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Conjugation of overpartitions and some applications of over q-binomial coefficients

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Abstract. We study the conjugation of overpartitions and give the generating function for the number of self-conjugate overpartitions of an integer. Following the recent introduction of over q-binomial coefficients, we obtain the over q-analogue of the Chu-Vandermonde identity. Consequently a new generating function for the number of overpartitions is proved. We also give a new over q-analogue of the Chu-Vandermonde identity.

Keywords: partition, overpartition, Durfee Square

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1. Introduction

A partition of a positive integer n is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We call the summands λ_i *parts*. For example, there are 7 partitions of 5:

$$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1).$$

An overpartition of n is an integer partition of n in which the last occurrence of a part may be overlined [3]. The number of overpartitions of n is denoted by $\overline{p}(n)$. For example, $\overline{p}(4) = 14$, where the overpartitions of 4 are

$$(4),\ (\overline{4}),\ (3, 1),\ (\overline{3}, 1),\ (3, \overline{1}),\ (\overline{3}, \overline{1}),\ (2, 2),\ (2, \overline{2}),\ (2, 1, 1),$$
$$(\overline{2}, 1, 1),\ (2, 1, \overline{1}),\ (\overline{2}, 1, \overline{1}),\ (1, 1, 1, 1),\ (1, 1, 1, \overline{1}).$$

It is well known that

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} = \frac{(-q; q)_\infty}{(q; q)_\infty}, \tag{1.1}$$

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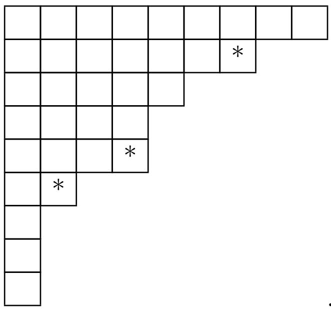
where $(A;q)_0 = 1$ and

$$(A;q)_n := (1-A)(1-Aq)\cdots(1-Aq^{n-1}) = \prod_{j=0}^{n-1}(1-Aq^j),$$

$$(A;q)_\infty := \lim_{n\to\infty}(A;q)_n = \prod_{j=0}^{\infty}(1-Aq^j).$$

The *Young diagram* (or Ferrers board) of an overpartition is the same as that of the underlying ordinary partition with the exception that the last block of an overlined part is marked. For example, the Young diagram of $\lambda = (9, \overline{7}, 5, 4, \overline{4}, \overline{2}, 1, 1, 1)$ is



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The *Durfee square* of an overpartition is the largest square that can fit into its Ferrers board.

We emphasize that a Durfee square of length s consists of $s \cdot s = s^2$ unit squares in the Young diagram of an overpartition such that s is maximal. So it's generating function is

$$q^{1+\cdots+1} \equiv q^{s\cdot s} = q^{s^2}. \tag{1.2}$$

Analogously one may consider *Durfee rectangles* of side lengths s and t, when necessary, and apply the generating function q^{st}.

The *conjugate* overpartition of λ is denoted by λ' and is obtained by reading the columns of the Ferrers board of λ. Thus for example, $\lambda' = (9, \overline{6}, 5, \overline{5}, 3, 2, \overline{2}, 1, 1)$.

A second method of obtaining the conjugate of an overpartition is as follows.

The conjugate of $\lambda = (p_1, p_2, \ldots, p_k)$ is given by $\lambda' = (q_1, q_2, \ldots, q_s)$, where

$$q_j = \begin{cases} \overline{|\{r : p_r \geq j\}|} & \text{if } j = p_i \text{ is overlined} \\ |\{r : p_r \geq j\}| & \text{otherwise.} \end{cases}$$

In other words, let the overlined parts of λ be $u_1 > u_2 > \cdots > u_t$, and let the underlying ordinary partition be $f(\lambda)$. Then λ' is obtained by overlining the parts of the partition $f(\lambda)$ that are in positions $u_1, u_2, \ldots, u_t$.

