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Students' non-development in high school geometry*

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Abstract

Earlier findings suggest that there is a gap between the knowledge of students entering the university and the expectations and prerequisites of the universities' curriculum. In this paper we investigate this gap in the Hungarian mathematics education. In particular we concentrate on the geometrical knowledge of the Hungarian high school students. For measuring their levels of geometrical understanding we used the Usiskin test for the framework of the Van Hieles. The test was filled in by 342 students from five different high schools. The results show that there is no improvement during the high school years, the average score of the Usiskin test is between 2.03 and 2.17 on all grades.

Keywords: van Hiele levels, math education, high shool, understanding geometry, development

MSC: AMS classification numbers

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1. Introduction and problem statement

Earlier findings show that there is a gap between the knowledge of students entering the university and the expectations and prerequisites of the universities' curriculum [8]. These prerequisites are based on the National Core Curriculum (NCC) [20] for high schools. The admission process to the university is strongly based on a final exam that every student has to take at the end of secondary school from five subjects: mathematics is compulsory. They get a grade 1–5 for the exam and this grade is put into their transcripts. At the same time their score in percentages counts at the admission points to the university. Students can choose between two levels, medium and raised. The tasks in the final exam are mainly standard tasks and can be anticipated, even on the raised level, hence, can be practised. Thus a student can practice to the final exam without gaining deeper understanding. This final exam has a high impact on secondary education and on the transition from secondary to tertiary education in Hungary. Not only students are ranked and can get admission to universities based on their final exam-results, high schools are also ranked based on the average scores of their students on final exams [19]. Most students successfully pass their final exam in mathematics [7], however, it is a general observation that the knowledge of students entering university is deficient. This suggests that there is a gap between the final exams and the NCC [8].

We would like to argue that the strong external influence of the final exam distorts the original conceptions of the NCC. German universities struggled with a similar problem [3]: there was a big difference between the knowledge of the students entering the university and the knowledge required by the university. After several conciliations between the universities and each province's secondary schools, this problem seems to be being solved in Germany. In this paper we investigate this gap in the Hungarian math education.

Understandably, achieving good results on the final exam becomes a crucial aspect in high school mathematics education – sometimes even more important than aspects set up by the NCC. This implies that teachers will concentrate more on the topics and the tasks which occur in the final exam than on other topics that are in the curriculum. The entry system allows students to enter the university in the absence of the required knowledge [8]. We chose to examine this problem focusing on students' geometrical thinking due to the great proportion of geometry in the curriculum and the final exam. Geometry holds a central role in science, has several applications in everyday life, and in arts as well [5, 12]. Geometry itself is a separate high school subject in Greece, for example. In Hungary usually thirty percent of the final exam tasks are geometric flavoured. This is a significant proportion. Geometry is a substantial part of secondary mathematics education as well. It occupies approximately thirty-five percent of the high school mathematics material, similarly to its proportion in the final exam.

Hence it is natural to consider to investigate the geometrical understanding of Hungarian high school students. The aim of this research was to investigate students' Van Hiele levels to follow their development, especially to see whether or not this development is parallel to the requirements of the NCC. In particular, we were interested if students from grade 12 have achieved level 4, the level of proofs. In our study we use the Usiskin test for the framework of the Van Hieles [15]. The test was filled in by 342 students from five different high schools. The results show that there is no improvement during the high school years, the average score of the Usiskin test is between 2.03 and 2.17 on all grades.

2. Description of geometrical understanding in the National Core Curriculum

There are several ways of thinking about geometry, there are different ways people think about it and there are several ways to structure geometry and how to teach geometry. The Van Hieles elaborated one possible way of structuring and describing people's understanding of geometry: focusing on understanding of geometrical shapes and structures, they distinguished five different levels of geometrical understanding. These levels are: visualization, analysation, abstraction, deduction and rigor (they are explained down below). According to the van Hiele theory, a student moves sequentially from the initial level (Visualization) to the highest level (Rigor). Students cannot achieve one level of thinking successfully without having passed through the previous levels [15].

The van Hieles' theory has been applied to clarify students' difficulties with the higher order cognitive processes. In order to succeed in high school geometry, higher order cognitive processes are indispensable. [20] According to the theory if students are not taught at the proper Van Hiele level, then they will face difficulties and they cannot understand geometry. This makes measuring students' Van Hiele level necessary. A possible validated tool for this measurement is the test elaborated by Usiskin in 1982.

