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Ruled like surfaces in three dimensional Euclidean space

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Abstract. In this paper, we introduce ruled like surfaces in three-dimensional Euclidean space, E^3. To form a ruled like surface in E^3, we consider a base curve $\gamma(s)$ and a director curve $X(s)$. Let parameter s be the angle between the tangent of $\gamma(s)$ and $X(s)$ when $X(s)$ lie on rectifying plane or in the osculating plane. Whereas, if $X(s)$ is in the normal plane, then parameter s will be the angle between the normal of $\gamma(s)$ and position vector of $X(s)$ at the corresponding point in E^3. Then we investigate some characterizations of such types of surfaces (say $\mathbb{S}(s,v)$). Moreover, we find the condition for the existence of Bertrand mate of $\gamma(s)$ in $\mathbb{S}(s,v)$. Finally, as examples, we construct the surfaces $\mathbb{S}(s,v)$ by using a straight line, circle and helix in E^3.

Keywords: Bertrand curve, Frenet frame, rectifying plane, osculating plane, normal plane, ruled surfaces

AMS Subject Classification: 53A05, 53A04

1. Introduction

Ruled surfaces are one of the basic and useful types of surfaces in differential geometry. Ruled surfaces are in the class of those surfaces which are broadly used in CAD systems. Ruled surfaces were introduced by G. Monge as a solution of a partial differential equation. Different properties depending upon geodesic curvature and the second fundamental form of ruled surfaces in E^3 were studied in [1]. Whereas the ruled surfaces generated by some special curves like circular

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helices, circular slant helices and Salkowski curves were considered in [15]. In [18], authors derived the isogeodesic surface pencil so that the geodesic curve is a directrix of the ruled surface.

The notion of pitch function for ruled surfaces was introduced by H. R. Müller in 1951. The pitch function and angle function of the pitch for non-developable ruled surfaces in E^3 and E^3_1 were further generalized in [9, 10]. For any non-developable ruled surface, if the base curve is a striction line and the directrix is a spherical curve, then the spherical Frenet frame can be obtained by using directrix. This spherical Frenet frame brings out three functions along the base curve on E^3, known as structural functions. In [19], authors studied the properties of non-developable ruled surfaces using structure functions. Ruled surfaces were also studied in Minkowski space [7, 17] and in three-dimensional Lie groups [16].

The idea of the Bertrand curve was given by Saint Venant in 1845 by the question "for any surface generated by a curve $\gamma(s)$, does there exist any other curve whose normal coincides with the normal of the initial curve". Bertrand answered this question in 1850 [4] by the condition, "a curve $\gamma(s)$ on E^3 is a Bertrand curve if and only if there exists a linear relationship with constant coefficients between the curvature and torsion of the original curve". In [3, 5, 11], authors studied the Bertrand curve in Minkowski space and three-dimensional sphere.

We organize our article as follows: Section 2, discusses some basic results of curves and surfaces in E^3. Ruled like surfaces, which are the core of our research article, are also defined in the same section. In Section 3, we talk about various characterizations of our surfaces, normal of the surface, Gaussian curvature, mean curvature etc. In Section 4, the conditions are obtained for the Bertrand mate of the curve $\gamma(s)$, which lie in the normal ruled like surface formed by $\gamma(s)$. In the final section, as examples, the surfaces are constructed using a straight line, plane curve circle and space curve helix.

2. Preliminaries and some results

Let $\gamma(s)$ be a unit speed space curve in R^3 with Frenet frame $\{T, N, B\}$ along $\gamma(s)$. Then, we know that

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N,$$

where κ is a curvature and τ is a torsion of $\gamma(s)$.

Definition 2.1 ([6])**.** Let $\gamma(s)$ be a smooth curve on E^3. Then $\gamma(s)$ is said to be a Bertrand curve if there exists another curve $\beta(\bar{s} = \phi(s))$ in E^3 such that the normals of $\gamma(s)$ and $\beta(\bar{s} = \phi(s))$ are linearly dependent to each other at corresponding points. Here ϕ is a bijection from $\gamma(s)$ to $\beta(\bar{s})$ and $\beta(\bar{s})$ is the Bertrand mate of $\gamma(s)$.

Definition 2.2 ([8])**.** The parametric representation of a ruled surface $\mathbb{S}(s, v)$ in E^3 is $\mathbb{S}(s, v) = \gamma(s) + v\delta(s)$, where $\gamma(s)$ is a space curve, $\delta\colon I \to \mathbb{R}^3 - \{0\}$ is a smooth map and I is an open interval or a unit circle. The curves $\gamma(s)$ and $\delta(s)$ are known as the base and director curves, respectively. The map $v \to \gamma(s)+v\delta(s)$ is known as a ruling of $\mathbb{S}(s,v)$.

Let $\mathbb{S}(s,v)$ be a ruled surface in E^3, then the various quantities associated with the surface are defined as follows:

(A). Unit surface normal: $\hat{N} = \frac{\mathbb{S}_s \times \mathbb{S}_v}{\|\mathbb{S}_s \times \mathbb{S}_v\|}$, where $\mathbb{S}_s = \frac{\partial \mathbb{S}}{\partial s}$ and $\mathbb{S}_v = \frac{\partial \mathbb{S}}{\partial v}$.

(B). First fundamental form: $I = \mathbb{E}ds^2 + 2\mathbb{F}dsdv + \mathbb{G}dv^2$, where $\mathbb{E} = \langle \mathbb{S}_s, \mathbb{S}_s\rangle$, $\mathbb{F} = \langle \mathbb{S}_s, \mathbb{S}_v\rangle$ and $\mathbb{G} = \langle \mathbb{S}_v, \mathbb{S}_v\rangle$.

(C). Second fundamental form: $II = \mathbb{L}ds^2 + 2\mathbb{M}dsdv + \mathbb{N}dv^2$, where $\mathbb{L} = \langle \mathbb{S}_{ss}, \hat{N}\rangle$, $\mathbb{M} = \langle \mathbb{S}_{sv}, \hat{N}\rangle$ and $\mathbb{N} = \langle \mathbb{S}_{vv}, \hat{N}\rangle$.

If K is a Gaussian curvature, H is a mean curvature and λ is a distribution parameter of $\mathbb{S}(s,v)$, then from [13]

(D). $K = \frac{\mathbb{L}\mathbb{N}-\mathbb{M}^2}{\mathbb{E}\mathbb{G}-\mathbb{F}^2}$, $H = \frac{\mathbb{E}\mathbb{N}+\mathbb{G}\mathbb{L}-2\mathbb{F}\mathbb{M}}{2(\mathbb{E}\mathbb{G}-\mathbb{F}^2)}$ and $\lambda = \frac{det(\gamma'(s),\delta(s),\delta'(s))}{\|\delta'(s)\|}$.

The second Gaussian curvature K_{II} of $\mathbb{S}(s,v)$ in E^3 is defined by replacing the components of the first fundamental form $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ by the components of the second fundamental form $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{N}$ in Brioschi's formulae respectively. In [2], the second Gaussian curvature of a surface is defined as

$$K_{II} = \frac{1}{(\mathbb{L}\mathbb{N}-\mathbb{M}^2)^2}\left(\begin{vmatrix} -\frac{1}{2}\mathbb{L}_{vv}+\mathbb{M}_{sv}-\frac{1}{2}\mathbb{N}_{ss} & \frac{1}{2}\mathbb{L}_s & \mathbb{M}_s-\frac{1}{2}\mathbb{L}_v \\ \mathbb{M}_v-\frac{1}{2}\mathbb{N}_s & \mathbb{L} & \mathbb{M} \\ \frac{1}{2}\mathbb{N}_v & \mathbb{M} & \mathbb{N} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\mathbb{L}_v & \mathbb{N}_s \\ \frac{1}{2}\mathbb{L}_v & \mathbb{L} & \mathbb{M} \\ \frac{1}{2}\mathbb{N}_s & \mathbb{M} & \mathbb{N}) \end{vmatrix}\right).$$

Let $\beta(s)$ be a curve in $\mathbb{S}(s,v)$, then the normal curvature κ_n, geodesic curvature κ_g and geodesic torsion τ_g of $\beta(s)$ [1] are given by

$$\kappa_n = \langle \hat{N}, T'\rangle, \quad \kappa_g = \langle \hat{N}\times T, T'\rangle, \quad \text{and} \quad \tau_g = \langle \hat{N}\times \hat{N}', T'\rangle.$$

The curve $\gamma(s)$ in $\mathbb{S}(s,v)$ can be characterized on the basis of the values of κ_g, κ_n and τ_g. That is
(1) $\gamma(s)$ will be a geodesic if and only if $\kappa_g = 0$.
(2) $\gamma(s)$ will be a asymptotic line if and only if $\kappa_n = 0$.
(3) $\gamma(s)$ will be a principal line if and only if $\tau_g = 0$.