For instance, given $\lambda = (9, \overline{7}, 5, 4, \overline{4}, \overline{2}, 1, 1, 1)$, then to obtain λ', the 2nd, 4th and 7th parts of the conjugate of $f(\lambda) = (9, 7, 5, 4, 4, 2, 1, 1, 1)$ will be overlined. That is, $\lambda' = (9, \overline{6}, 5, \overline{5}, 3, 2, \overline{2}, 1, 1)$.

Definition 1.1. An overpartition is said to be *self-conjugate* if it is identical with its conjugate.

For example, it may be verified that $\lambda = (7, \overline{6}, 4, 4, 2, \overline{2}, 1)$ is self-conjugate.

One of our main results is the following:

Theorem 1.2. *Let $\overline{sc}(N)$ be the number of self-conjugate overpartitions of N. Then*

$$\sum_{N=0}^{\infty} \overline{sc}(N) q^N = 1 + \sum_{j=1}^{\infty} 2q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_j}.$$

The q-binomial coefficients (or Gaussian polynomials) are defined, for non-negative integers m, n, as

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \frac{(1-q^{m+n})(1-q^{m+n-1})\cdots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}. \tag{1.3}$$

These polynomials have many important applications in Combinatorics, Number Theory and Physics [1]. In partition theory, Eqn (1.3) is interpreted as the generating function for the number of partitions fitting inside an $m \times n$ rectangle, i.e., partitions that have parts of size $\leq m$ and a number of parts $\leq n$.

Recently, Dousse and Kim [5] introduced the *over q-binomial coefficient* which is an overpartition analogue of the q-binomial coefficients, defined by

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} := \sum_{k=0}^{\min\{m,n\}} q^{\binom{k+1}{2}} \frac{(q;q)_{m+n-k}}{(q;q)_k (q;q)_{m-k} (q;q)_{n-k}}. \tag{1.4}$$

This function is interpreted as the generating function for the number of overpartitions fitting inside an $m \times n$ rectangle. The over q-binomial coefficients have many properties similar to those of ordinary q-binomial coefficients [4, 5].

In 2003 Prellberg and Stanton [8] published a proof of the monotonicity conjecture which states the coefficients of the function

$$(1-q)\frac{1}{(q^n;q)_n} + q$$

are non-negative, for all positive integers n. This conjecture was originally formulated by Friedman et al. [6].

Subsequently, Dousse and Kim [4] formulated the following analogous conjecture based on the geometry of over q-binomial coefficients.

Conjecture 1.3. *For all positive integers n, the coefficients of*

$$(1-q)\frac{(-q^n;q)_n}{(q^n;q)_n} + q$$

are non-negative.

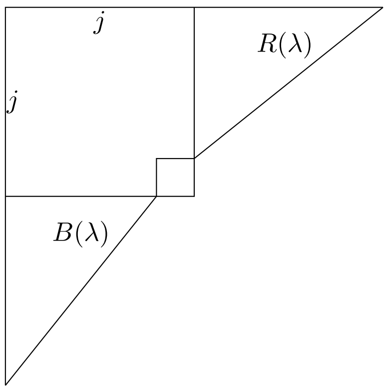
Conjecture 1.3 is an over q-analogue of the monotonicity conjecture. We will indicate a possible path to realizing a combinatorial proof of this conjecture in Section 4.

In Section 2, we give a proof of Theorem 1.2. In Section 3, we prove a certain over q-binomial coefficient identity, and establish an over q-analogue of the Chu-Vandermonde identity. We end the section with an alternative summative generating function for the number of overpartitions of n.

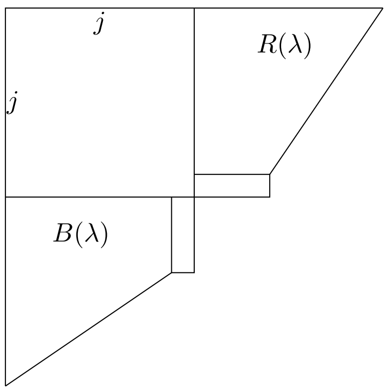
2. Proof of Theorem 1.2

Suppose we have a self-conjugate overpartition λ of N with a Durfee square of length $j > 0$. Then in the Ferrers graph of λ the j^{th} part of λ, i.e., the last part of the Durfee square, may be overlined or not. There are two cases to consider (see the diagrams below).

