most effective figures of mathematical history, reported on his revolutionary results in the field of hyperbolic geometry in a letter written to his father, Farkas Bolyai November 3rd 1823. The quoted sentences are taken from this.

"...from nothing I have created a new different world; all other things that I have sent to you are just a house of cards compared to a tower." [3]; (translation by Prékopa)

From the letter, we can draw two conclusions.

One of them is that J. Bolyai was aware of the scientific significance of hyperbolic geometry itself, and he wanted to let his father know about it by connecting creation and the creation of the new world with mathematical activity. This sentence refers to the parallelly existing geometries and mathematical worlds and to the genesis of these worlds at the same time.

The other conclusion is that the sentence referred to texts, which were known for both of them; what is more, they even talked about them with high probability. One of the possible sources is a contemporary Hungarian poem. One of the stanzas of Mihály Csokonai Vitéz's poem, titled To Solitude goes like this: [7]

> "In you, the poet's fancy flashes bright
> As rapid lightning in a murky night,
> While he creates new things by power of thought
> And fashions worlds undreamt-of out of nought."

It cannot be known if Bolyai knew the poem, but he doubtlessly knew Csokonai's first appearing biography. In the registry about the library of the Bolyais, the work of Márton Domby, who was the first biography writer of Csokonai, can be found. In the volume stands the following entry: "A book of János Bolyai. A gift from his father" [8]. Therefore, this book was read by János Bolyai, which can be almost surely declared. In the introduction, he encountered these lines:
"I would say first: These are unuseful materials [namely: the poems in the book], but secondly even for that reason are these brilliant, because you can see from them, *how the Genius creates a new world out of nothing...*" (italics mine). [9]

With this, Domby obviously referred to the poem titled To Solitude, with the difference that creation was interpreted as ability, not only by poets, but also by geniuses.

Bolyai's sentence can have other sources too, for instance, Shakespeare[1] or Rousseau[2], but Bolyai's sentence is so specific, and originally in Hungarian it equals almost literally with the two texts above to the extent, that it can be stated with high probability that Csokonai's poem, or rather Márton Domby's Csokonai-

[1] "The poet's eye, in a fine frenzy rolling,
Doth glance from heaven to earth, from earth to heaven,
And as imagination bodies forth
The forms of things unknown, the poet's pen
Turns them to shapes, and gives to airy nothing
A local habitation and a name."[35]

[2] "[...] the true genius that creates and produces anything from nothing." [34]

biography inspired it[3]. From both of the poems' context, it turns out that János Bolyai understood the significance of his own result, and he considered the role of the creator artist to be his own.

3.2. The mathematical-philosophical significance of Bolyai's work

As we have written in 1.c, creation in mathematics has different levels. Every new theorem, new connection, new theory is a creation. But it has a highest level when we create other type of mathematics compared to the one that we have in our culture. For 2000 years Euclid provided mathematics. Every new thing was adapted to this system until the appearance of János Bolyai. Then, this highest creation happened for the first time. Therefore, it is about the creation of a new world which repeated itself later, on the basis of Cantor, in set theory, but there along with preliminaries, as Kurt Gödel had already proved his famous incompleteness theorem by that time in 1931, which provided an interpretational frame to the achievement of Bolyai [12, 17]. The theorem plainly means that in any axiom system in which the infinite appears, so, at least the natural numbers are included, such a statement can be formulated that can neither be proved nor denied in the given system. Namely, an unexpected, new situation has been created, as up to that time something was either true or false, but from that point on there are things about which free decision is possible as the statement can neither be deduced nor rejected. At this point, a new way of creation appears, a new kind of mathematics can be constructed to the axiom system by attaching its statement or its rejection to it (one of them is usually the *old* world and the other one is the new one). Therefore, mathematics is not universal; there are more mathematics next to each other. In addition, János Bolyai, respectively Lobachevsky at around the same time, found this first case of an *undecided theorem*.