2.1. Level 1: Visualization

At this initial stage, students recognize figures only by appearance and they usually think about space only as something that exists around them. Geometric concepts are viewed as undivided, whole entities rather than as having components or attributes. For example, geometric figures are recognized by their whole physical appearance, not by their parts or properties, so the properties of a figure are not detected. A person functioning at this level makes decisions based on perception, not reasoning. On the other hand, they can learn geometric vocabulary, identify specified shapes, reproduce a given figure. However, a person at this stage would not recognize the part of the figures, thus, they cannot identify the properties of these parts.

2.2. Level 2: Analysation

At this level an analysis of geometric concepts begins. For example, students can connect a collection of properties to figures, but at this point they see no relationship between these properties. Figures are recognized as having parts and are recognized by their parts. Usually they know a list of properties, but they cannot decide which properties are necessary and which are sufficient to describe the object. Interrelationships between figures are still not seen, and definitions are not yet understood at this level.

2.3. Level 3: Abstraction

At level 2 students perceive relationships between properties and between figures, they are able to establish the interrelationships of properties both within figures (e.g., in a quadrilateral, opposite angels being equal necessitates opposite sides being equal) and among figures (a rectangle is a parallelogram because it has all the properties of a parallelogram). So, at this level, class inclusion is understood, and definitions are meaningful. They are also able to give informal arguments to justify their reasoning. However, a student at this level does not understand the role and significance of formal deduction.

2.4. Level 4: Deduction

The 4th level is the level of deduction: students can construct smaller proofs (not just memorize them), understand the role of axioms, theorems, postulates and definitions, and recognize the meaning of necessary and sufficient conditions. The possibility of developing a proof in more than one way is also seen and distinctions between a statement and its converse can be made at this level.

2.5. Level 5: Rigor

This level is the most abstract of all. A person at this stage can think and construct proofs in different kind of geometric axiomatic systems. So, students at this level can understand the use of indirect proof and proof by contra-positive and can understand non-Euclidean systems.

The existence of Level 0 – the level of pre- recognition is also proposed [6]. Students at this level notice only a subset of the visual characteristics of a shape. As a result, they are not able to distinguish between certain figures. Progress from one level to the next is more dependent on educational experiences, than on age or maturation. Some experiences can facilitate progress within a level or to a higher level.There is some logic behind this kind of structuring. Although there might be other levelings, but these levels should be achieved by everybody independently of the manner in which they learned geometry.

The logic of this structure is also confirmed by the observation that the Van Hiele levels can be recognized in the Hungarian National Core Curriculum [20] step by step. The following sentences and requirements connecting to different grades are from the NCC.

- Grade 1–4: "The creation, recognition and characteristics of triangles, squares, rectangles, polygons and circles."
- Grade 5–8 "Triangles and their categories. Quadrilaterals, special quadrilateral (trapezoids, parallelograms, kites, rhombuses). Polygons, regular polygons. The circle and its parts. Sets of points that meet given criteria."
- Grade 9–12.: "The classification of triangles and quadrilaterals. Altitudes, centroid, incircle and circumcircle of triangles. The incircle and circumcircle of regular polygons. Thales' theorem."
  "Remembering argumentation, refutations, deductions, trains of thought; applying them in new situations, remembering proof methods is important."
  "Generalization, concretization, finding examples and counterexamples (confirming general statements by deduction; proving, disproving: demonstrating errors by supplying a counterexample); declaring theorems and proving them (directly and indirectly) is also necessary."

The levels correspond to age groups: a 4^{th} grader (10 years old) has to reach level 1, a 6^{th} grader (12 years old) should reach level 2, an 8^{th} grader (14 years old) should be on at least level 3, and finally at grade 12 students (18 years old) have to reach level 4, which means they have to reach the level of deductions – students have to be able to construct smaller proofs, understand the role of axioms, theorems, postulates and definitions.