- Case I: when the j^{th} part is overlined (i.e., j^{th} part $= j$ and $(j+1)^{th}$ part $< j$);



- Case II: when the j^{th} part is not overlined (i.e., j^{th} part $\geq j$ and $(j+1)^{th}$ part $\leq j$).



In the first case, the Durfee square is generated by q^{j^2} (from (1.2)). Since λ is self-conjugate the overpartition $R(\lambda)$ represented by the boxes on the right of the Durfee square is the conjugate of the overpartition $B(\lambda)$ represented by the boxes below the Durfee square. Therefore each of these overpartitions is generated by $\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}$, which is the generating function for the number of overpartitions with at most $j-1$ parts (cf. Eqn (1.1)). Adding the rows of $B(\lambda)$ to the corresponding columns of $R(\lambda)$ gives an overpartition into even parts. So for each $j \geq 1$ we deduce that these overpartitions are generated by

$$q^{j^2}\prod_{r=1}^{j-1}\frac{1+q^{2r}}{1-q^{2r}} = q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}}.$$

Similarly in the second case, the Durfee square is generated by q^{j^2} while the overpartitions represented on the right of and below the Durfee square are generated by $\frac{(-q^2;q^2)_j}{(q^2;q^2)_j}$. Thus such overpartitions are generated, for all $j \geq 1$, by

$$q^{j^2}\prod_{r=1}^{j}\frac{1+q^{2r}}{1-q^{2r}} = q^{j^2}\frac{(-q^2;q^2)_{j}}{(q^2;q^2)_{j}}.$$

Hence, with 1 counting the empty (self-conjugate) overpartition, we have

$$\begin{aligned}
\sum_{N=0}^{\infty}\overline{sc}(N)q^N &= 1+\sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}} + \sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j}}{(q^2;q^2)_{j}} \\
&= 1+\sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}}\left(1+\frac{1+q^{2j}}{1-q^{2j}}\right) \\
&= 1+\sum_{j=1}^{\infty}q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j}}\left(1-q^{2j}+1+q^{2j}\right) \\
&= 1+\sum_{j=0}^{\infty}2q^{j^2}\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j}}.
\end{aligned}$$

This completes the proof.

3. Over q-binomial coefficients

Basic properties of over q-binomial coefficients are given in [4, 5]. Several of these properties resemble those of ordinary q-binomial coefficients. For example, we have the symmetry property,

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} m+n \\ m \end{bmatrix}}. \tag{3.1}$$

We recall the following series-product identities, known respectively, as Cauchy's Identity and q-Binomial Theorem (see [2, 7]).

Theorem 3.1. *For $|q|, |z| < 1$ we have*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{3.2}$$

$$\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k = \prod_{k=1}^{n} (1 + zq^k). \tag{3.3}$$

The following theorem was proved combinatorially in [4]. Here we will give an algebraic proof.

Theorem 3.2 (Dousse-Kim [4])**.** *For every positive integer m, we have*

$$\sum_{k=0}^{\infty} \overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} z^k q^k = \frac{(-zq^2;q)_{m-1}}{(zq;q)_m}. \tag{3.4}$$

Algebraic Proof. We will use the fact that

$$(q;q)_{m+k} = (q;q)_m (q^{m+1};q)_k. \tag{3.5}$$

We simplify the left-hand side of (3.4) (with $m-1$ replaced by m). Note that in the second equality below we set $\min(m,k) = m$ since $0 \le j \le m$ but $j \le k \to \infty$.