In case of ruled surface $\mathbb{S}(s,v)$, the position vector of unit director curve $\delta(s)$ can be written as [1]

$$\delta(s) = f_1T + f_2N + f_3B, \tag{2.1}$$

where $\{T,N,B\}$ is a Frenet frame along $\gamma(s)$ and f_i, $i \in \{1,2,3\}$, are fixed components, i.e., $f_1^2+f_2^2+f_3^2 = 1$.

In equation (2.1), it is clear that the components f_i of the director curve are fixed. Now, consider $\delta(s)$ lie on the normal plane of $\gamma(s)$, such that the angle between $\delta(s)$ and N is arc length parameter s at the corresponding point. Then the parametrization of $\mathbb{S}(s,v)$ is

$$\mathbb{S}(s,v) = \mathbb{S}^n(s,v) = \gamma(s) + v(\cos(s)N + \sin(s)B). \tag{2.2}$$

Obviously, the parametrized surface formed in (2.2), is not a ruled surface. Because the components $f_1 = 0$, $f_2 = \cos(s)$ and $f_3 = \sin(s)$ are not fixed. Similarly, we can construct the surfaces

$$\mathbb{S}(s,v) = \mathbb{S}^o(s,v) = \gamma(s) + v(\cos(s)T + \sin(s)N), \tag{2.3}$$

and

$$\mathbb{S}(s,v) = \mathbb{S}^r(s,v) = \gamma(s) + v(\cos(s)T + \sin(s)B), \tag{2.4}$$

by taking $\delta(s)$ in osculating plane $\{T,N\}$, and rectifying plane $\{T,B\}$ respectively, such that the angle between $\delta(s)$ and T is s at corresponding point. Here, we define the definition of a ruled like surface.

Definition 2.3. A surface $\mathbb{S}(s,v)$ with parametrization given by any one of the equations (2.2), (2.3) and (2.4) is said to be a ruled like surface generated by a curve $\gamma(s)$ on E^3. The surface $\mathbb{S}^n(s,v)$ is said to be a normal ruled like surface of $\gamma(s)$. Similarly, $\mathbb{S}^o(s,v)$ and $\mathbb{S}^r(s,v)$ are named as osculating ruled like surface and rectifying ruled like surface of $\gamma(s)$ on E^3.

3. Some characterization of ruled like surfaces

For any surface in E^3, unit surface normal, Gaussian curvature and Mean curvature are some basic properties that help to understand the surface. In this section, all these mentioned properties of ruled like surfaces generated by a space curve and a plane curve in E^3 are studied.

3.1. Normal ruled like surfaces

Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by space curve $\gamma(s)$ on E^3. Then the partial derivative of (2.2), gives us

$$\begin{cases} \mathbb{S}^n_s(s,v) = (1 - v\kappa\cos(s))T - v(1+\tau)\sin(s)N + v(1+\tau)\cos(s)B, \\ \mathbb{S}^n_v(s,v) = \cos(s)N + \sin(s)B, \end{cases}$$

as a natural frame $\{\mathbb{S}^n_s(s,v), \mathbb{S}^n_v(s,v)\}$ of tangent space on $\mathbb{S}^n(s,v)$. Also,

$$\| \mathbb{S}^n_s(s,v) \times \mathbb{S}^n_v(s,v) \|^2 = v^2(1+\tau)^2 + (1 - v\kappa\cos(s))^2 = 0,$$

if and only if $\tau = -1$ and $v = \frac{1}{\kappa\cos(s)}$, for all $s \in \mathbb{R} - \{(2n-1)\frac{\pi}{2}\}, n$ is an integer. Thus the singularity of $\mathbb{S}^n(s,v)$ can be removed by considering either $\tau \neq -1$ or $v \neq \frac{1}{\kappa\cos(s)}$, for all $s \in \mathbb{R} - \{(2n-1)\frac{\pi}{2}\}, n$ is an integer.

From now on, we will take only those ruled like surfaces that are generated by curves with $\tau(s) \neq -1$. The unit surface normal $\hat{N}^n$ of $\mathbb{S}^n(s,v)$ generated by a curve $\gamma(s)$ with $\tau(s) \neq -1$ is obtained as follows:

$$\hat{N}^n = \frac{-v(1+\tau)T - \sin(s)(1 - v\kappa\cos(s))N + \cos(s)(1 - v\kappa\cos(s))B}{\sqrt{v^2(1+\tau)^2 + (1 - v\kappa\cos(s))^2}}. \tag{3.1}$$

The coefficients of first and second fundamental forms of surface $\mathbb{S}^n(s,v)$ are

$$\begin{cases} \mathbb{E} = v^2(1+\tau)^2 + (1 - v\kappa\cos(s))^2, \\ \mathbb{F} = 0, \\ \mathbb{G} = 1, \end{cases}$$

and,

$$\begin{cases} \mathbb{L} = \frac{1}{\sqrt{\mathbb{E}}}\{v^2(1+\tau)(\kappa'\cos(s) - \kappa(2+\tau)\sin(s)) \\ \qquad\qquad -(1 - v\kappa\cos(s))(\kappa\sin(s)(1 - v\kappa\cos(s)) - v\tau')\}, \\ \mathbb{M} = \frac{1+\tau}{\sqrt{\mathbb{E}}}, \\ \mathbb{N} = 0, \end{cases}$$

respectively. Therefore the Gaussian curvature K and mean curvature H of the surface are given by

$$\begin{cases} K = -\frac{(1+\tau)^2}{\mathbb{E}^2}, \\ H = \frac{1}{2\mathbb{E}^{\frac{3}{2}}}\{v^2(1+\tau)(\kappa'\cos(s) - \kappa(2+\tau)\sin(s)) \\ \qquad\qquad -(1 - v\kappa\cos(s))(\kappa\sin(s)(1 - v\kappa\cos(s)) - v\tau')\}. \end{cases} \tag{3.2}$$

If $\gamma(s)$ is a plane curve, then for a normal ruled like surface of $\gamma(s)$ the unit surface normal $\hat{N}^n$, the Gaussian and the mean curvatures can be obtained simply by substituting $\tau = 0$, in equations (3.1) and (3.2), respectively. Here we discuss only the second Gaussian curvature K_{II} of $\mathbb{S}^n(s,v)$ generated by a plane curve. The second Gaussian curvature of $\mathbb{S}^n(s,v)$ is computed as:

$$\begin{aligned} K_{II} = &-\frac{\mathbb{L}_v\mathbb{E}_v}{4} + \frac{\mathbb{L}}{4\mathbb{E}}\left(\frac{\mathbb{E}_v^2}{2} - \mathbb{E}\mathbb{E}_{vv}\right) + \frac{\mathbb{E}_{vs}}{2\sqrt{\mathbb{E}}} - \frac{\mathbb{E}_s\mathbb{E}_v}{2\mathbb{E}^{\frac{3}{2}}} \\ &+ \frac{1}{\sqrt{\mathbb{E}}}\big\{\kappa'\cos(s) + \kappa\sin(s)\big(1 - \kappa^2\cos^2(s)\big)\big\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}_v &= 2\{v - \kappa\cos(s)(1 - v\kappa\cos(s))\}, \\ \mathbb{E}_s &= 2v(\kappa\sin(s) - \kappa'\cos(s))(1 - v\kappa\cos(s)), \\ \mathbb{E}_{vv} &= 2(1 + \kappa^2\cos^2(s)), \\ \mathbb{L} &= \frac{1}{\sqrt{\mathbb{E}}}\Big\{v^2(2\kappa\sin(s) + \kappa'\cos(s)) - \kappa\sin(s)(1 - v\kappa\cos(s))^2\Big\}, \end{aligned}$$

$$\mathbb{L}_v = \frac{1}{\mathbb{E}}\left[2\sqrt{\mathbb{E}}\big\{v(2\kappa\sin(s) + \kappa'\cos(s)) + \kappa^2\sin(s)\cos(s)(1 - v\kappa\cos(s))\big\} - \frac{1}{2}\mathbb{L}\mathbb{E}_v\right].$$

From all the above discussions, we obtain the following theorems and corollary.

