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# Short remark to the Rimán's Theorem

**Tamás Glavosits**

Institute of Mathematics, Department of Applied Mathematics University of Miskolc,
Miskolc-Egyetemváros, Hungary
tamas.glavosits@uni-miskolc.hu

**Abstract.** In this paper the general solution of the functional equation $f(x+y) = g(x) + h(y)$ $((x, y) \in D)$ is given with unknown functions $f\colon D_{x+y} \to Y$, $g\colon D_x \to Y$, $h\colon D_y \to Y$ where $D \subseteq \mathbb{G}^2$ is a nonempty, open set, $(\mathbb{G}, \leqslant)$ is an ordered, dense, Abelian group, the topology on $\mathbb{G}$ is generated by the open intervals of $\mathbb{G}$, the sets $D_x$, $D_y$, $D_{x+y}$ are defined by $D_x := \{u \in \mathbb{G} \mid \exists v \in \mathbb{G} : (u, v) \in D\}$, $D_y := \{v \in \mathbb{G} \mid \exists u \in \mathbb{G} : (u, v) \in D\}$, $D_{x+y} := \{z \in \mathbb{G} \mid \exists (u, v) \in D : z = u + v\}$, and $Y(+)$ is an Abelian group.

The main result of the article is a common generalization of similar results by L. Székelyhidi and J. Rimán. Analogous theorem concerning logarithmic functions is also shown.

*Keywords:* additive functional equations, logarithmic functional equations, Pexider generalizations, restricted functional equations, Archimedean ordered Abelian groups, dense ordered groups, general solution of functional equations

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## 1. Introduction

The main purpose of this article is to prove the generalization of J. Rimán's Extension Theorem [21]. Now, we give a non-exhaustive overview of the most important steps of the theory of Extension and Uniqueness Theorems concerning restricted Pexider additive functional equations.

In the sequel we will use the notations

$$\begin{aligned} D_x &:= \{u \in X \mid \exists v \in \mathbb{G} : (u, v) \in D\}, \\ D_y &:= \{v \in Y \mid \exists u \in \mathbb{G} : (u, v) \in D\}, \\ D_{x+y} &:= \{z \in X \mid \exists (u, v) \in D : z = u + v\} \end{aligned}$$

where $D \subseteq \mathbb{G}^2 := \mathbb{G} \times \mathbb{G}$ and $\mathbb{G}(+)$ is a grupoid.

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The early results were grouped around the following problem. Let be $D \subseteq \mathbb{R}^2$, $f\colon D_x \cup D_y \cup D_{x+y} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) + f(y) \quad ((x,y) \in D). \tag{RestAdd}$$

The functional equation (RestAdd) is said to be restricted additive functional equation. The problem is to find a function $F\colon \mathbb{R} \to \mathbb{R}$ such that

$$F(x+y) = F(x) + F(y) \quad (x, y \in \mathbb{R}), \tag{1.1}$$

and $F(x) = f(x)$ for all $x \in \mathcal{D}_f$ (see [14] Part IV. Geometry, Section Extension of Functional equation p. 447–460). The function $F$ is said to be additive extension of the function $f$ from the set $D$ to the $\mathbb{R}^2$.

If a function $F\colon \mathbb{R} \to \mathbb{R}$ satisfies the equation (1.1), then the function $F$ is said to be Cauchy-additive function (see A. E. Legendre [18], C. F. Gauss [9]). A. L. Cauchy first found the continuous solutions of equation (1.1) [5].

In [4] $D = (\mathbb{R}_+ \cup \{0\})^2$ $(\mathbb{R}_+ := \{x \in \mathbb{R}_+ \mid x > 0\})$. The solution of equation (RestAdd) is $f(x) = F(x)$ for all $x \in \mathbb{F}_+$ where the function $F$ is a Cauchy-additive function.

In [2] the concept of quasi-extension can be found. The situation is that $D \subseteq \mathbb{R}^2$ is a nonempty connected open set, and the functions $f$ satisfies the functional equation (RestAdd) for all $(x,y) \in D$ then there exists an additive function $F$ and exist constants $C_1$, $C_2 \in \mathbb{R}$ such that

$$\begin{aligned} f(z) &= F(z) + C_1 + C_2 && (z \in D_{x+y}), \\ f(u) &= F(u) + C_1 && (u \in D_x), \\ f(v) &= F(v) + C_2 && (v \in D_y). \end{aligned} \tag{1.2}$$

If the function $f$ and the additive function $F$ is in the form of (1.2), then the function $F$ is said to be quasi extension of the function $f$.

In [6] $D = \mathbb{R}_+^2$ or $D$ is circle neighbourhood of the point $(0,0) \in \mathbb{R}^2$. In these case $f$ has additive extension.

In [23] $D$ is an open subset of $\mathbb{R}^2$. The author of this paper has shown that the set $D$ is a countable disjoint union of connected open sets, that is $D = \bigcup_i D^i$. The sets $D_i$ is said to be components of the set $D$. For all $i$ there exists an additive function $F_i\colon \mathbb{R} \to \mathbb{R}$ and constants $C_1^i$, $C_2^i \in \mathbb{R}$ such that

$$\begin{aligned} f(z) &= F_i(z) + C_1^i + C_2^i && (z \in D_{x+y}^i), \\ f(u) &= F_i(u) + C_1^i && (u \in D_x^i), \\ f(v) &= F_i(v) + C_2^i && (v \in D_y^i). \end{aligned} \tag{1.3}$$

If $i \neq j$, then the obtained functions $F_i$, and $F_j$, as well as the obtained constants $C_1^i$, and $C_1^j$, or $C_2^i$, and $C_2^j$ are not necessarily different depending on whether $D_x^i \cap D_x^j \neq \emptyset$ or $D_y^i \cap D_y^j \neq \emptyset$ or $D_{x+y}^i \cap D_{x+y}^j \neq \emptyset$.