$$\begin{aligned}
\sum_{k=0}^{\infty} \overline{\begin{bmatrix} m+k \\ k \end{bmatrix}} z^k q^k &= \sum_{k=0}^{\infty} \sum_{j=0}^{\min(m,k)} q^{\binom{j+1}{2}} \frac{(q;q)_{m+k-j}}{(q;q)_j (q;q)_{m-j} (q;q)_{k-j}} (zq)^k \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} \sum_{k=j}^{\infty} \frac{(q;q)_{m+k-j}}{(q;q)_{k-j}} (zq)^k \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} (zq)^j \sum_{k=0}^{\infty} \frac{(q;q)_{m+k}}{(q;q)_k} (zq)^k \\
&= \sum_{j=0}^{m} \frac{(zq)^j q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} (q;q)_m \sum_{k=0}^{\infty} \frac{(q^{m+1};q)_k}{(q;q)_k} (zq)^k \quad \text{(by Eqn (3.5))} \\
&= \sum_{j=0}^{m} \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}} (zq)^j q^{\binom{j+1}{2}} \cdot \frac{(zq^{m+2};q)_\infty}{(zq;q)_\infty} \quad \text{(by Eqn (3.2))} \\
&= \frac{(zq^{m+2};q)_\infty}{(zq;q)_\infty} (-zq \cdot q;q)_m \quad \text{(by Eqn (3.3))} \\
&= \frac{(-zq^2;q)_m}{(zq;q)_{m+1}}.
\end{aligned}$$

This completes the proof. $\square$

3.1. An over q-analogue of Chu-Vandermonde identity

Consider the classical combinatorial identity,

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}, \quad n \geq 0. \tag{3.6}$$

It is known that this identity has the following q-analogue [2]:

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

We state a new over q-analogue of (3.6) using the over q-binomial coefficients.

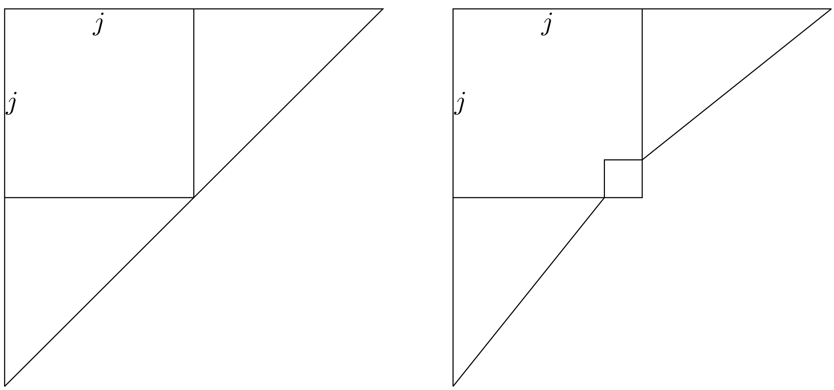
Proposition 3.3. *For any non-negative integer n, we have*

$$\overline{\begin{bmatrix} 2n \\ n \end{bmatrix}} = \sum_{j=0}^{n} q^{j^2} \left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2 \right).$$

Proof. It is clear that overpartitions fitting inside an $n \times n$ square are generated by (cf. Eqn (1.4))

$$\overline{\begin{bmatrix} n+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} 2n \\ n \end{bmatrix}}.$$

Now assume the overpartitions have Durfee squares of length j. Such overpartitions may be represented by either of the following diagrams depending on whether the last part of the Durfee square is not overlined or overlined, respectively.



In either diagram the Durfee square is generated by q^{j^2}, $j \geq 0$.

In the second diagram the subdiagram attached to the right side of the Durfee square represents an overpartition fitting inside an $(n-j) \times (j-1)$ rectangle, and the subdiagram attached below the Durfee square represents an overpartition fitting inside a $(j-1)\times(n-j)$ rectangle. So both subdiagrams are generated by

$$\begin{aligned}
&\overline{\begin{bmatrix}(n-j)+(j-1)\\ j-1\end{bmatrix}} \times \overline{\begin{bmatrix}(j-1)+(n-j)\\ n-j\end{bmatrix}}\\
&= \overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}} \times \overline{\begin{bmatrix}n-1\\ n-j\end{bmatrix}}\\
&= \overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}} \times \overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}} \quad \text{(by symmetry, (3.1))}\\
&= \overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}^2.
\end{aligned}$$