Theorem 3.1. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a space curve $\gamma(s)$, $s \in I \subset \mathbb{R}$. Then the surface is singular if and only if $\tau(s) = -1$, where $\tau(s)$ is a torsion of $\gamma(s)$.*

Theorem 3.2. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a space curve $\gamma(s)$ with $\tau(s) \neq -1$. Then $\mathbb{S}^n(s,v)$ is neither a part of a sphere nor a plane.*

Corollary 3.3. *The Gaussian curvature and the mean curvature of a normal ruled like surface are related by $aH + bK = 0$, where $a = 2(1+\tau)^2$ and $b = \mathbb{E}\mathbb{L} = \sqrt{\mathbb{E}}\Big\{v^2(2\kappa\sin(s) + \kappa'\cos(s)) - \kappa\sin(s)(1 - v\kappa\cos(s))^2\Big\}$.*

Theorem 3.4. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by $\gamma(s)$ with $\tau(s) \neq -1$. Then $\mathbb{S}^n(s,v)$ is a minimal surface if and only if $\gamma(s)$ is a straight line.*

Proof. Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then from second part of equation (3.2), we have

$$\begin{aligned}
&v^2(1+\tau)(\kappa'\cos(s) - 2\kappa\sin(s)) = (1 - v\kappa\cos(s))(\kappa\sin(s)(1 - v\kappa\cos(s)) - v\tau') \\
&\implies v^2\big\{(1+\tau)(\kappa'\cos(s) - 2\kappa\sin(s)) - \kappa^3\sin(s)\cos^2(s) - \kappa\tau'\cos(s)\big\} \\
&\qquad\qquad\qquad\qquad + v\big(\tau' + 2\kappa^2\sin(s)\cos(s)\big) + \kappa\sin(s) = 0.
\end{aligned}$$

Now, comparing the coefficients of v on both sides, we get

$$\begin{cases}
(1+\tau)(\kappa'\cos(s) - 2\kappa\sin(s)) - \kappa^3\sin(s)\cos^2(s) - \kappa\tau'\cos(s) = 0, \\
\tau' + 2\kappa^2\sin(s)\cos(s) = 0, \\
\kappa\sin(s) = 0.
\end{cases} \tag{3.3}$$

Because $s \in I \subset \mathbb{R}$, therefore $\sin(s) \neq 0\ \forall\ s$. Thus, from the last part of (3.3), $\kappa = 0$. Hence $\gamma(s)$ is a straight line.

Conversely, assume that $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a straight line. Then taking $\kappa = 0$ and $\tau = 0$ in second part of equation (3.2), we have $H = 0$. Hence $\mathbb{S}^n(s,v)$ is a minimal surface. □

3.2. Osculating and rectifying ruled like surfaces

In this section, the coefficients of the first and the second fundamental forms, the Gaussian and the mean curvatures of osculating and rectifying ruled like surfaces are studied.

Let $\gamma(s)$ be a space curve in E^3 and $\mathbb{S}^o(s,v)$, $\mathbb{S}^r(s,v)$ are osculating and rectifying ruled like surfaces, respectively. Then natural frame $\{\mathbb{S}^o_s(s,v), \mathbb{S}^o_v(s,v)\}$ of

$\mathbb{S}^o(s,v)$, and $\{\mathbb{S}^r_s(s,v), \mathbb{S}^r_v(s,v)\}$ of $\mathbb{S}^r(s,v)$ are

$$\begin{cases}\mathbb{S}^o_s(s,v) = (1 - v(1+\kappa)\sin(s))T + v(1+\kappa)\cos(s)N + v\tau\sin(s)B,\\ \mathbb{S}^0_v(s,v) = \cos(s)T + \sin(s)N,\end{cases}$$

and,

$$\begin{cases}\mathbb{S}^r_s(s,v) = (1 - v\sin(s))T + v(\kappa\cos(s) - \tau\sin(s))N + v\cos(s)B,\\ \mathbb{S}^r_v(s,v) = \cos(s)T + \sin(s)B,\end{cases}$$

respectively. First, we will discuss various properties of $\mathbb{S}^o(s,v)$ in E^3. The unit surface normal for $\mathbb{S}^o(s,v)$ is obtained by using the relation $\hat{N} = \frac{\mathbb{S}^o_s \times \mathbb{S}^o_v}{\|\mathbb{S}^o_s \times \mathbb{S}^o_v\|}$, where

$$\mathbb{S}^o_s \times \mathbb{S}^o_v = -\tau v\sin^2(s)T + \tau v\sin(s)\cos(s)N + (\sin(s) - v(1+\kappa))B,$$
$$\| \mathbb{S}^o_s \times \mathbb{S}^o_v \|^2 = v^2\tau^2\sin^2(s) + (\sin(s) - v(1+\kappa))^2.$$

Now, $\| \mathbb{S}^o_s \times \mathbb{S}^o_v \|^2 = 0$ if and only if any one of the following conditions holds:
(1) $v = 0$ and $s = n\pi$, where n is an integer,
(2) $\tau = 0$ and $v = \frac{\sin(s)}{1+\kappa}$.

Therefore, if $\gamma(s)$ is neither a plane curve nor a straight line, then $\mathbb{S}^o(s,v)$, $s, v \in I$(open interval) $\subset \mathbb{R}$, have singularity only at $v = 0$ and $s = n\pi$, where n is an integer. The parametrization for $\mathbb{S}^o(s,v)$ can be further modified by removing $v = 0$.

But just for convenience we are considering the surface $\mathbb{S}^o(s,v)$ with parameters $s, v \in I$(open interval) $\subset \mathbb{R}$ such that $v > 1$ i.e., $v = (1, |a|)$, where $1 < |a| \in \mathbb{R}$. Thus the surface $\mathbb{S}^o(s,v)$ is now a regular surface for all $s \in I$, and $v = (1, |a|)$. The unit surface normal $\hat{N}^o$ of $\mathbb{S}^o(s,v)$, is obtained as

$$\hat{N}^o = \frac{-\tau v\sin^2(s)T + \tau v\sin(s)\cos(s)N + (\sin(s) - v(1+\kappa))B}{\sqrt{\tau^2 v^2\sin^2(s) + (\sin(s) - v(1+\kappa))^2}}. \tag{3.4}$$

The components of the first and second fundamental forms, the Gaussian and mean curvatures of $\mathbb{S}^o(s,v)$ are

$$\mathbb{E} = \cos^2(s) + (\sin(s) - v(1+\kappa))^2 + \tau^2 v^2\sin^2(s),\ \mathbb{F} = \cos(s),\ \mathbb{G} = 1.$$

$$\begin{cases}\mathbb{L} = \frac{1}{\sqrt{\mathbb{EG}-\mathbb{F}^2}}\{\tau v\sin(s)\big[v\kappa' + \cos(s)\big(\kappa - v\tau^2\sin(s)\big)\big]\\ \qquad + (\sin(s) - v(1+\kappa))(v\tau(2+\kappa)\cos(s) + v\tau'\sin(s))\},\\ \mathbb{M} = \frac{\tau\sin^2(s)}{\sqrt{\mathbb{EG}-\mathbb{F}^2}}, \quad \mathbb{N} = 0.\end{cases} \tag{3.5}$$

$$K^o = -\frac{\tau^2\sin^4(s)}{(\mathbb{EG} - \mathbb{F}^2)^2}, \quad H^o = \frac{\mathbb{L}}{2(\mathbb{EG} - \mathbb{F}^2)} - \cos(s)\sqrt{-K^o},$$

respectively. Similarly, for surface $\mathbb{S}^r(s,v)$, $\| \mathbb{S}^r_s \times \mathbb{S}^r_v \|^2 = v^2(\kappa\cos(s) - \tau\sin(s))^2 + (v - \sin(s))^2 = 0$ if and only if it satisfies any one of the following conditions:

(1) $v = 0$ and $s = n\pi$, where n is an integer,
(2) $\kappa\cos(s) - \tau\sin(s) = 0$ and $v = \sin(s)$.

