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# What positive integers $n$ can be presented in the form $n = (x + y + z)(1/x + 1/y + 1/z)$?

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**Abstract**

This paper shows that the equation in the title does not have positive integer solutions when $n$ is divisible by 4. This gives a partial answer to a question by Melvyn Knight. The proof is a mixture of elementary $p$-adic analysis and elliptic curve theory.

*Keywords:* Elliptic curves, $p$-adic numbers

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## 1. Introduction

According to Bremner, Guy, and Nowakowski [1], Melvyn Knight asked what integers $n$ can be represented in the form

$$n = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right), \tag{1.1}$$

where $x, y, z$ are integers. In the same paper [1], the authors made an extension study of (1.1) in integers when $n$ is in the range $|n| \leq 1000$. Integer solutions are found except for 99 values of $n$. The question becomes more interesting if we ask for positive integer solutions, which was also briefly discussed in [1, Section 2]. In this paper, we will prove the following theorem:

**Theorem 1.1.** *Let $n$ be a positive integer. Then equation*

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=n$$

*does not have positive integer solutions if* $4 \mid n$.

This theorem gives the first parametric family when (1.1) does not have positive integer solutions. The proof technique is a nice combination of $p$-adic analysis and elliptic curve theory, which was successfully applied to prove the insolubility of the equation

$$(x+y+z+w)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)=n$$

for the families $n=4m^2$, $4m^2+4$, $m \in \mathbb{Z}$ and $m \not\equiv 2 \pmod 4$, see [2].

## 2. The Hilbert symbol

Let $p$ be a prime number, and let $a$, $b \in \mathbb{Q}_p$. The Hilbert symbol $(a,b)_p$ is defined as

$$(a,b)_p=\begin{cases}1, & \text{if the equation } aX^2+bY^2=Z^2 \text{ has a solution}\\ & (X,Y,Z)\neq(0,0,0) \text{ in } \mathbb{Q}_p^3,\\ -1, & \text{otherwise.}\end{cases}$$

The symbol $(a,b)_\infty$ is defined similarly but $\mathbb{Q}_p$ is replaced by $\mathbb{R}$. The following properties of the Hilbert symbol are true, see Serre [3, Chapter III]:

(i) For all $a$, $b$, and $c$ in $\mathbb{Q}_p^*$, then

$$(a,bc)_p=(a,b)_p(a,c)_p,$$
$$(a,b^2)_p=1.$$

(ii) For all $a$ and $b$ in $\mathbb{Q}_p^*$, then

$$(a,b)_\infty \prod_{p \text{ prime}} (a,b)_p=1.$$

(iii) Let $p$ be a prime number, and let $a$ and $b$ in $\mathbb{Q}_p^*$. Write $a=p^\alpha u$ and $b=p^\beta v$, where $\alpha=v_p(a)$ and $\beta=v_p(b)$. Then

$$(a,b)_p=(-1)^{\frac{\alpha\beta(p-1)}{2}}\left(\frac{u}{p}\right)^\beta\left(\frac{v}{p}\right)^\alpha \quad \text{if } p\neq 2,$$
$$(a,b)_p=(-1)^{\frac{(u-1)(v-1)}{4}+\frac{\alpha(v^2-1)}{8}+\frac{\beta(u^2-1)}{8}} \quad \text{if } p=2,$$

where $\left(\frac{u}{p}\right)$ denotes the Legendre symbol.

# 3. A main theorem

**Theorem 3.1.** *Let $n$ be a positive integer divisible by 4. Let $u$ and $v$ be nonzero rational numbers such that*

$$v^2 = u(u^2 + (n^2 - 6n - 3)u + 16n).$$

*Then*

$$u > 0.$$

Let $A = n^2 - 6n - 3$, $B = 16n$ and $D = n - 1$. Then

$$A^2 - 4B = (n - 9)(n - 1)^2.$$

Now

$$v^2 = u(u^2 + Au + B). \tag{3.1}$$

The proof of Theorem 3.1 is achieved by means of the following three lemmas.

**Lemma 3.2.** *If $p$ is an odd prime number, then*

$$(u, -D)_p = 1.$$

*Proof.* Let $r = v_p(u)$. Then $u = p^r s$, where $s \in \mathbb{Z}_p$, and $p \nmid s$.