Similarly for the first diagram, the subdiagram on the right side of the Durfee square represents an overpartition fitting inside an $(n-j)\times j$ rectangle, and the subdiagram below the Durfee square is an overpartition fitting inside a $j\times(n-j)$ rectangle. Thus they are generated by

$$\overline{\begin{bmatrix}(n-j)+j\\ j\end{bmatrix}} \times \overline{\begin{bmatrix}j+(n-j)\\ n-j\end{bmatrix}} = \overline{\begin{bmatrix}n\\ j\end{bmatrix}} \times \overline{\begin{bmatrix}n\\ n-j\end{bmatrix}} = \overline{\begin{bmatrix}n\\ j\end{bmatrix}} \times \overline{\begin{bmatrix}n\\ j\end{bmatrix}} = \overline{\begin{bmatrix}n\\ j\end{bmatrix}}^2.$$

Hence the generating function for the number of partitions into at most n parts with part-sizes $\leq n$, and Durfee square of length j, is given by

$$q^{j^2}\left(\overline{\begin{bmatrix}n\\ j\end{bmatrix}}^2 + \overline{\begin{bmatrix}n-1\\ j-1\end{bmatrix}}^2\right).$$

Lastly, we sum over $0\leq j\leq n$ to obtain the stated identity. $\square$

Proposition 3.3 may be regarded as a 'finite' version of the following identity which provides another generating function for $\overline{p}(n)$ (cf. Eqn (1.1)).

Corollary 3.4. *We have*

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{j=0}^{\infty} 2q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_j}\right)^2 (1+q^{2j}).$$

Proof. Let $n\to\infty$ in Proposition 3.3. Then

$$\begin{aligned}
\lim_{n\to\infty}\overline{\begin{bmatrix}2n\\ n\end{bmatrix}} &= \lim_{n\to\infty}\sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{2n-k}}{(q;q)_k(q;q)_{n-k}(q;q)_{n-k}}\\
&= \sum_{k=0}^{\infty}\frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k}\cdot\frac{1}{(q;q)_\infty}\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}.
\end{aligned}$$

Proceeding to the limit we also have

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=0}^{n} q^{j^2}\left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2\right) &= \sum_{j=0}^{\infty} q^{j^2}\left(\left(\frac{(-q;q)_j}{(q;q)_j}\right)^2 + \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2\right) \\ &= \sum_{j=0}^{\infty} q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2\left(\left(\frac{1+q^j}{1-q^j}\right)^2+1\right) \\ &= \sum_{j=0}^{\infty} q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2 \cdot \frac{2(1+q^{2j})}{(1-q^j)^2} \\ &= \sum_{j=0}^{\infty} 2q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_j}\right)^2 (1+q^{2j}).\end{aligned}$$

Hence the result. $\square$

Remark 3.5. Note that Corollary 3.4 may also be proved by pure combinatorial reasoning by splitting the set of overpartitions into two classes, in the spirit of the proof of Theorem 1.2.

Direct Combinatorial proof of Corollary 3.4. It is clear that

$$\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{1}{(q;q)_k}q^{\frac{k(k+1)}{2}} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

Now suppose we have an overpartition λ of N with Durfee square of side j. Separate λ into two classes as in the proof of Theorem 1.2 (see Section 2). In both cases, the Durfee square is obviously generated by q^{j^2}. However, in the first case the top-right and bottom-left subdiagrams are generated by $\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}$. Such overpartitions are generated by

$$q^{j^2}\frac{(-q;q)_{j-1}^2}{(q;q)_{j-1}^2}.$$

Similarly, in the second case, the top-right and bottom-left subdiagrams are generated by $\frac{(-q;q)_j}{(q;q)_j}$. Hence this class is generated by

$$q^{j^2}\frac{(-q;q)_j^2}{(q;q)_j^2}.$$

Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}(n)q^n &= \sum_{j=0}^{\infty} q^{j^2}\frac{(-q;q)_{j-1}^2}{(q;q)_{j-1}^2} + \sum_{j=0}^{\infty} q^{j^2}\frac{(-q;q)_j^2}{(q;q)_j^2} \\ &= \sum_{j=0}^{\infty} q^{j^2}\left(\left(\frac{(-q;q)_j}{(q;q)_j}\right)^2 + \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\right)^2\right)\end{aligned}$$

$$= \sum_{j=0}^{\infty} 2q^{j^2} \left(\frac{(-q;q)_{j-1}}{(q;q)_j} \right)^2 (1+q^{2j}). \qquad \square$$

4. Remarks on Conjecture 1.3

The combinatorial proof of the following lemma is given in [5]. For completeness we provide an algebraic proof below.