3. The survey

A survey of high-school students was held during the 2015/2016 academic year. Participants were 342 students from five different high-schools: one from Budapest and four from Miskolc. The schools were selected from a list that either had an agreement with our university or showed earlier a willingness to participate in research experiments. We omitted the schools with a special math program and schools founded by our university. Among the schools there was one music conservatory, and four standard high-schools such that three of them is considered as an average high-school, and one of them is in the top forty by the official ranking of the Hungarian Ministry of National Resources [19]. Four schools are founded by the government, one by the church. The data from Miskolc was collected by two colleagues from Miskolc: Csenge Edőcsény and Ákos Győry. There was 62 students from the music conservatory and the other 280 students followed the normal curriculum. Also there were 32 pupils who belonged to the Arany János Tehetséggondozó Program (AJTP) which is a talent care program for pupils coming from socially handicapped families, mostly from small villages. Out of the 280 students there were 91 from grade 9, 103 from grade 10, 27 from grade 11, and 59 from grade 12. Among the 62 music conservatory students 18 9th graders, 17 10th graders, 15 11th graders, and 59 12th graders participated in our research.

The measuring of the levels was carried out by means of the Usiskin-test [15], which is a 25 item multiple-choice test with 5 foils per item. The test contains five questions per level and to fulfill correctly a level one has to answer correctly to at least 3 or 4 questions – depending on which scoring system is used – out of the five questions. We distinguish two kinds of scoring system: we called them "strong" version and a "weak" version. In the "weak" version one has to answer correctly to at least 3 questions from the five to fulfill correctly a level, while in the "strong" version at least 4 good answer is needed. To reach a level one has to fulfill correctly all the previous levels, too. That means if a person completed correctly level 1, 2, 3, 5 but not level 4, then this person is on level 3 according to the test. In general if a person met the criteria of passing each level up to and including level n, but not level $n+1$, then the person is assigned to level n. This test was used in more than forty countries [1, 2, 4, 9–11, 13, 14, 16–18, 22], and this test is tested and used continuously from 1982.

There are 35 minutes for the test independently of age and grade. In our experiment the students had to complete the test either on paper or online decided by the teacher of the class.

4. The results

The following table shows the results of the high-school students. On the table A, B, C, D, E letters denote the schools. The abbreviation n.o.p. denote the number of participants. By strong version we mean that the text filler has to answer four questions correctly out of the five to fill correctly a certain level and by weak version we mean that the text filler has to answer only three questions correctly out of the five.

	A	**B**	**C**	**D**	**E**	**total**
mean	1,42	1,26	1,40	1,00	0,67	1,21
dev.	1,35	1,38	1,16	1,05	0,97	1,23
n.o.p.	24	27	30	10	18	109

Table 1: Grade 9 – strong version

	A	**B**	**C**	**D**	**E**	**total**
mean	2,29	2,22	2,13	2,10	1,17	2,03
dev.	0,95	1,69	1,22	1,20	1,34	1,35
n.o.p.	24	27	30	10	18	109

Table 2: Grade 9 – weak version

	A	**B**	**C**	**D**	**E**	**total**
mean	1,18	1,13	1,54	1,80	1,00	1,31
dev.	1,26	1,18	1,14	1,32	1,12	1,19
n.o.p.	22	32	39	10	17	120

Table 3: Grade 10 – strong version

	A	**B**	**C**	**D**	**E**	**total**
mean	1,18	2,16	2,21	2,40	1,59	2,05
dev.	1,10	1,30	1,10	0,97	1,12	1,16
n.o.p.	22	32	39	10	17	120

Table 4: Grade 10 – weak version

	A	**B**	**C**	**D**	**E**	**total**
mean	2,86	-	1,38	1,14	0,47	1,26
dev.	1,21	-	1,39	0,90	0,74	1,33
n.o.p.	7	0	13	7	15	42

Table 5: Grade 11 – strong version

	A	**B**	**C**	**D**	**E**	**total**
mean	4,00	-	2,23	2,43	1,13	2,17
dev.	1,29	-	1,48	1,27	0,92	1,54
n.o.p.	7	0	13	7	15	42

Table 6: Grade 11 – weak version

	A	**B**	**C**	**D**	**E**	**total**
mean	0,70	1,75	2,11	0,87	1,00	1,14
dev.	1,02	1,36	1,05	1,19	1,04	1,21
n.o.p.	23	12	9	15	12	71

Table 7: Grade 12 – strong version

	A	**B**	**C**	**D**	**E**	**total**
mean	2,47	2,75	2,89	1,00	1,17	2,04
dev.	1,04	1,42	0,60	1,95	1,11	1,34
n.o.p.	23	12	9	15	12	71