Case 1: $r < 0$. Then from (3.1), we have

$$v^2 = p^{3r} s(s^2 + p^{-r}A + Bp^{-2r}).$$

Therefore $2v_p(v) = 3r$, hence $2 \mid r$. Now

$$(p^{-3r/2}v)^2 = s(s^2 + p^{-r}A + Bp^{-2r}). \tag{3.2}$$

Note that $p \nmid s$. Taking (3.2) modulo $p$ gives $s$ is a square modulo $p$. Hence $s \in \mathbb{Z}_p^2$. We also have $2 \mid r$, so $u = 2^r s \in \mathbb{Q}_p^2$. Therefore $(u, -D)_p = 1$.

Case 2: $r = 0$.

If $p \nmid D$, then both $u$ and $-D$ are units in $\mathbb{Z}_p$. Therefore $(u, -D)_p = 1$.

If $p \mid D$, then $n \equiv 1 \pmod p$. Hence $A = n^2 - 6n - 3 \equiv -8 \pmod p$ and $B = 16n \equiv 16 \pmod p$. Thus

$$\begin{aligned} v^2 &\equiv u(u^2 - 8u + 16) \pmod p \\ &\equiv u(u-4)^2 \pmod p. \end{aligned} \tag{3.3}$$

If $u \equiv 4 \pmod p$, then $u \in \mathbb{Z}_p^2$. Hence $(u, -D)_p = 1$. If $u \not\equiv 4 \pmod p$, then from (3.3), we have

$$u \equiv \left(\frac{v}{u-4}\right)^2 \pmod p.$$

Therefore $u \in \mathbb{Z}_p^2$. Hence $(u, -D)_p = 1$.

Case 3: $r > 0$. Then (3.1) becomes

$$v^2 = p^r s(p^{2r} s^2 + Ap^r s + B). \tag{3.4}$$

If $p \mid B$, then $p \mid n$. Therefore $-D = 1 - n \equiv 1 \pmod{p}$. Hence $-D \in \mathbb{Z}_p^2$. Thus $(u, -D)_p = 1$.
If $p \nmid B$, then from (3.4) we have $r = 2v_p(v)$. Thus $2 \mid r$.

If $p \nmid D$, then both $s$ and $-D$ are units in $\mathbb{Z}_p$. Therefore $(s, -D)_p = 1$. Hence

$$(u, -D)_p = (p^r s, -D)_p = (s, -D)_p = 1.$$

If $p \mid D$, then $n \equiv 1 \pmod{p}$. Therefore $A = n^2 - 6n - 3 \equiv -8 \pmod{p}$ and $B = 16n \equiv 16 \pmod{p}$. Let $\omega = p^{\frac{-r}{2}} v$. Because $r = 2v_p(v)$, we have $p \nmid \omega$. From (3.4) we have

$$\omega^2 \equiv s(p^{2r} s^2 - 8p^r s + 16) \equiv 16s \pmod{p},$$

so that

$$s \equiv (\omega/4)^2 \pmod{p}.$$

Thus $s \in \mathbb{Z}_p^2$. Hence $(s, -D)_p$=1. Note that $2 \mid r$, therefore

$$(u, -D)_p = (p^r s, -D)_p = (s, -D)_p = 1. \qquad \square$$

**Lemma 3.3.** *We have*

$$(u, -D)_2 = 1.$$

*Proof.* Let $n = 4k$, where $k \in \mathbb{Z}^+$.

If $2 \mid k$, then $-D = 1 - 4k \equiv 1 (\mathrm{mod}\ 8)$. Therefore $-D \in \mathbb{Z}_2^2$. Hence $(u, -D)_2 = 1$. So we only need to consider the case $2 \nmid k$. Let $r = v_2(u)$. Then $u = 2^r s$, where $2 \nmid s$.

Case 1: $2 \mid r$. Then

$$\begin{aligned}(u, -D)_2 &= (2^r s, 1 - 4k)_2 \\ &= (s, 1 - 4k)_2 \\ &= (-1)^{\frac{(s-1)(1-4k-1)}{4}} \\ &= 1.\end{aligned}$$

Case 2: $2 \nmid r$. We show that this case is not possible.

If $r < 0$, then from (3.1), we have

$$v^2 = 2^{3r} s(s^2 + 2^{-r} As + 2^{-2r} B).$$

Therefore $3r = 2v_2(v)$. Hence $2 \mid r$, a contradiction.