Half a century after this came the hypothesis of Cantor, being at the same time Hilbert's first problem in 1900, among the range of problems to be urgently solved at the beginning of the XX. Century [16]. *Is there cardinality between the uncountable and the continuum? Cantor's hypothesis was that there is not.*

Finally, to this, an unexpected answer arrived. In 1938 Gödel proved with the help of the sets, the so called *constructive sets of Gödel* – named after him – that Cantor's hypothesis cannot be denied [10, 11]. Paul Cohen admitted that

[3]In Hungarian the similarity of the three texts is more visible:
Bolyai's sentence: "*A semmiből egy új, más világot teremtettem.*"
Csokonai's poem:

> Tebenned úgy csap a poéta széjjel,
> Mint a sebes villám setétes éjjel;
> *Midőn teremt új dolgokat*
> *S a semmiből világokat.*

and Domby's sentence: "Erre én is azt mondom először, hogy igen is: Tanúság nélkül való, és haszontalan Matériák ezek [értsd: a könyvben közölt versek]: de másodszor azt, hogy éppen ezért remekek ezek, mivel ezekből látod, *miképpen teremt a' Genie semmiből is Világokat...*"

its rejection cannot be denied either, having done that with the method of forcing created by him, meaning that numerosity can exist between the two [6]. With this, it turned out that in the axiom system of Zermelo–Frankel the continuum hypothesis can neither be proved nor denied, therefore two different set theories can be imagined, according to our decision (like during the time of Bolyai the Eucledian and the non-Eucledian geometry). Thus, we have found *the second case of an undecided theorem.* Ever since not another one's fact of existence has reached us.

From then on, in case of a statement there are three opportunities instead of the – up to this time known – two. In this case, mathematics goes on in both directions, in one of them there will be an axiom, in the other one its rejection will be the axiom, so, these are two new worlds. This revelation was revolutionary, and it shook the – up to that time formed – picture about mathematics radically. However, it merely has had an effect on school mathematics until now. We think, it can have a positive effect if teachers themselves have a deeper understanding of this situation and also if they accept it.

3.3. The history of effect of the sentence

In Rózsa Péter's Playing with Infinity [29] two passages deal with Bolyai's results and works. In one of the passages, the following is circumscribed as the two most important and essential attributes of the geometry of Bolyai: 1) The *opportunity of choice.* Going towards (in those days) the most modern results of logic, the text is about the choices of axiomatisation, and taking the fact into consideration that the author formulates statements and opinions consequently on mathematics in general throughout the book, the notion of choice can grow into the symbol of *freedom* referring to mathematics in general. 2) Even if experienced reality can motivate the birth of mathematical notions, it is not identical with mathematical reality.

The other passage is shorter but equally important from the point of the examination of the text's cultural context. The following passage brings the connection of mathematical creation on surface:

"What is all this really about? Man created the natural number system for his own purposes, it is his own creation; it serves the purposes of counting and the purposes of the operations arising out of counting. Nevertheless, once created, he has no further power over it. The natural number series exists; it has acquired an independent existence. No more alterations can be made; it has its own laws and its own peculiar properties, properties such as man never even dreamed of when he created it. The sorcerer's apprentice stands in utter amazement before the spirits he has raised. The mathematician 'creates a new world out of nothing' and then this world gets hold of him with its mysterious, unexpected regularities. He is no longer a creator but a seeker; he seeks the secrets and relationships of the world, which he has raised." [29]

In this passage, the activity of the creator (inventor) and the researcher (discoverer) is connected on the basis of the famous sentence quoted from János Bolyai.

To Goethe's poem, titled The Sorcerer's Apprentice, a whole chapter's title refers in Rózsa Péter's text. The poem itself tells a story, in which the sorcerer's apprentice – left alone – tries to bring the broom to life, he succeeds, but the broom gets loose and does not obey to him, but the spell supposed to reverse this act is not known by the apprentice, that is why, in the end, the apprentice has to be rescued by the sorcerer because everything would be flooded if the broom did not obey again.