Lemma 4.1 (Dousse-Kim [5])**.** *For every positive integer n, we have*

$$\frac{(-zq;q)_n}{(zq;q)_n} = 1 + \sum_{k\geq 1} z^k q^k \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right).$$

Proof. We have that the right-hand side

$$\begin{aligned}
&= 1 + \sum_{k\geq 1} z^k q^k \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \sum_{k\geq 1} z^k q^k \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \\
&= \sum_{k\geq 0} z^k q^k \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \sum_{k\geq 0} z^{k+1} q^{k+1} \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} \\
&= \frac{(-zq^2;q)_{n-1}}{(zq;q)_n} + zq\frac{(-zq^2;q)_{n-1}}{(zq;q)_n} \quad \text{(by Theorem 3.2)} \\
&= \frac{(-zq^2;q)_{n-1}}{(zq;q)_n}(1+zq) = \frac{(-zq;q)_n}{(zq;q)_n}. \qquad \square
\end{aligned}$$

This result enables the translation of the coefficients of the conjectured generating function into the coefficients of generating functions of overpartitions.

Set $z = q^{n-1}$ in Lemma 4.1 to get

$$\begin{aligned}
\frac{(-q^n;q)_n}{(q^n;q)_n} &= 1 + \sum_{k\geq 1} q^{k(n-1)} q^k \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right) \\
&= 1 + \sum_{k\geq 1} q^{kn} \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1-q)\frac{(-q^n;q)_n}{(q^n;q)_n} + q &= 1 + \sum_{k\geq 1} q^{kn}(1-q) \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right), \qquad (4.1) \\
&= 1 + \sum_{k\geq 1} q^{kn} \left(\overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}} - q\overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} \right)
\end{aligned}$$

$$+\sum_{k\geq 1} q^{kn} \overline{\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}} - \sum_{k\geq 1} q^{kn+1} \overline{\begin{bmatrix} n+k-1 \\ k \end{bmatrix}}. \tag{4.2}$$

It is clear that the right-hand-sides of (4.1) and (4.2) enumerate overpartitions. Hence one approach to proving Conjecture 1.3 combinatorially relies on interpreting the right-hand-side of the equation as the generating function of a non-vacuous union of certain sets of restricted overpartitions.

It is hoped that this will enhance the discovery of a purely combinatorial proof of the conjecture.

Lastly, we lend credence to the conjecture by providing the results of a computational study of the actual coefficients of the associated generating function.

Let $[q^N]f(q)$ denote that coefficient of q^N in the Maclaurin series expansion of $f(q)$.

For all $n > 2$, the terms of the number sequences $S(N,n) = [q^N]\Big((1-q)\frac{(-q^n;q)_n}{(q^n;q)_n} + q\Big)$, $N > 0$, are mostly positive, assuming zero values for few initial values of N. The following properties of the sequences were discovered using the computer algebra system Maple [9].

1. $S(0,n) = 1$ for all $n > 0$,
2. $S(1,1) = 2,\ S(N,1) = 0$ for $N > 1$,
3. $S(N,2) \in \{0,2\}$ for all $N > 0$,
4. $S(N,3) \in \{0,2\}$, $1 \leq N \leq 11$,
   $S(12,3) = 4,\ S(13,3) = S(14,3) = 0$,
   and $S(N,3) > 1$ for $N \geq 15$,
5. $S(N,n) \in \{0,2\}$, $1 \leq n \leq m$, and $S(N,n) \geq 2$, $n \geq m$, where $m = 3n + 2$, $n \geq 4$.

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