Because $-1 \leq \sin(s) = v \leq 1$, therefore the surface $\mathbb{S}^r(s,v)$ is regular $\forall\ s \in I(\textit{open interval}) \subset \mathbb{R}$ and $v = (1, |a|)$, where $|a|$ is some real number greater then one. The unit surface normal, the Gaussian curvature and the mean curvature of $\mathbb{S}^r(s,v)$ are given by the following relations

$$\hat{N}^r = \frac{v\sin(s)(\kappa\cos(s)-\tau\sin(s))T+(v-\sin(s))N+v\cos(s)(\tau\sin(s)-\kappa\cos(s))B}{\sqrt{(v-\sin(s))^2+v^2(\kappa\cos(s)-\tau\sin(s))^2}}, \tag{3.6}$$

$$K^r = -\frac{\sin^2(s)(\kappa\cos(s)-\tau\sin(s))^2}{(\mathbb{E}\mathbb{G}-\mathbb{F}^2)^2}, \quad H^r = \frac{\mathbb{L}}{2(\mathbb{E}\mathbb{G}-\mathbb{F}^2)} - \cos(s)\sqrt{-K^r}. \tag{3.7}$$

where,

$$\mathbb{E}\mathbb{G} - \mathbb{F}^2 = (v - \sin(s))^2 + v^2(\kappa\cos(s) - \tau\sin(s))^2,$$

and,

$$\begin{aligned}\mathbb{L} = \frac{1}{\sqrt{\mathbb{E}\mathbb{G}-\mathbb{F}^2}}\{&\left[-v^2(\kappa\cos(s) - \tau\sin(s))^2(\kappa\sin(s) + \tau\cos(s))\right]\\ &+ (v - \sin(s))(v(\kappa\cos(s) - \tau\sin(s))' + \kappa(1 - v\sin(s)) - \tau v\cos(s))\}.\end{aligned}$$

Thus, we have the following theorems:

Theorem 3.5. *Let $\gamma(s)$ be a space curve with $\tau \neq 0$ and surfaces $\mathbb{S}^o(s,v)$, $\mathbb{S}^r(s,v)$, $s \in I(\textit{open interval}) \subset \mathbb{R}$, $1 < v \in J(\textit{open interval}) \subset \mathbb{R}$ are generated by $\gamma(s)$. Then at points $s = n\pi$, the surfaces are flat.*

Theorem 3.6. *Let $\gamma(s)$ be a plane curve and $\mathbb{S}^o(s,v)$, $s \in I(\textit{open interval}) \subset \mathbb{R}$, $1 < v \in J(\textit{open interval}) \subset \mathbb{R}$ is an osculating surface. Then $\mathbb{S}^o(s,v)$ is flat and minimal in E^3.*

Theorem 3.7. *Let $\mathbb{S}^r(s,v)$, $s \in I(\textit{open interval}) \subset \mathbb{R}$, $1 < v \in J(\textit{open interval}) \subset \mathbb{R}$ be a rectifying ruled like surfaces generated by $\gamma(s)$; $s \in I$. Then $\mathbb{S}^r(s,v)$ is a flat and minimal surface if and only if it is generated by a straight line.*

4. Characterizations of curves in normal ruled like surface $\mathbb{S}^n(s,v)$

Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then the different properties of $\gamma(s)$ in $\mathbb{S}^n(s,v)$ like, whether $\gamma(s)$ is a geodesic or not and asymptotic curve of $\mathbb{S}^n(s,v)$ or not are studied. Also, we find the condition for Bertrand mate of $\gamma(s)$ to lie on $\mathbb{S}^n(s,v)$.

Theorem 4.1. *Let $(\gamma(s), \beta(\bar{s}))$ be a Bertrand couple in E^3 and $\mathbb{S}^n(s,v)$ be a normal ruled like surface of unit speed space curve $\gamma(s)$ with $\tau(s) \neq 0$. Then unit speed* *curve* $\beta(\bar{s})$ *with* $\bar{\kappa}(\bar{s}) \neq 0$ *lies on* $\mathbb{S}^n(s(\bar{s}), v(\bar{s}))$ *if and only if the parameters* $s(\bar{s})$ *and* $v(\bar{s})$ *satisfies the following conditions*

$$\begin{cases} \sin(s)\ddot{v} + 2b\cos(s)(1+\tau)\dot{v} \\ \qquad + b^2\Big(\cos(s)(\eta ab(1+\tau)+\tau') - \sin(s)(1+\tau)^2\Big)v = 0, \\ -2\kappa\cos(s)\dot{v} + (\kappa\sin(s)(2+\tau) - \cos(s)(\kappa' + ab\eta\kappa))v + \eta ab^2 = 0, \\ \frac{ds}{d\bar{s}} = \frac{1}{\sqrt{(1-\eta\kappa)^2+\eta^2\tau^2}} = b \quad \textit{and} \quad \eta(\kappa^2-\tau^2)' = 2(\kappa' - \frac{a}{b}), \end{cases} \tag{4.1}$$

where $\epsilon = \pm 1$, $\eta \neq 0$ *is an arbitrary constant and* $a = \epsilon\sqrt{(\kappa'^2+\tau'^2)}$.

Proof. Let $(\gamma(s), \beta(\bar{s}))$ be a Bertrand couple in E^3 and $\gamma(s)$ be a space curve with $\tau(s) \neq 0$. Then

$$\beta(\bar{s}) = \gamma(s) + \eta(s)N, \tag{4.2}$$

where η is a smooth function on E^3 and N is a normal vector field of Frenet frame $\{T, N, B\}$ along $\gamma(s)$ on E^3. The derivative of equation (4.2), with respect to $\bar{s}$, gives the relation

$$\bar{T}(\bar{s}) = ((1-\eta(s)\kappa)T + \eta' N + \eta\tau B)\frac{ds}{d\bar{s}}, \tag{4.3}$$

where $\bar{T}$ is a tangent vector field of $\beta(\bar{s})$ in E^3. The scalar product of equation (4.3) with N, implies that $\eta(s) = constant \neq 0$. Now differentiating the equation (4.3) with respect to $\bar{s}$, and then taking the scalar product of differential equation with $\bar{T}$, $\bar{B}$, we have

$$(1-\eta\kappa)\frac{d^2s}{d\bar{s}^2} = \frac{\eta\kappa'}{((1-\eta\kappa)^2+\eta^2\tau^2)} \quad \text{and} \quad \eta\tau\frac{d^2s}{d\bar{s}^2} = -\frac{\eta\tau'}{((1-\eta\kappa)^2+\eta^2\tau^2)}.$$

$$\Longrightarrow \frac{d^2s}{d\bar{s}^2} = \frac{\epsilon\eta\sqrt{(\kappa'^2+\tau'^2)}}{((1-\eta\kappa)^2+\eta^2\tau^2)^{\frac{3}{2}}}, \tag{4.4}$$

where $\epsilon = \pm 1$. Also

$$\frac{ds}{d\bar{s}} = \frac{1}{\sqrt{((1-\eta\kappa)^2+\eta^2\tau^2)}} \Longrightarrow \frac{d^2s}{d\bar{s}^2} = \frac{\eta\kappa' - \eta^2(\kappa\kappa'+\tau\tau')}{((1-\eta\kappa)^2+\eta^2\tau^2)^2}. \tag{4.5}$$

Thus from (4.4) and (4.5), we get $\eta(\kappa^2-\tau^2)' = 2(\kappa' - \frac{a}{b})$, where $a = \epsilon\sqrt{(\kappa'^2+\tau'^2)}$ and $b = \frac{1}{\sqrt{((1-\eta\kappa)^2+\eta^2\tau^2)}}$.