It is also worth mentioning that if $D^i_x \cap D^i_{x+y} \neq \emptyset$ then $C^i_2 = 0$; if $D^i_y \cap D^i_{x+y} \neq \emptyset$ then $C^i_1 = 0$; if $D^i_x \cap D^i_y \neq \emptyset$ then $Ci_1 = Ci_2$ for all component $D^i$. If the point $(0,0)$ is an inner point of a component $D^i$ then $D^i_x \cap D^i_y \cap D^i_{x+y} \neq \emptyset$ thus $C^i_1 = C^i_2 = 0$.

In [21] J. Rimán studied restricted Pexider additive functional equations in the form

$$f(x+y) = g(x) + h(y) \quad ((x,y) \in D) \tag{RestPexAdd}$$

where the set $D$ is a connected open subset of the set $\mathbb{R}^2$, $E = E(+)$ is an Abelian group, and the unknown functions $f\colon D_{x+y} \to E$, $g\colon D_x \to E$, $h\colon D_y \to E$ satisfy the equation (RestPexAdd) for all $(x,y) \in D$. The solution of equation (1.4) is

$$\begin{aligned} f(z) &= F(z) + C_1 + C_2 && (z \in D_{x+y}),\\ g(u) &= F(u) + C_1 && (u \in D_x),\\ h(v) &= F(v)C_2 && (v \in D_y), \end{aligned} \tag{1.4}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is an additive function, $C_1$, $C_2 \in E$ are constants.

In [1] $D = H(I)$ where $I$ is a nonempty open interval of the real line and the set $H(I)$ is defined by

$$H(I) := \{(x,y) \in \mathbb{R}^2 \mid x, y, x+y \in I\}.$$

The set $H(I)$ is a hexagon, sometimes a triangle or the emptyset.

M. Kuczma in his book [16] investigated both of Pexider type functional equations and additive functional equations, but did not consider restricted Pexider-additive functional equations. He used Jensen functions for his Extension Theorem and gave the solution of equation (RestAdd) (Theorem 13.6.1), where $D$ is a nonempty, connected, open subset of $\mathbb{R}^{2N} := \mathbb{R}^N \times \mathbb{R}^N$ and $D_x \cup D_y \cup D_{x+y} \subseteq \mathcal{D}_f$. He showed that the solution of equation (RestAdd) is in the form of (1.2) where $F\colon \mathbb{R}^N \to \mathbb{R}^N$ is an additive function, $C_1$, $C_2 \in \mathbb{R}^N$ are constants. The extension was brought back to the theory of Jensen functions.

An $X = X(+)$ Abelian group is said to be uniquely 2-divisible, if for all $x \in X$ there uniquely exists an $y \in X$ such that $y + y := 2y = x$. This element $y \in X$ is denoted by $y = \frac{1}{2}x$. A nonempty set $A \subseteq X$ is said to be midconvexe, if $\frac{x+y}{2} \in A$ for all $x, y \in A$. Let $Y = Y(+)$ be also a uniquely 2-divisible Abelian group. A function $j\colon A \to Y$ is said to be Jensen [7, 15, 16] if

$$j\left(\frac{x+y}{2}\right) = \frac{j(x)+j(y)}{2} \quad (x, y \in A).$$

The way outlined by M. Kuczma is not suitable for us, since we do not want to deal with either 2-divisible or p-divisible groups, and we do not think that the vector space structure is necessary for an additive extension theorem.

In the article [20] an extension theorem for restricted Pexider additive functional equation can be found, where $D \subseteq (\mathbb{R}^N)^2$ is a nonempty, connected, open set.

In the book [3] several functional equations can be found in more general abstract algebraic settings .

Concerning the Extension Theorems see also [8, 13, 17].

## 2. Some necessary concepts and results

Now, we review the concepts and results which will be used in the sequel.

- If $\mathbb{G}(+,\leqslant)$ is an ordered group, $\alpha$, $\beta \in \mathbb{G}$ such that $\alpha < \beta$ then the set $]\alpha,\beta[ := \{x \in \mathbb{G} \mid \alpha < x < \beta\}$ is said to be open interval.
- An ordered group $\mathbb{G}(+ \leqslant)$ is said to be dense (in itself) if $]\alpha,\beta[ \neq \emptyset$ for all $\alpha$, $\beta \in \mathbb{G}$ with $\alpha < \beta$.
- An ordered group $\mathbb{G}(+,\leqslant)$ is said to be Archimedean ordered if for all $x$, $y \in \mathbb{G}_+$ there exists a positive integer $n$ such that $y < nx := x + \cdots + x$.
- An ordered field $\mathbb{F}(+,\cdot,\leqslant)$ is said to be Archimedean ordered if $\mathbb{F}(+,\leqslant)$ is an Archimedean ordered group.