Rózsa Péter says about the apprentice: "The sorcerer's apprentice stands in utter amazement before the spirits he has raised". Here an important moment's dramatization happens from a psychological point of view by paralleling the mathematician with the sorcerer's apprentice. With this parallel, Rózsa Péter depicts the moment when a mathematical creation is born. If we compare this step with the text of the Goethe-poem, then we can state the following: Mathematical creation is in such a relation to a person – to the creator itself – as a sorcerer's apprentice to a ghost. It is about a larger existence than the apprentice here, to whom laws do not apply, even though the sorcerer's apprentice himself created them. These mathematical creations can be monumental things like hyperbolic geometry, or much less monumental, but yet similar creations in nature like natural numbers that Rózsa Péter mentions, too. The common feature that makes the basis of the parallel is the fact that the creator does not have power over the creature. If we observe this simile from the viewpoint of knowledge and cognition, then it sheds a light on the fact that the construction of mathematical knowledge is an act of invention and discovery at the same time, but also on the other fact that mathematics is not a closed and finite set of knowledge.

We are convinced that this textual tradition contributed to the history of mathematics education in Hungary. From this symbolic sentence of the history of mathematics, and from the context of this sentence by Bolyai, some specific features of mathematics education in the Hungarian culture of didactics can be detected:
The concept of creation is a key part of mathematical activity. But it is not only connected to mathematical activity, but also to school mathematics from the beginning of the second half of the 20th century. The movement of complex mathematics education in the 20th century, based on T. Varga's complex mathematics education experiment belongs to the guided discovery method in mathematics education and partially also to the inquiry based mathematics education.

Erika Jakucs formulated the following sentences in the documentary film of Tamás Varga in 2019 [37, 38]:
"Teaching math is like a game field. Like creating a bord game. We create the aboard and the figures, and set down the rules, and then, we have to play by these rules. Still, math opens many areas, where we can figure out what the rules are, what type of objects do we have to use for these rules to follow through, and we must follow through these rules. Therefore in math, we can create small enclosed worlds or systems. There we can learn that if we agree to certain rules – and that's the key word – then we must adhere to these rules. Alternatively, agree to certain amendments. For me this is important, because all natural sciences work with

given materials, while math creates its own resource materials for any situations. You have a greater freedom."

Tamás Varga was one of the organisers of the aspirations of mathematics education in the previous century, and was one leading figure of an extensive international school reform [13] One of his principles is that creation during mathematical activity can be recognised starting from early childhood. The curriculum of the reform and the methodological materials connected to it were created with his collaborators in a way that this principle was all along taken into consideration. Of course, it does not mean the creation of the new mathematical world, but the understanding of it within the environment of school mathematics, mainly among the frames of conceptualisation taking place during games, which were based on experience [15].

In Bolyai's sentence – according to the concept of Romanticism – only the chosen ones and geniuses are blessed with the ability of creation who are separated from the public. By the 20th century, the concept of creation was deprived of this romantic genius-cult and one of the key points of Tamás Varga's reform was that he did not think in elite education but he intended to make mathematics and the possibility of experiencing mathematical creation within the frames of public education available for everyone [5].

Rózsa Péter writes about the fact in Playing with Infinity (first in [28]) that there are no two cultures (humanities and science) but art and mathematics are related. With this, she refers to the text written by C. P. Snow [32, 36]. Rózsa Péter collaborated with Tamás Varga, and this approach to culture can be felt on the works by Tamás Varga, too.

3.4. Some phenomena that can be indirectly connected to this tradition

István Lénárt created the so-called Lénárt Sphere, which is suitable for visualising and teaching spherical and hyperbolic geometry in the junior section of elementary school [22].

With this teaching aid and the methodological construction connected to it, Lénárt made not only Bolyai's elementary steps of his mathematical results ostensive, but also the historical significance of mathematical result itself. Students can experience the opportunity of the *freedom of choice* between geometrical worlds.

In the environment of Hungarian mathematical talent development, which is known inter-nationally, creation appears to be a key part of mathematical activity. In this field, Lajos Pósa is the leading person in Hungary. He and his foundation organise this activity in a non-regular way, but in form of weekend or summer camps. (About the theoretical background see Dániel Katona's works, for example [18], whose PhD-research is about the Pósa-method.)

Not necessarily the Pósa-idea of talent management should be thought about to detect the notion of creation. About the creation of real numbers using geometry, the number line, see yet [25, 26] and similarly about the creation of rational numbers beginning from fractions see [27].

The Hungarian project is the carrier of the written tradition from a cultural point of view (led by Ö. Vancsó) which deals with the idea and background of the Complex Mathematics Education of T. Varga, which has rooted in this tradition in Hungary[4].