Let $\beta(\bar{s})$ be a curve on surface $\mathbb{S}^n(s(\bar{s}), v(\bar{s}))$. Then $\beta(\bar{s})$ is given by

$$\beta(\bar{s}) = \mathbb{S}^n(s(\bar{s}), v(\bar{s})); \quad \bar{s} \mapsto (s(\bar{s}), v(\bar{s})). \tag{4.6}$$

Differentiating (4.6), two times with respect to $\bar{s}$, we have

$$\bar{\kappa}(\bar{s})\bar{N}(\bar{s}) = \mathbb{S}^n_v(s(\bar{s}), v(\bar{s}))\ddot{v} + 2\mathbb{S}^n_{sv}(s(\bar{s}), v(\bar{s}))\dot{s}\dot{v} + \mathbb{S}^n_{vv}(s(\bar{s}), v(\bar{s}))\dot{v}^2$$

$$+\,\mathbb{S}^n_{ss}(s(\bar{s}),v(\bar{s}))\dot{s}^2+\mathbb{S}^n_{s}(s(\bar{s}),v(\bar{s}))\ddot{s}, \tag{4.7}$$

where $\ddot{v}=\frac{d^2v}{d\bar{s}^2}$, $\dot{v}=\frac{dv}{d\bar{s}}$, $\ddot{s}=\frac{d^2s}{d\bar{s}^2}$ and $\dot{s}=\frac{ds}{d\bar{s}}$. The partial derivatives of $\mathbb{S}^n(s(\bar{s}),v(\bar{s}))$ with respect to s and v, are

$$\mathbb{S}^n_{v}(s(\bar{s}),v(\bar{s}))=(\cos(s)N+\sin(s)B), \tag{4.8}$$

$$\mathbb{S}^n_{sv}(s(\bar{s}),v(\bar{s}))=-\kappa\cos(s)T-\sin(s)(1+\tau)N+\cos(s)(1+\tau)B, \tag{4.9}$$

$$\mathbb{S}^n_{s}(s(\bar{s}),v(\bar{s}))=(1-v\kappa\cos(s))T-v\sin(s)(1+\tau)N+v\cos(s)(1+\tau)B, \tag{4.10}$$

$$\begin{aligned}\mathbb{S}^n_{ss}(s(\bar{s}),v(\bar{s}))&=v(\kappa(2+\tau)\sin(s)-\kappa'\cos(s))T\\&\quad+v\big(\tau'\cos(s)-(1+\tau)^2\sin(s)\big)B,\\&\quad+\big(\kappa(1-v\kappa\cos(s))-v\tau'\sin(s)-v(1+\tau)^2\cos(s)\big)N.\end{aligned} \tag{4.11}$$

Now, using the equations (4.8)–(4.11), in equation (4.7), and the fact that $\bar{N}$ and N are collinear, we get

$$\begin{cases}\sin(s)\ddot{v}+2\cos(s)(1+\tau)\dot{s}\dot{v}+\cos(s)(1+\tau)v\ddot{s}\\ \qquad +v\big(\tau'\cos(s)-(1+\tau)^2\sin(s)\big)\dot{s}^2=0,\\ -2\kappa\cos(s)\dot{v}\dot{s}+(1-v\kappa\cos(s))\ddot{s}+v(\kappa(2+\tau)\sin(s)-\kappa'\cos(s))\dot{s}^2=0.\end{cases} \tag{4.12}$$

Substituting $\dot{s}$ and $\ddot{s}$ from (4.3) and (4.4), in equation (4.12), we obtained the required conditions.

Conversely, Let $\beta(\bar{s})$ is a curve on surface $\mathbb{S}^n(s(\bar{s}),v(\bar{s}))$ such that the map $\bar{s}\mapsto(s(\bar{s}),v(\bar{s}))$, satisfies the equation (4.1). Then, on substituting (4.8)–(4.11), in equation (4.7), we obtain

$$\begin{aligned}\bar{\kappa}(\bar{s})\bar{N}(\bar{s})&=\{\sin(s)\ddot{v}+2\cos(s)(1+\tau)\dot{s}\dot{v}+\cos(s)(1+\tau)v\ddot{s}\\&\quad+v\big(\tau'\cos(s)-(1+\tau)^2\sin(s)\big)\dot{s}^2\}B+\{\cos(s)\ddot{v}-2\sin(s)(1+\tau)\dot{s}\dot{v}\\&\quad-\sin(s)(1+\tau)v\ddot{s}+\big(\kappa(1-v\kappa\cos(s))-v\tau'\sin(s)-v(1+\tau)^2\cos(s)\big)\dot{s}^2\}N\\&\quad\{-2\kappa\cos(s)\dot{v}\dot{s}+(1-v\kappa\cos(s))\ddot{s}+v(\kappa(2+\tau)\sin(s)-\kappa'\cos(s))\dot{s}^2\}T.\end{aligned}$$

As $\langle\bar{N},T\rangle=0$ and $\langle\bar{N},B\rangle=0$, hence $\bar{N}$ and N are collinear. Therefore, $\beta(\bar{s})$ is a Bertrand mate of $\gamma(s)$. □

Theorem 4.2. *Let $(\gamma(s),\beta(\bar{s}))$ be a Bertrand couple in E^3 and $\beta(\bar{s})$ is lying on normal ruled like surface $\mathbb{S}^n(s,v)$ of $\gamma(s)$ with $\tau(s)\neq 0$. Then the map $\bar{s}\mapsto v(\bar{s})$ satisfies the relation*

$$v=\begin{cases}\frac{\kappa\sin(s)\cos(s)b\left(\kappa-(\kappa(1-\eta\kappa)-\eta\tau^2)b^3\right)-\eta(1+\tau)ab}{\kappa\sin(s)(\kappa^2\cos^2(s)+(1+\tau)(2+\tau))+\cos(s)(\kappa\tau'-\kappa'(1+\tau))} & \text{if } \sin(s)\neq 0 \text{ and } \cos(s)\neq 0,\\ -\lambda\frac{\eta ab}{\kappa(2+\tau)} & \text{if } \sin(s)=\pm1=\lambda \text{ and } \cos(s)=0,\\ -\lambda\frac{\eta ab(1+\tau)}{\tau'\kappa-\kappa'(1+\tau)} & \text{if } \cos(s)=\pm1=\lambda \text{ and } \sin(s)=0,\end{cases}$$

where $\epsilon=\pm1$, $\eta\neq0$ is an arbitrary constant, $a=\epsilon\sqrt{(\kappa'^2+\tau'^2)}$ and $b=\frac{1}{\sqrt{(1-\eta\kappa)^2+\eta^2\tau^2}}$.

Proof. Let $(\gamma(s), \beta(\bar{s}))$ be a Bertrand couple in E^3 and $\beta(\bar{s})$, lying on normal ruled like surface $\mathbb{S}^n(s,v)$ of $\gamma(s)$ with $\tau(s) \neq 0$. Then, substituting (4.8)–(4.11), in (4.7), and taking the scalar product with T, N and B, we have

$$\begin{cases} -2\kappa\cos(s)\dot{v}\dot{s} + (1 - v\kappa\cos(s))\ddot{s} + v(\kappa(2+\tau)\sin(s) - \kappa'\cos(s))\dot{s}^2 = 0, \\ \cos(s)\ddot{v} - 2\sin(s)(1+\tau)\dot{s}\dot{v} - \sin(s)(1+\tau)v\ddot{s} \\ \qquad + \big(\kappa(1 - v\kappa\cos(s)) - v\tau'\sin(s) - v(1+\tau)^2\cos(s)\big)\dot{s}^2 = \bar{\kappa}\langle N, \bar{N}\rangle, \\ \sin(s)\ddot{v} + 2\cos(s)(1+\tau)\dot{s}\dot{v} + \cos(s)(1+\tau)v\ddot{s} \\ \qquad + v\big(\tau'\cos(s) - (1+\tau)^2\sin(s)\big)\dot{s}^2 = 0. \end{cases} \tag{4.13}$$