Now, we review some properties of open intervals ([10, 12]). The open intervals are

- translation invariant, that is, if $\mathbb{G}(+,\leqslant)$ is an ordered, dense, Abelian group, then $\gamma + ]\alpha,\beta[ = ]\gamma+\alpha,\gamma+\beta[$ for all $\alpha$, $\beta$, $\gamma \in \mathbb{G}$ such that $\alpha < \beta$.
- additive, that is, if $\mathbb{G}(+,\leqslant)$ is an ordered, dense, Abelian group, then $]\alpha,\beta[ + ]\gamma,\delta[ = ]\alpha+\gamma,\beta+\delta[$ for all $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{G}$ with $\alpha < \beta$ and $\gamma < \delta$.
- homothety invariant, that is, if $\mathbb{F}(+,\cdot,\leqslant)$ is an ordered field, then $\gamma \cdot ]\alpha,\beta[ = ]\gamma\alpha,\gamma\beta[$ for all $\alpha$, $\beta$, $\gamma \in \mathbb{F}$ with $\alpha < \beta$ and $\gamma > 0$.
- multiplicative, that is, if $\mathbb{F}(+,\cdot,\leqslant)$ is an ordered field, then $]\alpha,\beta[ \cdot ]\gamma,\delta[ = ]\alpha\gamma,\beta\delta[$ for all $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{F}$ with $0 < \alpha < \beta$ and $0 < \gamma < \delta$.

If $\mathbb{G}(+,\leqslant)$ is an ordered group, $x \in \mathbb{G}$, (or $x := (x_1,x_2) \in \mathbb{G}^2$), $\varepsilon \in \mathbb{G}_+$, then define the set $B(x,\varepsilon)$ by $B(x,\varepsilon) := ]x-\varepsilon, x+\varepsilon[$, ($B(x,\varepsilon) := ]x_1-\varepsilon, x_1+\varepsilon[ \times ]x_2-\varepsilon, x_2+\varepsilon[$) respectively. The set $B(x,\varepsilon)$ is said to be open neighbourhood of the point $x$ with radius $\varepsilon$.

A function $a\colon X \to Y$ is said to be additive if $X(+)$ and $Y(+)$ are algebraic structures, and

$$a(x+y) = a(x) + a(y) \quad (x, y \in X).$$

A function $l\colon X \to Y$ is said to be logarithmic if $X(\cdot)$ and $Y(+)$ are algebraic structures, and

$$l(xy) = l(x) + l(y) \quad (x, y \in X).$$

Concerning the additive and logarithmic functions see [3, 16].

# 3. Extension Theorem for Pexider additive functional equation

We shall use the Existence Theorem for additive functions [10] according to which if $\mathbb{G}(+,\leqslant)$ is an Archimedean ordered, dense, Abelian group, $Y(+)$ is a group, $\varepsilon\in\mathbb{G}_+$, and the function satisfy the equation (RestAdd) where $D:=]0,\varepsilon[^2$ then there exists an additive function $a\colon\mathbb{G}\to Y$ which extends the function $f$ from $]-2\varepsilon,2\varepsilon[$ to $\mathbb{G}$.

**Theorem 3.1.** *If $\mathbb{G}(+,\leqslant)$ is an Archimedean ordered, dense, Abelian group, $Y(+)$ is an Abelian group, $x_0$, $y_0\in\mathbb{G}$, $\varepsilon\in\mathbb{G}_+$, and the functions $f\colon B(x_0+y_0,2\varepsilon)\to Y$, $g\colon B(x_0,\varepsilon)\to Y$, $h\colon B(y_0,\varepsilon)\to Y$ satisfies the functional equation (RestPexAdd) then there exists an additive function $a\colon\mathbb{G}\to Y$ and exist constants $C_1$, $C_2\in Y$ such that the functions $f$, $g$, $h$ are in the form of (1.4).*

***Proof.*** By the translation invariant property of the open intervals we have that

$$\begin{aligned}B(x_0,\varepsilon)&=x_0+B(0,\varepsilon),\\ B(y_0,\varepsilon)&=y_0+B(0,\varepsilon),\\ B(x_0+y_0,2\varepsilon)&=x_0+y_0+B(0,2\varepsilon).\end{aligned}$$

Define the functions $F\colon B(0,2\varepsilon)\to Y$, $G\colon B(0,\varepsilon)\to Y$, $H\colon B(0,\varepsilon)\to Y$ by

$$\begin{aligned}F(w)&=f(x_0+y_0+w)&&(w\in B(0,2\varepsilon)),\\ G(u)&=g(x_0+u)&&(u\in B(0,\varepsilon)),\\ H(v)&=h(y_0+v)&&(v\in B(0,\varepsilon)).\end{aligned}\tag{3.1}$$

Then $F(0)=f(x_0+y_0)$, $G(0)=g(x_0)$, $H(0)=h(y_0)$ and

$$F(u+v)=G(u)+H(v)\quad(u,v\in B(0,\varepsilon)).$$

Thus we obtain that

$$\begin{aligned}F(u)&=G(u)+H(0)=G(u)+h(y_0)&&(u\in B(0,\varepsilon)),\\ F(v)&=G(0)+H(v)=g(x_0)+H(v)&&(v\in B(0,\varepsilon)),\end{aligned}\tag{3.2}$$

whence we obtain that

$$\begin{aligned}F(u)+F(v)&=G(u)+H(v)+g(x_0)+h(y_0)\\ &=F(u+v)+g(x_0)+h(y_0)\quad(u,v\in B(0,\varepsilon)).\end{aligned}$$