Table 8: Grade 12 – weak version

	n.o.p.	**mean (strong version)**	**mean (weak version)**
grade 9	109	1,21	2,03
grade 10	120	1,31	2,05
grade 11	42	1,26	2,17
grade 12	71	1,14	2,04

Table 9: cumulative results

Although the results are from different schools, the performances of the schools are similar and based on these results we can estimate the Van Hiele levels of students attending to other schools in the country. Based on this estimation most of the Hungarian high-school students are on the level of a primary school student in geometry. This raises the question how students can be successful on the final exam. This question requires a deeper investigation, a part of it could be the analysis of the geometry problems and their sample solutions in the final test. Reading through the past fifteen years' final exams it is reasonable to question the amount of geometrical proving skills needed to solve the tasks. Typical geometry flavoured tasks are the following ones [21]:

Problem 4.1 (A tipical final exam task – a "less difficult" one)**.** *The ending point of a straight line that closes at* $6.5°$ *to the horizontal is* 124 *meters higher than its starting point. How long is the road? Justify your answer!*

Problem 4.2 (A tipical final exam task – a "difficult" one)**.** *A motion sensor is on the top of a* $4\,m$ *high vertical pole. The lamp connected to the sensor illuminates vertically downwards at a rotational cone of* $140°$*.*

a) *Make a sketch with the details.*
b) *How far is the farthest illuminated point from the lamp?*
c) *Does the sensor lamp illuminate an object on the ground* 15 *m from the bottom of the pole?*
d) *There are hooks on the pole, one per meter, in order to hang the motion detector lamp. Which hook should we use in order that the lamp illuminates at most* $100\,m^2$ *on the horizontal ground? (Numbering of the hooks starts from the bottom of the pole.)*

The first task is from May 2003 and the second one was from May 2006. On the first task students could reach 3 points, while with the second one they could get 17 out of 115 points on the exam. In the latter case pupils get only 2 points for noticing that the flat section is a triangle and they get 15 points for the calculations. So a possible answer could be obtained answering the quesiton: Does the final exam require the 4^{th} van Hiele level at all?

5. Discussion

The tables show the results of the van-Hiele tests from five Hungarian high-schools. The sample naturally does not cover the whole country. It involves three schools that are average in the Hungarian rankings, one vocational music school and a non-elite, but fairly top ranked school. With these limitations we made the following observations. It can be read from the tables, that even considering the weaker criteria, in each grade the average performance of the students is around level 2 – which should be the level of a 6^{th} grader. There is no development in the level of understanding geometry from grade 9 to grade 12. Most of the students do not even reach the 3^{rd} level which should be the level of an 8^{th} grader according to the NCC. In the weak version 40.37% of the students reached the 3^{rd} Van Hiele level at grade 9, which is the level that the NCC suggests. In grade 12 45.07% of the students reached the 3^{rd} Van Hiele level and only 8.45% of these students reached the 4^{th} Van Hiele level, which is the level that a 12 grader should reach according to the NCC. Although there exist students who reach the required level, the geometrical thinking of the vast majority did not improve during their high-school years. Still, both groups passed the final exam with relatively good results. One can see that mathematical education in Hungary is in a controversial situation. On the one hand, students achieve a certain, well defined level, namely, they perform well on final exam. On the other hand, they do not reach the level of geometric understanding required by the NCC.

To look for the possible reasons it is worth examining the geometry content of the final exam. By its nature the final exam is predictable and a has a high impact on the curriculum and teacher activities in class.

The gap between the final exam's requirement and the NCC's requirement indicates further problems for higher education. Since the university education is built on the National Core Curriculum, not on the final exam, this gap results in a big difference between the knowledge of the students entering the university and the knowledge required by the universities. We see three kind of solution to this problem. The first one is to change the requirement system of the NCC and make it consistent with the requirement system of the final exam. It follows that universities would adopt to the new NCC and the standard of higher education would fall. The second solution is to change the entry system of the universities concerned, and make it obligatory to take the mathematics final exam on advanced level or reintroduce an entry exam for the universities. The third solution we imagine is to introduce bridging courses at the universities specialized to different topics and levels depending on their needs. We think that the latter solution would not work. It would be difficult and nearly impossible to bring the students from such a low level to the level where they understand the need for proof and where they can also construct easier ones in a half a year course.

References

[1] A. H. Abdullah, E. Zakaria: *Enhancing Students' Level of Geometric Thinking through Van Hiele's Phase-based Learning*, Indian Journal of Science and Technology 6.5 (2013), pp. 4432–4446.