Now, if both $\cos(s) \neq 0$ and $\sin(s) \neq 0$, then from second and third part of (4.13), we get

$$\begin{aligned} &2(1+\tau)\dot{v}\dot{s} + (1-\tau)v\ddot{s} + (-\kappa\sin(s)(1 - v\kappa\cos(s)) + v\tau')\dot{s}^2 \\ &= -\bar{\kappa}\sin(s)\langle N, \bar{N}\rangle, \end{aligned} \tag{4.14}$$

Using equations (4.3), (4.4) and (4.14) in the first part of (4.13), we obtain

$$v = \frac{\kappa\sin(s)\cos(s)b\big(\kappa - \bar{\kappa}\langle N, \bar{N}\rangle b\big) - \eta(1+\tau)ab}{\kappa\sin(s)(\kappa^2\cos^2(s) + (1+\tau)(2+\tau) + \cos(s)(\kappa\tau' - \kappa'(1+\tau)))}, \tag{4.15}$$

where $a = \epsilon\sqrt{(\kappa'^2 + \tau'^2)}$ and $b = \frac{1}{\sqrt{((1-\eta\kappa)^2 + \eta^2\tau^2)}}$. Also, if we differentiate (4.3) with respect to $\bar{s}$, and take the scaler product with the normal, then

$$\bar{\kappa}\langle N, \bar{N}\rangle = \frac{\kappa(1-\eta\kappa) - \eta\tau^2}{(1-\eta\kappa)^2 + \eta^2\tau^2} = b^2\big(\kappa(1-\eta\kappa) - \eta\tau^2\big). \tag{4.16}$$

Hence, equations (4.15) and (4.16) together prove the first part of the theorem. To prove the other two parts consider $\cos(s) = 0$, $\sin(s) = \pm 1 = \lambda$ and $\sin(s) = 0$, $\cos(s) = \pm 1 = \lambda$ in equation (4.13), we get

$$\begin{cases} \ddot{s} + \lambda\kappa(2+\tau)v\dot{s}^2 = 0, \\ -2\lambda(1+\tau)\dot{s}\dot{v} - \lambda(1+\tau)v\ddot{s} + (\kappa - v\tau'\lambda)\dot{s}^2 = \bar{\kappa}\langle N, \bar{N}\rangle, \\ \lambda\ddot{v} - v\lambda(1+\tau)^2\dot{s}^2 = 0, \end{cases} \tag{4.17}$$

and,

$$\begin{cases} -2\kappa\lambda\dot{v}\dot{s} + (1 - v\kappa\lambda)\ddot{s} - \kappa'\lambda v\dot{s}^2 = 0, \\ \lambda\ddot{v} + \big(\kappa(1 - v\lambda\kappa) - v(1+\tau)^2\lambda\big)\dot{s}^2 = \bar{\kappa}\langle N, \bar{N}\rangle, \\ 2\lambda(1+\tau)\dot{s}\dot{v} + \lambda(1+\tau)v\ddot{s} + \tau'\lambda v\dot{s}^2 = 0. \end{cases} \tag{4.18}$$

The second part of the theorem is proved by the first part of (4.17), (4.3) and (4.4). Whereas to prove the third part of the theorem, solve the first and third parts of (4.18) by replacing the values of $\dot{v}\dot{s}$, and then use equations (4.3) and (4.4) to get the required result. □

Theorem 4.3. *Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then $\gamma(s)$ is neither an asymptotic curve nor a geodesic of $\mathbb{S}^n(s,v)$.*

Proof. Let $\mathbb{S}^n(s,v)$ be a normal ruled like surface generated by a curve $\gamma(s)$. Then the unit surface normal of $\mathbb{S}^n(s,v)$ is given by the equation (3.1). Now from [14, p. 166], we have

$$\kappa_g = \kappa\langle N, \hat{N}\times T\rangle \ \ and \ \ \kappa_n = \kappa\langle N, \hat{N}\rangle. \tag{4.19}$$

Thus the unit surface normal $\hat{N}$ and $\hat{N}\times T$ along $\gamma(s)$, from (3.1) we have

$$\begin{cases} \hat{N}(s,0) = -\sin(s)N + \cos(s)B, \\ \hat{N}(s,0)\times T = \cos(s)N + \sin(s)B. \end{cases} \tag{4.20}$$

Therefore, from (4.19) and (4.20), $\kappa_g = \kappa\cos(s) \neq 0$ and $\kappa_n = -\kappa\sin(s) \neq 0$ for all s. Hence $\gamma(s)$ is neither an asymptotic curve nor a geodesic of $\mathbb{S}^n(s,v)$. □

Corollary 4.4. *The geodesic torsion of the curve $\gamma(s)$ on normal ruled like surface $\mathbb{S}^n(s,v)$ is given by $\tau_g = \kappa\cos(s)\sin(s)$.*

Proof. From relation $\tau^g_\Upsilon = \langle \hat{N}(s,0)\times \hat{N}_s(s,0), \kappa N\rangle$, we get the solution of this corollary by direct calculation. □

As we know $\mathbb{S}^n(s,v) = \gamma(s) + v(\cos(s)N + \sin(s)B)$, where $X(s) = \cos(s)N + \sin(s)B)$; $\langle X(s), X(s)\rangle = 1$ and $\langle T, X\rangle = 0$. Therefore, we can make another frame $\{T, X(s), T\times X = Y\}$ in $\mathbb{S}^n(s,v)$, such that the derivative of T, X and Y satisfies the equations

$$\begin{vmatrix} T' \\ X' \\ Y' \end{vmatrix} = \begin{vmatrix} 0 & \kappa\cos(s) & -\kappa\sin(s) \\ -\kappa\cos(s) & 0 & (1+\tau) \\ \kappa\sin(s) & -(1+\tau) & 0 \end{vmatrix} \begin{vmatrix} T \\ X \\ Y \end{vmatrix}, \tag{4.21}$$

and this frame coincides with the Darboux frame along $\gamma(s)$ in $\mathbb{S}^n(s,v)$.

Theorem 4.5. *Let $\mathbb{S}^n(s,v) = \gamma(s) + vX(s)$, where $X(s) = \cos(s)N + \sin(s)B$. Then orthogonal trajectory of $X(s)$ lies in $\mathbb{S}^n(s,v)$ if and only if $v = \frac{\kappa\cos(s)}{\kappa^2\cos^2(s)+(1+\tau)^2}$.*

Proof. Let $\delta(s)$ be an orthogonal trajectory of $X(s)$ lying on $\mathbb{S}^n(s,v)$. Then

$$\delta(s) = \gamma(s) + v(s)X(s) \ \ and \ \ \langle \delta'(s), X'(s)\rangle = 0$$

Also, from 4.21, we get

$$\begin{aligned} 0 =& \langle \delta'(s), X'(s)\rangle = \langle T, X'(s)\rangle + v\langle X'(s), X'(s)\rangle, \\ &\implies v = \frac{\kappa\cos(s)}{\kappa^2\cos^2(s) + (1+\tau)^2}. \end{aligned} \tag{4.22}$$

Equation (4.22) proves the first part of the theorem.

Now to prove converse part, let $\mathbb{S}^n(s,v)=\gamma(s)+vX(s)$, with $X(s)=\cos(s)N+\sin(s)B$ and $v=\frac{\kappa\cos(s)}{\kappa^2\cos^2(s)+(1+\tau)^2}$. Then by taking

$$\delta(s)=\gamma(s)+\frac{\kappa\cos(s)}{\kappa^2\cos^2(s)+(1+\tau)^2}(\cos(s)N+\sin(s)B),$$

it is easy to prove that $\langle\delta'(s),X'(s)\rangle=0$(use Frenet frame of $\gamma(s)$). Hence $\delta(s)$ is an orthogonal trajectory of $\gamma(s)$ in $\mathbb{S}^n(s,v)$. $\square$

Note. Similar way, we can also study the characterizations of curves lying on osculating and rectifying ruled like surfaces.