Define the function $\varphi\colon B(0,\varepsilon)\to Y$ by

$$\varphi(x):=F(x)-(g(x_0)+h(y_0))\quad(x\in B(0,2\varepsilon))\tag{3.3}$$

Then

$$\varphi(x+y)=\varphi(x)+\varphi(y)\quad(x,y\in B(0,\varepsilon)),$$

whence by the Extension Theorem [10] we obtain that there exists an additive function $a\colon \mathbb{G} \to Y$ such that

$$\varphi(x) = a(x) \quad (x \in B(0, 2\varepsilon)). \tag{3.4}$$

Then by equations (3.1), (3.3), and (3.4) we have that

$$\begin{aligned} f(x_0 + y_0 + w) &\overset{(3.1)}{=} F(w) \overset{(3.3)}{=} \varphi(w) + (g(x_0) + h(y_0)) \\ &\overset{(3.4)}{=} a(w) + (g(x_0) + h(y_0)) \quad (w \in B(0, 2\varepsilon)). \end{aligned} \tag{3.5}$$

By equations (3.1), (3.2), (3.3) and (3.4) we have that

$$\begin{aligned} g(x_0 + u) &\overset{(3.1)}{=} G(u) \overset{(3.2)}{=} F(u) - h(y_0) \overset{(3.3)}{=} \varphi(u) + g(x_0) \\ &\overset{(3.4)}{=} a(w) + g(x_0) \quad (w \in B(0, \varepsilon)). \end{aligned} \tag{3.6}$$

By equations (3.1), (3.2), (3.3) and (3.4) we have that

$$\begin{aligned} h(y_0 + v) &\overset{(3.1)}{=} H(v) \overset{(3.2)}{=} F(u) - g(x_0) \overset{(3.3)}{=} \varphi(v) + h(y_0) \\ &\overset{(3.4)}{=} a(u) + h(y_0) \quad (w \in B(0, \varepsilon)). \end{aligned} \tag{3.7}$$

Take the substitutions: $w \longleftarrow w - (x_0 + y_0)$ in (3.5), $u \longleftarrow u - x_0$ in (3.6), $v \longleftarrow v - y_0$ in (3.7), and define the constants $c\ d \in Y$ by $c := g(x_0) - a(x_0)$, $d := h(y_0) - a(y_0)$ thus the translation invariant property of the intervals we obtain equation (1.4) which was to be prooved. □

We shall use the Existence Theorem for logarithmic functions in [10] according to which if $\mathbb{F}(+, \cdot, \leqslant)$ is an Archimedean ordered field, $Y(+)$ is a group, $\varepsilon \in \mathbb{F}$ such that $\varepsilon > 1$, and the function $f\colon ]\varepsilon^{-2}, \varepsilon^2[ \to Y$ satisfies the equation

$$f(xy) = f(x) + f(y) \quad (x, y \in ]\varepsilon^{-1}, \varepsilon[),$$

then there exists a logarithmic function $l\colon \mathbb{F}_+ \to Y$ which extends the function $f$ from $]\varepsilon^{-2}, \varepsilon^2[$ to the $\mathbb{F}_+^2$.

**Theorem 3.2.** *If $\mathbb{F}(+, \cdot, \leqslant)$ is an Archimedean ordered field, $Y(+)$ is an Abelian group, $x_0$, $y_0 \in \mathbb{F}_+$, $\varepsilon \in \mathbb{F}_+$, and $f\colon ]x_0y_0\varepsilon^{-2}, x_0y_0\varepsilon^2[ \to Y$, $g\colon ]x_0\varepsilon^{-1}, x_0\varepsilon[ \to Y$, $h\colon ]y_0\varepsilon^{-1}, y_0\varepsilon[ \to Y$ are functions such that*

$$f(xy) = g(x) + h(y) \quad (x \in ]x_0\varepsilon^{-1}, x_0\varepsilon[,\ y \in ]y_0\varepsilon^{-1}, y_0\varepsilon[),$$

*then there exists a logarithmic function $l\colon \mathbb{F}_+ \to Y$ and exist constants $C_1$, $C_2 \in Y$ such that*

$$\begin{aligned} f(w) &= l(w) + C_1 + C_2 && (w \in ]x_0y_0\varepsilon^{-2}, x_0y_0\varepsilon^2[), \\ g(u) &= l(u) + C_1 && (u \in ]x_0\varepsilon^{-1}, x_0\varepsilon[), \\ h(v) &= l(v) + C_2 && (v \in ]y_0\varepsilon^{-1}, y_0\varepsilon[). \end{aligned}$$

***Proof.*** The proof is analogues to the proof of the Theorem 3.1. □

# 4. Topology generated by the open intervals of an Archimedean ordered Abelian group

Let $\mathbb{G} = \mathbb{G}(+, \leqslant)$ be an ordered group, $X \in \{\mathbb{G}, \mathbb{G}^2\}$ and $D \subseteq X$. The set $D$ is said to be open if for every point $x$ in $D$ there exists an $\varepsilon \in \mathbb{G}_+$ such that $B(x, \varepsilon) \subseteq D$.

A subset $D \subseteq X$ is said to be well-chained, if for all $x, y \in D$ there exists a finite sequence $B_i := B(x_i, \varepsilon_i)$ $(i = 0, 1, \ldots, n)$ such that

- $B_i \subseteq D$ for all $i = 0, 1, \ldots, n$,
- $x \in B_0$, $y \in B_n$,
- $B_{i-1} \cap B_i \neq \emptyset$ for all $i = 1, \ldots, n$.