[2] R. Astuti, D. Suryadi, T. Turmudi: *Analysis on geometry skills of junior high school students on the concept congruence based on Van Hiele's geometric thinking level*, Journal of Physics Conference Series (2018), pp. 1–5, doi: https://doi.org/10.1088/1742-6596/1132/1/012036.

[3] I. Braun, J. E. Schröder: *Cooperation schule hochschule*, Baden-Württembergs: Hochschulen Baden-Württembergs, 2014.

[4] W. F. Burger, J. M. Shaughnessy: *Characterizing the van Hiele Levels of Development in Geometry*, Journal for Research in Mathematics Education 17.1 (1986), pp. 31–48, doi: https://doi.org/10.2307/749317.

[5] P. Bursill-Hall: *Why do we study geometry? Answers through the ages*, Cambridge: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 2002.

[6] D. Clements, M. T. Battista: *Geometry and spatial reasoning*, in: Jan. 1992, pp. 420–464.

[7] C. Csapodi, L. Koncz: *The efficiency of written final exam questions in mathematics based on voluntary data reports, 2012–2015*, Teaching Mathematics and Computer Science 14.1 (2016), pp. 63–81, doi: https://doi.org/10.5485/TMCS.2016.0417.

[8] É. Erdélyi, A. Dukán, C. Szabó: *The transition problem in Hungary: curricular approach*, Teaching Mathematics and Computer Science 17.1 (2019), pp. 1–16, doi: https://doi.org/10.5485/TMCS.2019.0454.

[9] T. Erdogan, S. Durmus: *The effect of the instruction based on Van Hiele model on the geometrical thinking levels of preservice elementary school teachers*, Procedia - Social and Behavioral Sciences 1.1 (2009), pp. 154–159, doi: https://doi.org/10.1016/j.sbspro.2009.01.029.

[10] K. Jones: *Issues in the teaching and learning of geometry*, in: Aspects of Teaching Secondary Mathematics: perspectives on practice. London, GB: Routledge, 2002, pp. 121–139, doi: https://doi.org/10.4324/9780203165874.

[11] G. Kospentaris, P. Spyrou: *Assessing the development of geometrical thinking from the visual towards the analytic-descriptive level*, Annales de didactique et de sciences cognitives 13.5 (2008), pp. 133–157.

[12] C. Mammana, V. V.: *Perspectives on the Teaching of Geometry for the 21st Century*, Dordrecht: Springer, 1998,
doi: `https://doi.org/10.1007/978-94-011-5226-6`.

[13] S. Senk: *Van Hiele levels and achievement in writing geometry proofs*, Journal for Research in Mathematics Education 20.3 (1989), pp. 309–321,
doi: `https://doi.org/10.2307/749519`.

[14] A. Ural: *Investigating 11th Grade Students' Van-Hiele Level 2 Geometrical Thinking*, Journal Of Humanities And Social Science 21.12 (2016), pp. 13–19,
doi: `https://doi.org/10.9790/0837-2112061319`.

[15] Z. Usiskin: *Van Hiele Levels and Achievement in Secondary School Geometry. CDASSG Project.* Chicago: Chicago Univ, IL., 1982.

[16] M. D. de Villers: *Some reflections on the Van Hiele theory*, in: June 2010.

[17] M. D. de Villers: *The role and function of a hierarchical classification of the quadrilaterals*, For the Learning of Mathematics 14.1 (1994), pp. 11–18.

[18] I. Vojkuvkova, J. Haviger: *The van Hiele Levels at Czech Secondary Schools*, Procedia - Social and Behavioral Sciences 171 (2015), pp. 912–918,
doi: `https://doi.org/10.1016/j.sbspro.2015.01.209`.

[19] www.eduline.hu: *A száz legjobb vidéki gimnázium és szakközépiskola - itt a teljes lista*, Eduline (2014).

[20] www.ofi.hu: *The Hungarian National Core Curriculum*, Teaching Mathematics and Computer Science (2012).

[21] www.oktatas.hu: *Központi írásbeli feladatsorok, javítási útmutatók*, www.oktatas.hu.

[22] I. Zachos: *Register of Educational Research in the United Kingdom – Problem Solving in Euclidean Geometry in Greek Schools*, in: vol. 10, 0615, London and New York: Routledge, 1995.