5. Examples for ruled like surfaces

In this section, we form the normal, osculating and rectifying ruled like surfaces generated from a straight line, circle and helix. Also, we plot the orthogonal trajectory of $X(s)=\cos(s)N+\sin(s)B$ in a normal ruled like surface.

Example 5.1. Let $\gamma(s)=(s,0,0)$ be a straight line in E^3. Then Frenet frame along $\gamma(s)$ can be taken as follows

$$T(s)=(1,0,0),\quad N(s)=(0,1,0),\quad B(s)=(0,0,1).$$

Then, the parametrization for normal, osculating and rectifying ruled like surfaces for a straight line are given by

$$\begin{cases}\mathbb{S}^n(s,v)=\Big(s,v\cos(s),v\sin(s)\Big),\ \forall\ s\in I, v\in J \text{ and } I,J\subset\mathbb{R},\\ \mathbb{S}^o(s,v)=\Big(s+v\cos(s),v\sin(s),1\Big),\ \forall\ s\in I\subset\mathbb{R}, v\in(1,b) \text{ and } 1<b\in\mathbb{R},\\ \mathbb{S}^n(s,v)=\Big(s+v\cos(s),1,v\sin(s)\Big)\ \forall\ s\in I\subset\mathbb{R}, v\in(1,b);\text{ and } 1<b\in\mathbb{R}.\end{cases}$$

Now, we will discuss these surfaces one by one.

Case 1. Consider the surface $\mathbb{S}^n(s,v)=(s,v\cos(s),v\sin(s))$, $\forall\ s\in I$ and $v\in J$; $I,J\subset\mathbb{R}$. Then the natural frame $\{\mathbb{S}^n_s(s,v),\mathbb{S}^n_v(s,v)\}$ on $\mathbb{S}^n(s,v)$ are

$$\mathbb{S}^n_s(s,v)=(1,-v\sin(s),v\cos(s)),\ \textit{and}\ \ \mathbb{S}^n_v(s,v)=(0,\cos(s),\sin(s)).$$

Therefore the unit surface normal of $\mathbb{S}^n(s,v)$ is $\hat{N}^n=\frac{1}{\sqrt{1+v^2}}(-v,\sin(s),\cos(s))$. The coefficients of first fundamental form are $E=(1+v^2)$, $F=0$ and $G=1$. Whereas coefficients of the second fundamental form are $\mathbb{L}=0$, $\mathbb{M}=\frac{1}{\sqrt{1+v^2}}$ and $\mathbb{N}=0$.

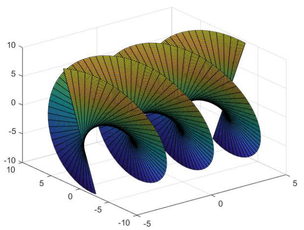
Thus the surface $\mathbb{S}^n(s,v)$ is minimal and a surface of negative Gaussian curvature in E^3.

Case 2. Let $\mathbb{S}^o(s,v) = (s + v\cos(s), v\sin(s), 1)$, $\forall\ s \in I, v \in (1,b), I \subset \mathbb{R}$ and $1 < b \in \mathbb{R}$. Then $\{\mathbb{S}^o_s(s,v), \mathbb{S}^0_v(s,v)\}$ is a natural frame of $\mathbb{S}^o(s,v)$ and $\mathbb{S}^o_s(s,v)$, $\mathbb{S}^0_v(s,v)$ are obtained as follows

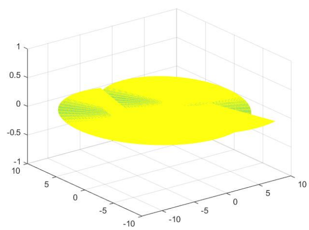
$$\mathbb{S}^o_s(s,v) = (1 - v\sin(s), v\cos(s), 0),\ \textit{and}\ \ \mathbb{S}^0_v(s,v) = (\cos(s), \sin(s), 0).$$

The unit surface normal $\hat{N}^o$ of $\mathbb{S}^o(s,v)$ is $\hat{N}^o = (0,0,1)$. Thus the first I and the second II fundamental forms of $\mathbb{S}^o(s,v)$ are $I = \big((s + v\cos(s)^2 + v^2\sin^2(s))\big)ds^2 + 2Fdsdv + dv^2$ and $II = 0$, respectively. Hence the surfaces of type $\mathbb{S}^o(s,v)$ generated by the straight line in E^3 are minimal and flat.

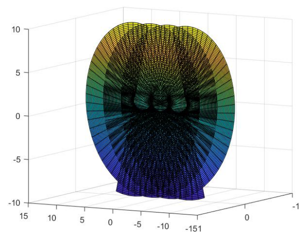
The nature of rectifying surface of a straight line is not much different as compared to the osculating surface. Because the rectifying and osculating ruled like surfaces of straight-line look the same. Therefore we give figures only for regular osculating surfaces and irregular rectifying surfaces in E^3.



(a) Normal ruled like surface of the straight line for $-5 < s < 5$ and $-10 < v < 10$.



(b) Osculating ruled like surface of the straight line for $-5 < s < 5$ and $1 < v < 10$.



(c) Rectifying ruled like surface of the straight line(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 1. Ruled like surfaces of a straight line.

Example 5.2. Let $\gamma(s) = (\cos(s), \sin(s), 0)$ be a circle in E^3. Then Frenet frame of $\gamma(s)$ on E^3 are

$$T(s) = (-\sin(s), \cos(s), 0),\ \ N(s) = (-\cos(s), -\sin(s), 0),\ \ B(s) = (0, 0, -1).$$

Therefore the ruled like surfaces of the circle are given by

$$\begin{cases} \mathbb{S}^n(s,v) = \Big(\cos(s) - v\cos^2(s), \sin(s) - v\sin(s)\cos(s), -v\sin(s)\Big), \\ \qquad\qquad\qquad\qquad \forall\ s \in I, v \in J \text{ and } I, J \subset \mathbb{R}, \\ \mathbb{S}^o(s,v) = \Big(\cos(s) - 2v\sin(s)\cos(s), \sin(s) + v(\cos^2(s) - \sin^2(s)), 0\Big), \\ \qquad\qquad\qquad\qquad \forall\ s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}, \\ \mathbb{S}^r(s,v) = \Big(\cos(s) - v\sin(s)\cos(s), \sin(s) + v\cos^2(s), -v\sin(s)\Big) \\ \qquad\qquad\qquad\qquad \forall\ s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}. \end{cases}$$

Thus, the unit surface normal of the surfaces from equations (3.1), (3.4) and (3.6) are

$$\begin{cases} N^n(s,v) = \frac{1}{\sqrt{v^2+(1-v\cos(s))^2}}\Big\{v\sin(s) + \sin(s)\cos(s)\big(1 - v\cos(s)\big), \\ \qquad -v\cos(s) + \sin^2(s)(1 - v\cos(s)), \cos(s)\big(1 - v\cos(s)\big)\Big\}, \\ N^o(s,v) = (0,0,-1), \\ N^r(s,v) = \frac{1}{\sqrt{(v-\sin(s))^2+v^2\cos(s)^2}}\Big\{v\sin^2(s)\cos(s) - \cos(s)(v - \sin(s)), \\ \qquad v\sin(s)\cos^2(s) - \sin(s)(v - \sin(s)), v\cos(s)\cos(s)\Big\}. \end{cases}$$

Similarly, the Gaussian and the mean curvatures for the surfaces can be obtained from (3.2), (3.5) and (3.7). Also, the orthogonal trajectory of $X(s) = \cos(s)N + \sin(s)B = (-\cos^2(s), -\sin(s)\cos(s), -\sin(s))$ from Theorem 4.5 is (see the Figure 2)

$$\delta(s) = \left(\frac{\cos(s)}{1+\cos^2(s)}, \frac{\sin(s)}{1+\cos^2(s)}, \frac{-\sin(s)\cos(s)}{1+\cos^2(s)}\right).$$

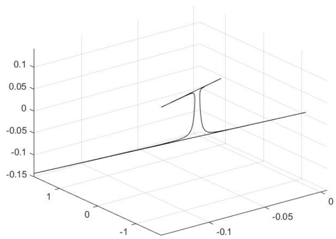
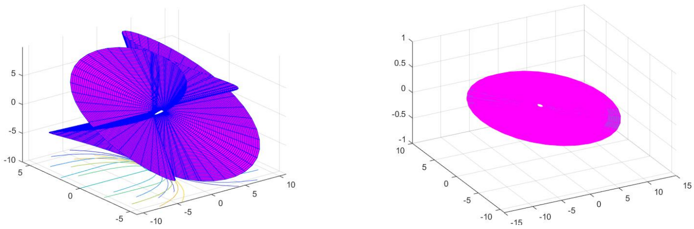


Figure 2. Orthogonal trajectory of $X(s)$ for $-5 < s < 5$ in Figure 3a.