A subset $C$ of a nonempty, open set $D \subseteq X$ is a component of $D$ if $C$ is a maximal (with respect the inclusion) well-chained, open subset of $D$.

A topological space $X(\mathcal{T})$ is said to be separable if there exists a subset $Y \subseteq X$ which is countable, infinite, and dense (in $X$).

**Theorem 4.1.** *If $\mathbb{G} = \mathbb{G}(+, \leqslant)$ is an ordered group, $X \in \{\mathbb{G}, \mathbb{G}^2\}$ and $D \subseteq X$ is a nonempty, well-chained, open set, then*

1. *$D$ is a disjoint union of its components;*
2. *If $X$ is separable then $D$ has countable components.*

***Proof.*** 1. Define the family $\mathcal{B}$ by

$$\mathcal{B} := \{B(x, \varepsilon) \subseteq D \mid x \in D, \varepsilon \in \mathbb{G}_+\} := \{B_\alpha \mid \alpha \in \Gamma\}.$$

Define the equivalence relation on $\mathcal{B}$ by $B_\alpha \sim B_\beta$ if and only if there exists a finite sequence $B_{\alpha_i}$ $(i = 0, 1, \ldots, n)$ such that $B_{\alpha_0} = B_\alpha$, $B_{\alpha_n} = B_\beta$ and $B_{\alpha_{i-1}} \cap B_{\alpha_i} \neq \emptyset$ for all $i = 1, 2, \ldots, n$. The set $\mathcal{B}$ is a disjoint union of its equivalence classes. The components of the set $D$ are the union of all balls $B_\alpha$ that belong to the same equivalence class.

2. Let $Y$ be a countable, dense subset of the set $X$, and let $B \subseteq X$ be a nonempty, open subset with components $\{D^i\}_{i \in I}$. Then for all $i \in I$ there exists a ball $B_i := B(x_i, \varepsilon_i)$ such that $B_i \subseteq D^i$. If $i \neq j$ then $B_i \cap B_j = \emptyset$. Since $Y$ is dense in $\mathbb{G}$ thus for all $i \in I$ there exists an $y_i \in Y$ such that $y_i \in B_i$. Define the function $\varphi \colon \{D^i\}_{i \in I} \to Y$ by $\varphi(D^i) := y_i$. Since the function $\varphi$ is injective thus the set $\{D^i\}_{i \in I}$ is countable. □

**Example 4.2.** If $\mathbb{G}(+, \leqslant)$ is a $p$- divisible, Archimedean ordered, Abelean group for a prime number $p$, then $\mathbb{G}$ is separable.

**Example 4.3.** Let $a \colon \mathbb{R} \to \mathbb{R}$ is a noncontinuous additive function. As it is well-known that the graph of $a$ is dense in $\mathbb{R}^2$ (with respect to usual topology on $\mathbb{R}^2$), but the restriction of the function $a$ to the set $\mathbb{Q}$ (where $\mathbb{Q}$ denotes the set of all rationals) is continuous with respect to the topology on the set $\mathbb{Q}(+)$ defined above, and the usual topology on the real line [11].

# 5. The generalization of Rimán's Extension Theorem

We shall use the Uniqueness Theorem for additive functions [10], according to which if $\mathbb{G}(+,\leqslant)$ is an Archimedean ordered, Abelian group, $a\colon \mathbb{G} \to Y$ is an additive function, $C \in Y$, and $]\alpha,\beta[ \subseteq Y$ is a nonempty interval such that $a(x) = C$ for all $x \in ]\alpha,\beta[$ then $a(x) = 0$ for all $x \in \mathbb{G}$ (and thus $C = 0$).

Now, we give the generalization of Rimán's Extension Theorem:

**Theorem 5.1.** *If $\mathbb{G}(+,\leqslant)$ be an Archimedean ordered, dense, Abelian group, and $D \subseteq \mathbb{G}^2$ is an open set with components $\{D^i \mid i \in I\}$ and $Y$ is an Abelian group then the functions $f\colon D_{x+y} \to Y$, $g\colon D_x \to Y$, $h\colon D_y \to Y$ if and only if are solutions of the functional equation (RestPexAdd) then there exists a family of additive functions $a_i\colon \mathbb{G} \to Y$ $(i \in I)$ and exist families of constants $C_1^i$, $C_2^i \in Y$ $(i \in I)$ such that*

$$\begin{aligned} f(z) &= a_i(z) + C_1^i + C_2^i && (z \in D_{x+y}^i),\\ g(u) &= a_i(u) + C_1^i && (u \in D_x^i),\\ h(v) &= a_i(v) + C_2^i && (v \in D_y^i) \end{aligned} \tag{5.1}$$

*with*

1. *if $D_{x+y}^i \cap D_{x+y}^j \neq \emptyset$, then $a_i = a_j$, and $C_1^i + C_2^i = C_1^i + C_2^i$;*
2. *if $D_x^i \cap D_x^j \neq \emptyset$, then $a_i = a_j$, and $C_1^i = C_1^j$;*
3. *if $D_y^i \cap D_y^j \neq \emptyset$, then $a_i = a_j$, and $C_2^i = C_2^j$*

*for all $i$, $j \in I$, $i \neq j$.*

***Proof.*** Let us assume that the functions $f$, $g$, $h$ satisfy the functional equation (RestPexAdd). By Theorem 3.1 we obtain that they are in the form of (5.1), and by Uniqueness Theorem [10] properties 1., 2., and 3. are fulfilled.

