Annales Mathematicae et Informaticae
***50** (2019) pp. 179–185*
DOI: 10.33039/ami.2019.12.001
http://ami.uni-eszterhazy.hu

A Marcinkiewicz–Zygmund type strong law of large numbers for non-negative random variables with multidimensional indices

Tibor Tómács*

Institute of Mathematics and Informatics
Eszterházy Károly University, Eger, Hungary
tomacs.tibor@uni-eszterhazy.hu

Submitted: September 2, 2019
Accepted: December 4, 2019
Published online: December 5, 2019

Abstract

In this paper a Marcinkiewicz–Zygmund type strong law of large numbers is proved for non-negative random variables with multidimensional indices, furthermore we give its an application for multi-index sequence of non-negative random variables with finite variances.

Keywords: Marcinkiewicz–Zygmund type strong law of large numbers, almost sure convergence, non-negative random variables, multidimensional indices

MSC: 60F15

1. Introduction

The Kolmogorov theorem and the Marcinkiewicz–Zygmund theorem are two famous theorems on the strong law of large numbers for X_n $(n \in \mathbb{N})$ sequence of independent identically distributed random variables (see e.g. Loève [8]). By Kolmogorov theorem, there exists a constant b such that $\lim_{n\to\infty} S_n/n = b$ almost surely if and only if $\mathrm{E}\,|X_1| < \infty$, where $S_n = \sum_{k=1}^{n} X_k$. If the latter condition is satisfied then $b = \mathrm{E}\,X_1$. By Marcinkiewicz–Zygmund theorem, if $0 < r < 2$ then

*The author's research was supported by the grant EFOP-3.6.1-16-2016-00001 ("Complex improvement of research capacities and services at Eszterhazy Karoly University").

$\lim_{n\to\infty}(S_n - bn)/n^{1/r} = 0$ almost surely if and only if $\mathrm{E}\,|X_1|^r < \infty$, where $b = 0$ if $0 < r < 1$, and $b = \mathrm{E}\,X_1$ if $1 \le r < 2$.

Etemadi [1] proved that the Kolmogorov theorem holds for identically distributed and pairwise independent random variables, furthermore Kruglov [7] extended the Marcinkiewicz–Zygmund theorem for pairwise independent case if $r < 1$.

Several papers are devoted to the study of the strong law of large numbers for multi-index sequence of random variables (see e.g. Gut [4], Klesov [5, 6], Fazekas [2], Fazekas, Tómács [3]). For example, Theorem 3.1 of Fazekas, Tómács [3] extends Theorem 2 of Kruglov [7] for multi-index case.

In this paper the main result is Theorem 3.1, which is a Marcinkiewicz–Zygmund type strong law of large numbers for non-negative random variables with multidimensional indices. It is a generalization of Theorem 3.1 of Fazekas, Tómács [3] in case $\mathbf{n} \to \infty$. Furthermore we give an application (see Theorem 4.1) for multi-index sequence of non-negative random variables with finite variances. A special case of this result gives Theorem of Petrov [9].

2. Notation

Let $\mathbb{N}^d$ be the positive integer d-dimensional lattice points, where d is a positive integer. For $\mathbf{n}, \mathbf{m} \in \mathbb{N}^d$, $\mathbf{n} \le \mathbf{m}$ is defined coordinate-wise, $(\mathbf{n}, \mathbf{m}] = (n_1, m_1] \times (n_2, m_2] \times \cdots \times (n_d, m_d]$ is a d-dimensional rectangle and $|\mathbf{n}| = n_1 n_2 \cdots n_d$, where $\mathbf{n} = (n_1, n_2, \ldots, n_d)$, $\mathbf{m} = (m_1, m_2, \ldots, m_d)$. $\sum_{\mathbf{n}}$ will denote the summation for all $\mathbf{n} \in \mathbb{N}^d$. We also use $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^d$ and $\mathbf{2} = (2, 2, \ldots, 2) \in \mathbb{N}^d$. Denote the integer part of x real number by $[x]$.

We shall say that $\lim_{\mathbf{n}\to\infty} a_{\mathbf{n}} = 0$, where $a_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{N}^d$) are real numbers, if for all $\delta > 0$ there exists $\mathbf{N} \in \mathbb{N}^d$ such that $|a_{\mathbf{n}}| < \delta$ $\forall \mathbf{n} \ge \mathbf{N}$.

We shall assume that random variables $X_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{N}^d$) are defined on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$. E and Var stand for the expectation and the variance.

Remark that a sum or a minimum over the empty set will be interpreted as zero (i.e. $\sum_{\mathbf{n}\in H} a_{\mathbf{n}} = \min_{\mathbf{n}\in H} a_{\mathbf{n}} = 0$ if $H = \emptyset$).