Example 5.3. Let $\gamma(s) = \frac{1}{\sqrt{2}}(\cos(s), \sin(s), s)$ be a circular helix in E^3. Then Frenet frame along $\gamma(s)$ are

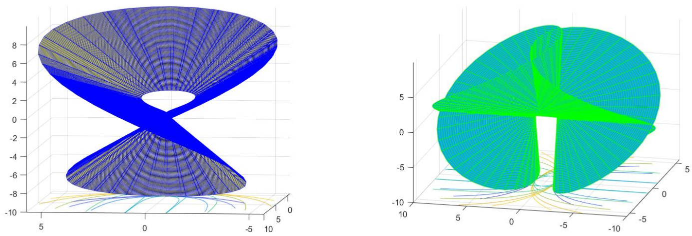
$$\begin{cases} T(s) = \frac{1}{\sqrt{2}}(-\sin(s), \cos(s), 1), \\ N(s) = (-\cos(s), -\sin(s), 0), \\ B(s) = \frac{1}{\sqrt{2}}(\sin(s), -\cos(s), 1). \end{cases}$$



(a) Normal ruled like surface of the circle for $-5 < s < 5$ and $-10 < v < 10$.

(b) Osculating ruled like surface of a circle for $-5 < s < 5$ and $1 < v < 10$.

Figure 3. Normal and osculating ruled like surfaces of the circle.



(a) Rectifying ruled like surface of the circle(regular) for $-5 < s < 5$ and $1 < v < 10$.

(b) Rectifying ruled like surface of the circle(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 4. Rectifying ruled like surfaces of the circle.

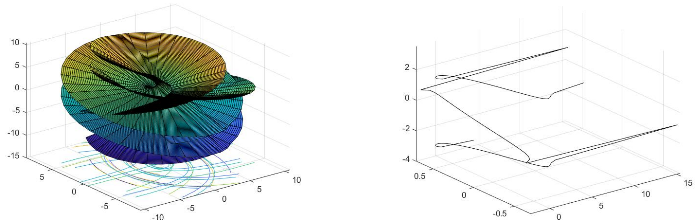
Thus, the ruled like surfaces of the circular helix are given by the following equations:

$$\begin{cases} \mathbb{S}^n(s,v) = \Big(\frac{\cos(s)}{\sqrt{2}} + v(-\cos^2(s) + \frac{\sin^2(s)}{\sqrt{2}}), \frac{\sin(s)}{\sqrt{2}} - v(\cos(s)\sin(s))(\frac{1+\sqrt{2}}{\sqrt{2}}), \\ \qquad\qquad \frac{s}{\sqrt{2}} + \frac{v\sin(s)}{\sqrt{2}}\Big), \forall\ s \in I, v \in J \text{ and } I, J \subset \mathbb{R}, \\ \mathbb{S}^o(s,v) = \Big(\frac{\cos(s)}{\sqrt{2}} - v(\frac{\sin(s)\cos(s)}{\sqrt{2}} + \sin(s)\cos(s)), \frac{\sin(s)}{\sqrt{2}} + v(\frac{\cos^2(s)}{\sqrt{2}} - \sin^2(s)), \\ \qquad\qquad \frac{s}{\sqrt{2}} + \frac{v\cos(s)}{\sqrt{2}}\Big),\ \forall\ s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}, \\ \mathbb{S}^r(s,v) = \frac{1}{\sqrt{2}}\Big(\cos(s) + v\sin(s)(\sin(s) - \cos(s)), \sin(s) + v\cos(s)\big(\cos(s) \\ \quad -\sin(s)\big), s + v(\sin(s) + \cos(s))\Big),\ \forall\ s \in I \subset \mathbb{R}, v \in (1,b) \text{ and } 1 < b \in \mathbb{R}. \end{cases}$$

The unit surface normal for these surfaces can be obtained by using equations (3.1), (3.4) and (3.6), respectively. Also, the orthogonal trajectory of $X(s) = \cos(s)N + \sin(s)B = (-\cos^2(s), -\sin(s)\cos(s), -\sin(s))$ from Theorem 4.5 is (see Figure 5b)

$$\delta(s) = \Big(\frac{1}{\sqrt{2}}\cos(s) + \frac{\sqrt{2}\cos(s)}{\cos^2(s) + (1+\sqrt{2})^2}\Big(\frac{\sin^2(s)}{\sqrt{2}} - \cos^2(s)\Big), \frac{1}{\sqrt{2}}\sin(s)$$

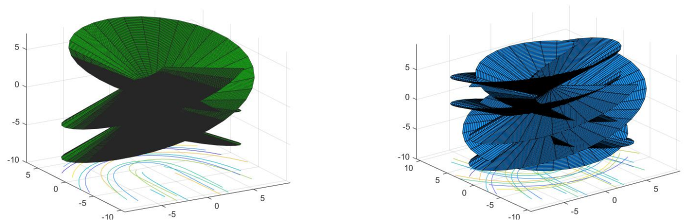
$$- \frac{(1+\sqrt{2})\sin(s)\cos^2(s)}{\cos^2(s)+(1+\sqrt{2})^2}, \frac{s}{\sqrt{2}} + \frac{\sin(s)\cos(s)}{\cos^2(s)+(1+\sqrt{2})^2}\Bigg).$$



(a) Normal ruled like surface of the helix for $-5 < s < 5$ and $-10 < v < 10$.

(b) Orthogonal trajectory of $X(s)$ for $-5 < s < 5$ in Figure 5a.

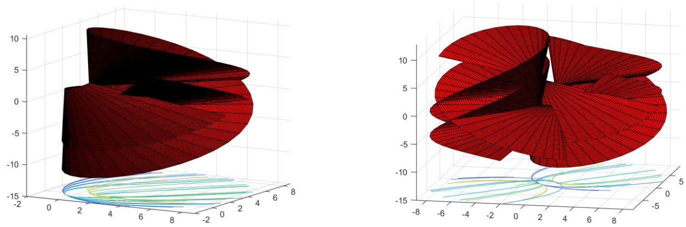
Figure 5. Normal ruled like surface of the helix and Orthogonal trajectory of $X(s)$.



(a) Osculating ruled like surface of the helix for $-5 < s < 5$ and $1 < v < 10$.

(b) Osculating ruled like surface of the helix(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 6. Osculating ruled like surfaces of the helix.



(a) Rectifying ruled like surface of the helix for $-5 < s < 5$ and $1 < v < 10$.

(b) Rectifying ruled like surface of the helix(Irregular) for $-5 < s < 5$ and $-10 < v < 10$.

Figure 7. Rectifying ruled like surfaces of the helix.

6. Conclusion

For normal ruled like surfaces, we consider only those surfaces which are generated by curves with $\tau(s) \neq -1$, therefore in the case of Salkowski curves [12] heaving $\tau(s) = \tan(s)$ regular normal ruled like surfaces are not possible with the same parametrization. Whereas in the case of rectifying ruled like surfaces generated by a curve, we got a case for some curve whose ratio of curvature and torsion holds the equation

$$\frac{\kappa}{\tau} = \begin{cases} \tan(s), & \text{if } s \neq (2n+1)\frac{\pi}{2}, \\ 0, & \text{if } s = (2n+1)\frac{\pi}{2}. \end{cases}$$

Thus exploring more details about this curve may give some new results. Furthermore, we believe that using this way of parametrization, one can find different surfaces in Minkowski space as well.

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