Conversely, let us assume that the functions $f$, $g$, $h$ are defined by equation (5.1), and the properties 1., 2., and 3. are fulfilled. These functions are well-defined, and they satisfy the functional equation (RestPexAdd). □

**Theorem 5.2.** *If $\mathbb{G}(+,\leqslant)$ be an Archimedean ordered, dense, Abelian group, and $D \subseteq \mathbb{G}^2$ is an open set with components $\{D^i \mid i \in I\}$ and $Y$ is an Abelian group. Define the set $D_0 := D_x \cup D_y \cup D_{x+x}$. The function $f\colon D_0 \to Y$ is satisfies functional equation (RestAdd) if and only if then there exists a family of additive functions $a_i\colon \mathbb{G} \to Y$ $(i \in I)$ and exist families of constants $C_1^i$, $C_2^i \in Y$ for all $i \in I$ such that*

$$\begin{aligned} f(z) &= a_i(z) + C_1^i + C_2^i && (z \in D_{x+y}^i),\\ g(u) &= a_i(u) + C_1^i && (u \in D_x^i),\\ h(v) &= a_i(v) + C_2^i && (v \in D_y^i) \end{aligned} \tag{5.2}$$

*with*

1. *If* $D^i_{x+y} \cap D^j_{x+y} \neq \emptyset$*, then* $a_i = a_j$*, and* $C^i_1 + C^i_2 = C^i_1 + C^i_2$*;*
2. *If* $D^i_x \cap D^j_x \neq \emptyset$*, then* $a_i = a_j$*, and* $C^i_1 = C^j_1$*;*
3. *If* $D^i_y \cap D^j_y \neq \emptyset$*, then* $a_i = a_j$*, and* $C^i_2 = C^j_2$*;*

*for all* $i$*,* $j \in I$*,* $i \neq j$*, moreover,*

4. *If* $D^i_{x+y} \cap D^i_x \neq \emptyset$*, then* $C^i_2 = 0$*;*
5. *If* $D^i_{x+y} \cap D^i_y \neq \emptyset$*, then* $C^i_1 = 0$*;*
6. *If* $D^i_y \cap D^i_y \neq \emptyset$*,* $C^i_2 = C^j_2$

*for all* $i \in I$*.*

***Proof.*** The proof can be easily obtained by Theorem 5.1 and the Uniqueness Theorem [10]. □

## 6. An application

Now we show a version of the well-known Rado-Baker functional equation [20].

It is worth mentioning that if $\mathbb{F}(+, \cdot, \leqslant)$ is an ordered field then $\mathbb{F}^2$ is a two-dimensional vector space over the ordered field $\mathbb{F}$ with the usual point-wise definition of vector operations. The set $C \subseteq \mathbb{F}^2$ is said to be

- convex if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$, and $\lambda \in ]0, 1[$;
- cone if $\lambda x \in C$ for all $\lambda \in \mathbb{F}_+$, and $x \in C$;
- convex cone if $\lambda x + \mu y \in C$ for all $x, y \in C$, and $\lambda, \mu \geqslant 0$ with $\lambda^2 + \mu^2 > 0$,

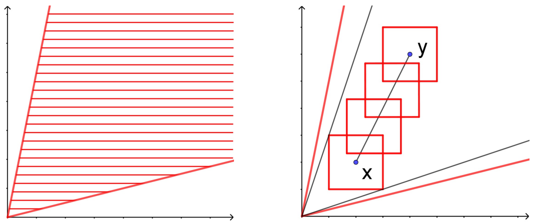
see Leonard Lewis [19], Rockafellar [22].

Let $\mathbb{F}(+, \cdot, \leqslant)$ be an Archimedean ordered field, $\alpha \in \mathbb{F}_+ \cup \{0\}$, $\beta \in \mathbb{F}_+ \cup \{+\infty\}$ such that $0 \leqslant \alpha < \beta \leqslant +\infty$. Define the set $C := C_{\alpha,\beta}$ by

$$C_{\alpha,\beta} \doteq \begin{cases} \{(x,y) \in \mathbb{F}^2_+ | \alpha x < y < \beta x\}, & \text{if } \alpha \in \mathbb{F}_+ \cup \{0\},\ \beta \in \mathbb{F}_+; \\ \{(x,y) \in \mathbb{F}^2_+ | \alpha x < y\}, & \text{if } \beta = +\infty. \end{cases}.$$

**Proposition 6.1.** *The set* $C = C_{\alpha,\beta}$ *is a nonempty, open, well-chained set.*

***Proof.*** Since $C_{\alpha,\beta}$ is a nonempty, open, convex cone thus it is a nonempty, well-chained, open set.




□

**Proposition 6.2.** *If $\mathbb{F}(+,\leqslant)$ is an Archimedean ordered field, $Y(+)$ is an abelian group, the functions $P,Q,R\colon \mathbb{F}_+\to Y$ are solutions of functional equation*

$$P(x+y)=Q(x)+R(y)\quad (x,y\in C(\alpha,\beta)) \tag{6.1}$$

*then there exists an additive function $a\colon \mathbb{F}\to Y$ and constants $C_1$, $C_2\in Y$ such that*

$$\begin{aligned} P(x)&=a(x)+C_1+C_2 && (x\in\mathbb{F}_+),\\ Q(x)&=a(x)+C_1 && (x\in\mathbb{F}_+),\\ R(x)&=a(x)+C_2 && (x\in\mathbb{F}_+). \end{aligned} \tag{6.2}$$

***Proof.*** Let $D:=C_{\alpha,\beta}$. Since $D_x=D_y=D_{x+y}=\mathbb{F}_+$ thus by Theorem 5.1 we obtain the statement. □

The following Theorem is a generalization of Rado–Baker Theorem [4], and if can be obtained from Proposition 6.2 as a simple consequence.