4. Tradition based on personal contacts

On examining the history of Tamás Varga's reform, one can conclude that the key element of the reform's spreading was engagement, which happened with the help of personal contacts. Many reports can be read how it happened.

Anna Kiss says in an interview about her own professional development as she talks about her participation in someone else's lesson as a listener: "Then it became obvious to me that I have to breathe from the air that determines the lesson, one has to experience it in one piece, how these discovery processes happen, and after that I have to find out what I can create in a way that is mine, too." (The source of the text is a book being about to be published.)

Thus, the functioning of the written tradition happens differently in a different environment. Below, I intend to report on a questionnaire's partial results. More specifically on those results that are in connection with the written and personally transmitted tradition and with the – in them appearing – notion of creation.

5. Questionnaire

This year, we examined a segment of teachers' beliefs related to mathematics as a school subject through an online questionnaire. We collected questions from a previous project LEMA [23] as a starting point, rephrased some of them and added further questions as well.

Now, we will only focus on some questions from the questionnaire, only on those ones, which give us information about the inquiry if the teachers' beliefs were influenced more by the written or the personal tradition, and also about the fact whether the notions of creation and freedom can be connected to school mathematics. The whole questionnaire can be the basis for further research[5].

[4] An international conference (Varga 100) was organised by this project in order to introduce that Varga's main ideas are also relevant today. You can read about it in two special issues of Teaching Mathematics and Computer Science (TMCS 2020/18.3). Another goal of this conference was to present the Legacy of T. Varga and to position it in the modern didactical trends. See K. Gosztonyi's plenary [14] and the panel plenary led by M. Artigue [2].

[5] A big and significant survey of this kind is the one carried out by the MTA Working Group on Mathematics in Public Education [19]. It worked with a large sample (4257 answers). The goal of this research was to find the biggest problems in mathematics education in Hungary. We did not want to draw up a complete belief map, as this can only be done through a more comprehensive and detailed test.

5.1. The sample

777 teachers took part in the questionnaire[6]. The distribution of participants by school type was as follows: 57 from lower primary (1–4. classes), 472 from upper primary school (5–8. classes), 64 from eight-year high school (5–12. classes), 79 from six-year high school (6–12. classes), 200 from four years high school (9–12. classes), 28 from vocational secondary school and 24 from other type (there are teachers who are employed in more than one schools). We do not have exact data, so we do not know how representative our questionnaire is, nevertheless, the total number of respondents is high, compared to previous research of this kind.

5.2. Questions

However, we asked not only closed-ended questions, but also open-ended questions where teachers could elaborate on the subject. This was important because we received several responses that were not closely related to the questions we asked, but clearly showed teachers' related areas of interest. Teachers were asked the following series of questions:

5.2.1. Introduction

"*Title: RESEARCH – subject-related questionnaire*
In the research below, we examine with Dr. Ödön Vancsó what picture exists in the head of practising teachers about school mathematics. It would help our work to a great extent if possibly the most teachers could fill in this form throughout the country, therefore, we are curious about your opinion, too..."

We had three questions about the years spent as a teacher (practice-time), where she/he got the university-degree and about other subjects.

Further questions and answers were as follows:

5.2.2. Questions on a five-point scale

"*At each question answers on a scale from 1 to 5 can be given.*
1 = I don't agree at all ... 5 = I completely agree."

For a significant number of questions, it is clear that on a five-point scale, answers are either "*agree*" or "*disagree*". Based on this data, we could say that there is a perception among the participants about the subject of mathematics. According to them:

1. Mathematics is an area where students can discover things.

2. In problemsolving, you don't need to know the one and only solution, you can experience choice and freedom.

[6] We would like to thank Gabriella Hajnal, Dániel Bebrevszky and Róza Hitérné Erdős for their help in sending out the questionnaire and all the participants for answering it.