3. The result

The following result is a generalization of Theorem 3.1 of Fazekas, Tómács [3] in case $\mathbf{n} \to \infty$.

Theorem 3.1. *Let $X_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{N}^d$) be a sequence of non-negative random variables, let $b_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{N}^d$) be a sequence of non-negative numbers, $B_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} b_{\mathbf{k}}$, $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$, $c > 0$, $K \in \mathbb{N}$ and $0 < r \le 1$. If*

$$B_{\mathbf{n}} - B_{\mathbf{m}} \le c(|\mathbf{n}| - |\mathbf{m}|) \quad \forall \mathbf{n}, \mathbf{m} \in \mathbb{N}^d, \mathbf{n} \ge \mathbf{m}, |\mathbf{n}| - |\mathbf{m}| \ge K \tag{3.1}$$

and

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathrm{P}\left(|S_{\mathbf{n}} - B_{\mathbf{n}}| > \varepsilon|\mathbf{n}|^{1/r}\right) < \infty \quad \forall\varepsilon > 0, \tag{3.2}$$

then

$$\lim_{\mathbf{n}\to\infty} \frac{S_{\mathbf{n}} - B_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} = 0 \quad \textit{almost surely.}$$

Proof. Let $\delta > 0$, $1 < \alpha < \left(\frac{\delta}{2c} + 1\right)^{1/3d}$ and $0 < \varepsilon < \frac{\delta}{2}\left(\frac{\delta}{2c} + 1\right)^{-1/r}$, which imply

$$\varepsilon\alpha^{3d/r} + c(\alpha^{3d} - 1) < \delta. \tag{3.3}$$

Let $k_n = [\alpha^n]$ $(n \in \mathbb{N})$ and $\mathbf{k_n} = (k_{n_1}, k_{n_2}, \ldots, k_{n_d})$, where $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$. It follows from the inequalities

$$\begin{aligned}
&\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathrm{P}\left(|S_{\mathbf{n}} - B_{\mathbf{n}}| > \varepsilon|\mathbf{n}|^{1/r}\right) \\
&\geq \sum_{\mathbf{n}} \sum_{\mathbf{h}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \frac{1}{|\mathbf{h}|} \mathrm{P}\left(|S_{\mathbf{h}} - B_{\mathbf{h}}| > \varepsilon|\mathbf{h}|^{1/r}\right) \\
&\geq \sum_{\mathbf{n}} \sum_{\mathbf{h}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \frac{1}{|\mathbf{k_{n+1}}|} \min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \mathrm{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon|\mathbf{k}|^{1/r}\right) \\
&= \sum_{\mathbf{n}} \frac{|\mathbf{k_{n+1}} - \mathbf{k_n}|}{|\mathbf{k_{n+1}}|} \min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \mathrm{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon|\mathbf{k}|^{1/r}\right)
\end{aligned}$$

and condition (3.2) that

$$\sum_{\mathbf{n}} \frac{|\mathbf{k_{n+1}} - \mathbf{k_n}|}{|\mathbf{k_{n+1}}|} \min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \mathrm{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon|\mathbf{k}|^{1/r}\right) < \infty. \tag{3.4}$$

Since $\lim_{n\to\infty}\left(1 - \frac{1}{\alpha^{n+1}} - \frac{1}{\alpha}\right) = 1 - \frac{1}{\alpha} > 0$, so $\left(1 - \frac{1}{\alpha^{n+1}} - \frac{1}{\alpha}\right) > \frac{\alpha-1}{2\alpha}$ except for finitely many $n \in \mathbb{N}$. This implies that there exists $\mathbf{N}_0 \in \mathbb{N}^d$ such that

$$\begin{aligned}
0 < \left(\frac{\alpha - 1}{2\alpha}\right)^d &< \prod_{i=1}^{d}\left(1 - \frac{1}{\alpha^{n_i+1}} - \frac{1}{\alpha}\right) = \prod_{i=1}^{d} \frac{\alpha^{n_i+1} - 1 - \alpha^{n_i}}{\alpha^{n_i+1}} \\
&\leq \prod_{i=1}^{d} \frac{[\alpha^{n_i+1}] - [\alpha^{n_i}]}{[\alpha^{n_i+1}]} = \frac{|\mathbf{k_{n+1}} - \mathbf{k_n}|}{|\mathbf{k_{n+1}}|} \quad \forall\mathbf{n} = (n_1, n_2, \ldots, n_d) \geq \mathbf{N}_0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{\alpha - 1}{2\alpha}\right)^d \sum_{\mathbf{n}\geq\mathbf{N}_0} \min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \mathrm{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon|\mathbf{k}|^{1/r}\right) \\
&\leq \sum_{\mathbf{n}\geq\mathbf{N}_0} \frac{|\mathbf{k_{n+1}} - \mathbf{k_n}|}{|\mathbf{k_{n+1}}|} \min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]} \mathrm{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon|\mathbf{k}|^{1/r}\right).
\end{aligned}$$