**Theorem 6.3.** *Let $\mathbb{F}(+,\cdot,\leqslant)$ be an Archimedean ordered field, $Y(+)$ be an Abelian group, $\alpha,\beta,\gamma,\delta\in\mathbb{F}$ such that $\alpha\delta-\beta\gamma\neq 0$. The functions $P,Q,R\colon\mathbb{F}_+\to Y$ if and only if satisfy the functional equation*

$$P((\alpha+\gamma)x+(\beta+\delta)y)=Q(\alpha x+\beta y)+R(\gamma x+\delta y),\quad (x,y\in\mathbb{F}_+) \tag{6.3}$$

*if they are of the form of (6.2) where $a\colon\mathbb{F}\to Y$ is an additive function, $C_1,C_2\in Y$ are constants.*

***Proof.*** Let us assume that the functions $P,Q,R\colon\mathbb{F}_+\to Y$ satisfy the functional equation (6.3) where az $\alpha,\beta,\gamma,\delta\in\mathbb{F}$ such that $\alpha\delta-\beta\gamma\neq 0$. Take the following substitution in (6.3):

$$\begin{gathered} P((\alpha+\gamma)x+(\beta+\delta)y)=Q(\underbrace{\alpha x+\beta y}_{u})+R(\underbrace{\gamma x+\delta y}_{v}),\\ x\leftarrow\frac{\delta u-\beta v}{\alpha\beta-\beta\gamma}>0\quad y\leftarrow\frac{\alpha v-\gamma u}{\alpha\delta-\beta\gamma}>0. \end{gathered} \tag{6.4}$$

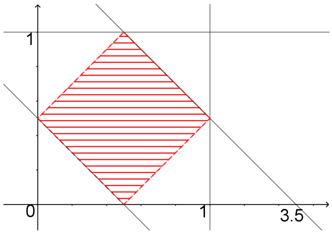
Thus we obtain that the functions $P,Q,R$ satisfy the equation (6.1) where the constants $\alpha\in\mathbb{F}_+\cup\{0\}$ and $\beta\in\mathbb{F}_+\cup\{+\infty\}$ are defined by

- $\alpha:=\frac{\gamma}{\alpha}$, $\beta:=\frac{\delta}{\beta}$ if $\alpha\delta-\beta\gamma>0$ and $\beta\neq 0$;
- $\alpha:=\frac{\gamma}{\alpha}$, $\beta:=+\infty$ if $\alpha\delta-\beta\gamma>0$ and $\beta=0$;
- $\alpha:=\frac{\delta}{\beta}$, $\beta:=\frac{\gamma}{\alpha}$ if $\alpha\delta-\beta\gamma<0$ and $\alpha\neq 0$;
- $\alpha:=\frac{\delta}{\beta}$, $\beta:=+\infty$ if $\alpha\delta-\beta\gamma<0$ and $\alpha=0$.

By Proposition 6.1 we obtain the statement. The converse statement is evident. □

# 7. Examples and problems

**Example 7.1.** Let $D := \{(x,y) \in \mathbb{R}^2 \mid |x-0.5| + |y-0.5| < 0.5\}$.




$$D_x = D_y = ]0,1[,$$
$$D_{x+y} = ]0.5, 1.5[.$$

Define the set $D_0$ by $D_0 := D_x \cup D_y \cup D_{x+y}$. By Theorem 5.1 we obtain that the general solution of the functional equation is

$$f(x) = a(x) \quad (x \in D_0 = ]0, 1.5[)$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a Cauchy additive function.

Let $D \subseteq \mathbb{R}^2$ be a well-chained open set with components $D^1$, $D^2$. By Theorem 5.1 we obtain that the general solution of functional equation (RestPexAdd) is in the form of (1.3).

$$f(z) = \begin{cases} a_1(z) + C_1^1 + C_2^1, & \text{if } z \in D^1_{x+y}; \\ a_2(z) + C_1^2 + C_2^2, & \text{if } z \in D^2_{x+y}; \end{cases}$$
$$g(u) = \begin{cases} a_1(u) + C_1^1, & \text{if } u \in D^1_x; \\ a_2(u) + C_1^2, & \text{if } u \in D^2_x; \end{cases}$$
$$h(v) = \begin{cases} a_1(v) + C_2^1, & \text{if } v \in D^1_y; \\ a_2(v) + C_2^2, & \text{if } v \in D^2_y, \end{cases}$$

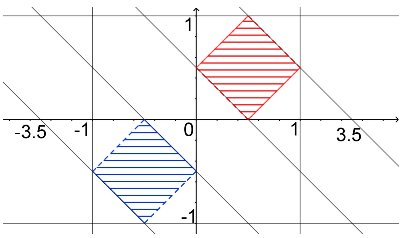
where $a_i$ is an additive function, $C_1^i$, $C_2^i$ are constants for all $i = 1, 2$.

The following two examples show how the structure of the general solution depends on the geometry of the sets $D^1$ and $D^2$.