Table 1

Question	1	2	3	4	5
If students get to grips with mathematical problems, they can often discover something new (connections and rules).	3 (0,4%)	25 (3,2%)	90 (11,6%)	310 (39,9%)	349 (44,9%)
In order to solve a mathematical task, one has to know the one and only correct procedure.	309 (39,8%)	237 (30,5%)	171 (22%)	45 (5,8%)	15 (1,9%)
In mathematics one finds the experience of freedom.	8(1%)	32 (4%)	163 (21%)	321 (41,3%)	253 (32,6%)
Every student can create or recreate parts of mathematics.	53 (6,8%)	132 (17%)	261 (33,6%)	226 (29,1%)	105 (13,5%)

5.2.3. Open-ended questions

The questionnaire also asked, where the respondents studied and how was their image of mathematics formed. Connections, significant teachers, personal student and teaching experiences dominated most in the formation of their image of mathematics.

“*Please write down some factors that influenced your opinion on mathematics as a school subject. These can be reading or other kinds of experiences, people, institutions, etc.*”

“*Other. In case of any other comments regarding the questionnaire that you would like us to know about, and that do not fit into any categories, please share with us here.*”

5.2.4. Results

1. When asked about creation, we did get the following answers.

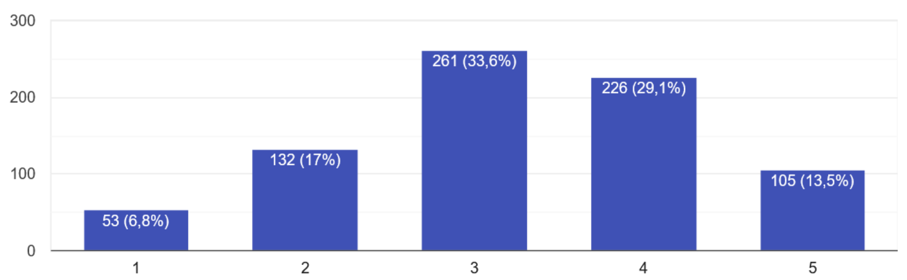

Figure 1. “Every student can create or recreate parts of mathematics.”

The answers were in the middle of the range, but we can say that creation as a belief is part of school mathematics in Hungary.

2. In the cultural tradition of mathematics education, there is a need for creation and artistic activity in relation to mathematics and mathematics in schools, but the practice seems to show something else. Creation is present, but not as prominently as one might expect based on the work of Tamás Varga and the leading didacticians and teachers of the 20th century.

3. Reading materials (books, journals) were much less frequent among responses, thus basic ideas in written didactic tradition influenced the teachers' image of mathematics through personal experiences. The written tradition and the transmission of practical experience are in harmony.

4. In conclusion (also based on the textual responses) we can say, that on the one hand there is an idyllic image of a school mathematics (that teachers consider to be good and that exists in their dreams as a content- and methodological environment). On the other hand, there is the reality that is much more prosaic and much less in line with teachers' image of mathematics. The tension between the two is evident from the textual responses.

To underline the last conclusion, we would like to quote two textual answers:

- "*In Hungarian public education, mathematics is taught in very heterogeneous groups. It would be nice if there were opportunities for discovery mathematics when the amount of practice added at home is negligible. (I teach in a high school, and the standards are going down there too. The amount of practice students do at home is not enough. But what can a district school say against the university's practise high schools in Budapest!*"

- "*I face it every day: classes of 33-36 students, maths lessons [45 minutes] for the whole class, 3 per week (this is in grades 9-10, before that there are smaller groups, but the number of lessons is still 3 per week), 24-26 teaching hours per week, plus free tutoring, specialised classes. Nevertheless, my enthusiasm is still there:) We compete in lessons, workshops, cutting, metalworking, playing free play...*"

6. Conclusions

Therefore, the philological idea that Bolyai's sentence was inspired by a poem or by a poetical biography, belongs to the history of the connection between creation and mathematics. With all its antecedents and effects, it can be proof for the fact that the *need for creation as a belief* is part of the Hungarian written didactic tradition. This belief is worth being taken into consideration when a current phenomenon of mathematics education is to be interpreted.

Based on the results of our questionnaire we can say that the concept of *creation* and another concept of the tradition, the *experience of freedom* is present among the beliefs of the mathematics teachers in Hungary. However, it must also be taken into account that reality and the day-to-day problems of school mathematics do not allow the basic ideas rooted in the tradition to develop sufficiently.

Nevertheless, this tension can be mildened by focusing on the common features existing in the tradition and among the beliefs. The personal experience seems to be the main source of the mathematics teachers' image of mathematics.

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