By this inequality and (3.4), it follows that

$$\sum_{\mathbf{n}\geq\mathbf{N}_0}\min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]}\mathrm{P}\left(|S_\mathbf{k}-B_\mathbf{k}|>\varepsilon|\mathbf{k}|^{1/r}\right)<\infty. \tag{3.5}$$

If $\mathbf{n}\geq\mathbf{N}_0$ then there exists $\mathbf{m_n}\in\mathbb{N}^d$ such that $\mathbf{m_n}\in(\mathbf{k_n},\mathbf{k_{n+1}}]$ and

$$\mathrm{P}\left(|S_{\mathbf{m_n}}-B_{\mathbf{m_n}}|>\varepsilon|\mathbf{m_n}|^{1/r}\right)=\min_{\mathbf{k}\in(\mathbf{k_n},\mathbf{k_{n+1}}]}\mathrm{P}\left(|S_\mathbf{k}-B_\mathbf{k}|>\varepsilon|\mathbf{k}|^{1/r}\right).$$

Therefore, by (3.5) we have

$$\sum_{\mathbf{n}\geq\mathbf{N}_0}\mathrm{P}\left(|S_{\mathbf{m_n}}-B_{\mathbf{m_n}}|>\varepsilon|\mathbf{m_n}|^{1/r}\right)<\infty. \tag{3.6}$$

By the Borel–Cantelli lemma, (3.6) implies that there exist $\mathbf{N}_1\in\mathbb{N}^d$ and $A\in\mathcal{F}$ such that $\mathbf{N}_1\geq\mathbf{N}_0$, $\mathrm{P}(A)=1$ and

$$\frac{|S_{\mathbf{m_n}}(\omega)-B_{\mathbf{m_n}}|}{|\mathbf{m_n}|^{1/r}}\leq\varepsilon\quad\forall\mathbf{n}\geq\mathbf{N}_1,\ \forall\omega\in A. \tag{3.7}$$

Henceforward let $\omega\in A$ be fixed.

If $\mathbf{n}\geq\mathbf{N}_1$ and $\mathbf{t}\in(\mathbf{k_{n+1}},\mathbf{k_{n+2}}]$, then by $\mathbf{t}\in(\mathbf{m_n},\mathbf{m_{n+2}}]$, (3.7) and

$$|\mathbf{m_{n+2}}|^{1/r}\geq|\mathbf{m_n}|^{1/r}\geq|\mathbf{m_n}|$$

we have

$$\begin{aligned}\frac{S_\mathbf{t}(\omega)-B_\mathbf{t}}{|\mathbf{t}|^{1/r}}&\geq\frac{S_{\mathbf{m_n}}(\omega)-B_{\mathbf{m_{n+2}}}}{|\mathbf{m_{n+2}}|^{1/r}}\\&=\frac{S_{\mathbf{m_n}}(\omega)-B_{\mathbf{m_n}}}{|\mathbf{m_n}|^{1/r}}\frac{|\mathbf{m_n}|^{1/r}}{|\mathbf{m_{n+2}}|^{1/r}}-\frac{B_{\mathbf{m_{n+2}}}-B_{\mathbf{m_n}}}{|\mathbf{m_{n+2}}|^{1/r}}\\&\geq-\varepsilon-\frac{B_{\mathbf{m_{n+2}}}-B_{\mathbf{m_n}}}{|\mathbf{m_n}|}.\end{aligned} \tag{3.8}$$

If $\mathbf{n}=(n_1,n_2,\dots,n_d)\geq\mathbf{N}_0$ and $\mathbf{m_n}=(\mathbf{m}_\mathbf{n}^{(1)},\mathbf{m}_\mathbf{n}^{(2)},\dots,\mathbf{m}_\mathbf{n}^{(d)})$ then

$$[\alpha^{n_i}]<\mathbf{m}_\mathbf{n}^{(i)}\leq[\alpha^{n_i+1}].$$

On the other hand $\mathbf{m}_\mathbf{n}^{(i)}\in\mathbb{N}$, hence we get

$$\alpha^{n_i}<\mathbf{m}_\mathbf{n}^{(i)}\leq\alpha^{n_i+1}. \tag{3.9}$$

This inequality implies

$$|\mathbf{m_{n+2}}|-|\mathbf{m_n}|>\prod_{i=1}^d\alpha^{n_i+2}-\prod_{i=1}^d\alpha^{n_i+1}$$