**Example 7.2.** Let

$$D^1 := \{(x,y) \in \mathbb{R}^2 \mid |x-0.5| + |y-0.5| < 0.5\},$$
$$D^2 := \{(x,y) \in \mathbb{R}^2 \mid |x+0.5| + |y+0.5| < 0.5\},$$

and let $D := D_1 \cup D_2$.




$$D^1_x = D^1_y = ]0,1[,$$
$$D^1_{x+y} = ]0.5, 1.5[,$$
$$D^2_x = D^2_y = ]-1,0[,$$
$$D^2_{x+y} = ]-1.5, -0.5[.$$

By Theorem 5.2 we have that since $D^1_{x+y} \cap D^1_x \neq \emptyset$ thus $C^1_2 = 0$. Since $D^1_{x+y} \cap D^1_y \neq \emptyset$ thus $C^1_1 = 0$. Since $D^2_{x+y} \cap D^2_x \neq \emptyset$ thus $C^2_2 = 0$. Since $D^2_{x+y} \cap D^2_y \neq \emptyset$ thus $C^2_1 = 0$. Whence we obtain that the general solution of equation (RestAdd) in this case is

$$f(z) = \begin{cases} a_1(z), & \text{if } z \in D^1_{x+y}; \\ a_2(z), & \text{if } z \in D^2_{x+y}; \end{cases}$$

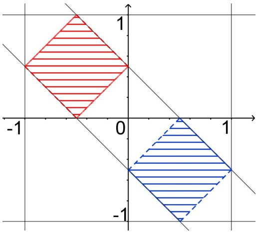
$$g(u) = \begin{cases} a_1(u), & \text{if } u \in D^1_x; \\ a_2(u), & \text{if } u \in D^2_x; \end{cases}$$

$$h(v) = \begin{cases} a_1(v), & \text{if } v \in D^1_y; \\ a_2(v), & \text{if } v \in D^2_y, \end{cases}$$

where $a_i$ is additive function for all $i = 1, 2$.

**Example 7.3.** Define the sets

$$\begin{aligned} D_1 &:= \big\{(x, y) \in \mathbb{R}^2 \mid |x + 0.5| + |y - 0.5| < 0.5\big\}, \\ D_2 &:= \big\{(x, y) \in \mathbb{R}^2 \mid |x - 0.5| + |y + 0.5| < 0.5\big\}, \\ D &:= D_1 \cup D_2 \end{aligned}$$




$$\begin{aligned} D^1_{x+y} &= D^2_{x+y} = \,]-0.5, 0.5[, \\ D^1_x &= D^2_y = \,]-1, 0[, \\ D^1_y &= D^2_x = \,]0, 1[. \end{aligned}$$

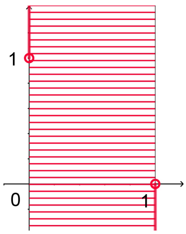
Since $D^1_{x+y} = D^2_{x+y}$ thus $a_1 = a_2$ and $C^1_1 + C^1_2 = C^2_1 + C^2_2$. Since $D^1_x = D^2_y$ thus $C^1_1 = C^2_2$, and $C^1_2 = C^2_1$. Since $D^1_{x+y} \cap D^1_x \neq \emptyset$ thus $C^1_1 + C^2_1 = C^1_1$. Since $D^1_{x+y} \cap D^1_y \neq \emptyset$ thus $C^1_1 + C^1_2 = C^1_2$. Consequently $C^i_1 = C^i_2 = 0$ for all $i = 1, 2$.

Whence we obtain that the general solution of equation (RestAdd) in this case is

$$\begin{aligned} f(z) &= a(z), \text{ if } z \in D^1_{x+y} = D^2_{x+y}; \\ g(u) &= a(u), \text{ if } u \in D^1_x = D^2_x; \\ h(v) &= a(v), \text{ if } v \in D^1_y = D^2_y. \end{aligned}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is an additive function.

**Example 7.4.** If $\mathbb{G} := (\mathbb{R}^2, +, \leqslant)$ where the addition is defined by the usual componentwise addition, and the ordering is the usual lexicographic ordering, that is, $(a_1, a_2) \leqslant (b_1, b_2)$ if and only if that either $a_1 < b_1$ or $a_1 = b_1$ and $b_1 \leqslant b_2$. Thus the group $\mathbb{G}(+, \leqslant)$ is an ordered Abelian group, but it is not an Archimedean ordered, because, for example, $(0,1) < (1,0)$, but there is no positive integer $n$ with $n(0,1) > (1,0)$.




This is the open interval $](0,1),(1,0)[$ in $\mathbb{G}$.

**Problem A.** Preserve the notations of Example 7.4, and let $Y(+)$ be an Abelian group. We want to know the general solution of functional equation (RestAdd) where $D := ](0,1),(1,0)[^2$.

**Problem B.** Preserve the notations of Example 7.4, and $Y(+)$ be an Abelian group. We also want to know the general solution of functional equation (RestPexAdd) where $D := ](0,1),(1,0)[^2$.

**Problem C.** In general, we also want to know the general solution of equation (RestAdd), or equation (RestPexAdd) in the case when $\mathbb{G}(+,\leqslant)$ is a nonarchimedean ordered Abelian group, $Y(+)$ be an Abelian group, $D \subseteq \mathbb{G}^2$ is a nonempty, well-chained, open set. The topology on $\mathbb{G}$ (or on $\mathbb{G}^2$) is generated by the open interval of $\mathbb{G}$ (or by the open rectangles of $\mathbb{G}^2$).

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