If $r \geq 0$, then (3.1) becomes

$$v^2 = 2^r s(2^{2r} s^2 + 2^r(16k^2 - 24k - 3)s + 2^6 k). \tag{3.5}$$

If $r \geq 7$, then

$$v^2 = 2^{r+6}s(2^{2r-6}s^2 + (16k^2 - 24k - 3)2^{r-6}s + k).$$

Therefore $r + 6 = 2v_2(v)$. Hence $2 \mid r$, a contradiction.

If $r < 7$, then $r \leq 5$. Let $\phi = \frac{v}{2^r}$. Then from (3.5), we have

$$\phi^2 = s(2^r s^2 + (16k^2 - 24k - 3)s + 2^{6-r}k). \tag{3.6}$$

If $r = 5$, then taking (3.6) modulo 8 gives $\phi^2 \equiv s(-3s + 2k) \pmod 8$. Hence $2sk \equiv \phi^2 + 3s^2 \equiv 4 \pmod 8$, which is not possible because $2 \nmid sk$.

If $r = 3$, then taking (3.6) modulo 8 gives $\phi^2 \equiv -3s^2 \pmod 8$. Hence $0 \equiv \phi^2 + 3s^2 \equiv 4 \pmod 8$, a contradiction.

If $r = 1$, then taking (3.6) modulo 8 gives $\phi^2 \equiv s(2 - 3s) \pmod 8$. So $2s \equiv 3s^2 + \phi^2 \equiv 4 \pmod 8$, which is not possible because $2 \nmid s$. □

**Lemma 3.4.**

$$(u, -D)_\infty = 1.$$

*Proof.* From the product formula for the Hilbert symbol, we have

$$(u, -D)_\infty \prod_{p \text{ prime}, \, p < \infty} (u, -D)_p = 1. \tag{3.7}$$

By Lemma 3.2, Lemma 3.3, and (3.7), we have $(u, -D)_\infty = 1$. □

To complete the proof of Theorem 3.1, we see that Lemma 3.4 shows that the equation $uX^2 + (1-n)Y^2 = Z^2$ has a solution $(X, Y, Z) \neq (0, 0, 0)$ in $\mathbb{R}^3$. Because $1 - n < 0$, we have $u > 0$. Hence Theorem 3.1 is proved.

# 4. A proof of Theorem 1.1

We follow [1, Section 2]. Write (1.1) as

$$x^2(y + z) + x(y^2 + (3 - n)yz + z^2) + yz = 0. \tag{4.1}$$

Hence

$$x = \frac{-y^2 + (n-3)yz - z^2 \pm \Delta}{2(y+z)},$$

where $\Delta$ satisfies

$$\Delta^2 = y^4 - 2(n-1)yz(y^2 + z^2) + (n^2 - 6n - 3)y^2z^2 + z^4. \tag{4.2}$$

Then (4.2) is birationally equivalent to the elliptic curve

$$v^2 = u(u^2 + (n^2 - 6n - 3)u + 16n), \tag{4.3}$$

and we can write out the maps between (4.1) and (4.3):

$$u = \frac{-4(xy + yz + zx)}{z^2}, \quad v = \frac{2(u - 4n)y}{z} - (n-1)u,$$

and

$$\frac{x,\, y}{z} = \frac{\pm v - (n-1)u}{2(4n - u)}.$$

Then the following is true.

**Proposition 4.1.** *The necessary and sufficient conditions for* (4.1) *to have positive integer solutions* $(x, y, z)$ *are* $n > 0$ *and* $u < 0$*.*

*Proof.* See Bremner, Guy, and Nowakowski [1, Section 2]. □

Now, let $n = 4k$, where $k \in \mathbb{Z}^+$. Assume there exists a positive integer solution $(x, y, z)$ to (1.1). Then Proposition 4.1 shows that $u < 0$. If $v = 0$, then (4.3) implies $u^2 + (n^2 - 6n - 3)u + 16n = 0$. Therefore $(n-9)(n-1)^3 = (n^2 - 6n - 3)^2 - 4 \times 16n$ is a perfect square. Hence $(n-9)(n-1)$ is a perfect square. Let $(n-9)(n-1) = m^2$. Then $(n-5)^2 - 16 = m^2$. The equation $X^2 - 16 = Y^2$ only has integer solutions $(X, Y) = (\pm 5, \pm 3)$. Thus $n - 5 = \pm 5$, giving no solutions $n = 4k$. Therefore $v \neq 0$. Hence $u$, $v \neq 0$. From Theorem 3.1, we have $u > 0$, contradicting Proposition 4.1. Hence (1.1) does not have solutions in positive integers.

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