$$
\begin{aligned}
&= (\alpha^d - 1) \prod_{i=1}^{d} \alpha^{n_i+1} \\
&> (\alpha^d - 1)\alpha^{n_1} \quad \forall \mathbf{n} = (n_1, n_2, \ldots, n_d) \geq \mathbf{N}_0.
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha^n = \infty$, therefore $\alpha^n \geq K(\alpha^d - 1)^{-1}$ except for finitely many values of $n \in \mathbb{N}$. Hence there exists $\mathbf{N}_2 \in \mathbb{N}^d$ such that $\mathbf{N}_2 \geq \mathbf{N}_1$ and

$$
|\mathbf{m}_{\mathbf{n}+\mathbf{2}}| - |\mathbf{m}_{\mathbf{n}}| > (\alpha^d - 1)\frac{K}{\alpha^d - 1} = K \quad \forall \mathbf{n} \geq \mathbf{N}_2.
$$

This inequality implies by (3.1), that

$$
B_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}} - B_{\mathbf{m}_{\mathbf{n}}} \leq c(|\mathbf{m}_{\mathbf{n}+\mathbf{2}}| - |\mathbf{m}_{\mathbf{n}}|) \quad \forall \mathbf{n} \geq \mathbf{N}_2. \tag{3.10}
$$

Using (3.9) we have

$$
\frac{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|}{|\mathbf{m}_{\mathbf{n}}|} \leq \prod_{i=1}^{d} \frac{\alpha^{n_i+3}}{\alpha^{n_i}} = \alpha^{3d} \quad \forall \mathbf{n} = (n_1, n_2, \ldots, n_d) \geq \mathbf{N}_2. \tag{3.11}
$$

Hence (3.8), (3.10), (3.11) and (3.3) imply, that if $\mathbf{n} \geq \mathbf{N}_2$ and $\mathbf{t} \in (\mathbf{k}_{\mathbf{n}+\mathbf{1}}, \mathbf{k}_{\mathbf{n}+\mathbf{2}}]$, then

$$
\begin{aligned}
\frac{S_{\mathbf{t}}(\omega) - B_{\mathbf{t}}}{|\mathbf{t}|^{1/r}} &\geq -\varepsilon - \frac{B_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}} - B_{\mathbf{m}_{\mathbf{n}}}}{|\mathbf{m}_{\mathbf{n}}|} \geq -\varepsilon - c\left(\frac{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|}{|\mathbf{m}_{\mathbf{n}}|} - 1\right) \\
&\geq -\varepsilon - c(\alpha^{3d} - 1) \geq -\varepsilon\alpha^{3d/r} - c(\alpha^{3d} - 1) > -\delta.
\end{aligned} \tag{3.12}
$$

If $\mathbf{n} \geq \mathbf{N}_2$ and $\mathbf{t} \in (\mathbf{k}_{\mathbf{n}+\mathbf{1}}, \mathbf{k}_{\mathbf{n}+\mathbf{2}}]$, then by $\mathbf{t} \in (\mathbf{m}_{\mathbf{n}}, \mathbf{m}_{\mathbf{n}+\mathbf{2}}]$, $|\mathbf{m}_{\mathbf{n}}|^{1/r} \geq |\mathbf{m}_{\mathbf{n}}|$, (3.7), (3.11), (3.10) and (3.3), we have

$$
\begin{aligned}
\frac{S_{\mathbf{t}}(\omega) - B_{\mathbf{t}}}{|\mathbf{t}|^{1/r}} &\leq \frac{S_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}}(\omega) - B_{\mathbf{m}_{\mathbf{n}}}}{|\mathbf{m}_{\mathbf{n}}|^{1/r}} \\
&= \frac{S_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}}(\omega) - B_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}}}{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|^{1/r}} \frac{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|^{1/r}}{|\mathbf{m}_{\mathbf{n}}|^{1/r}} + \frac{B_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}} - B_{\mathbf{m}_{\mathbf{n}}}}{|\mathbf{m}_{\mathbf{n}}|^{1/r}} \\
&\leq \frac{S_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}}(\omega) - B_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}}}{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|^{1/r}} \frac{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|^{1/r}}{|\mathbf{m}_{\mathbf{n}}|^{1/r}} + \frac{B_{\mathbf{m}_{\mathbf{n}+\mathbf{2}}} - B_{\mathbf{m}_{\mathbf{n}}}}{|\mathbf{m}_{\mathbf{n}}|} \\
&\leq \varepsilon\alpha^{3d/r} + c\left(\frac{|\mathbf{m}_{\mathbf{n}+\mathbf{2}}|}{|\mathbf{m}_{\mathbf{n}}|} - 1\right) \leq \varepsilon\alpha^{3d/r} + c(\alpha^{3d} - 1) < \delta.
\end{aligned}
$$

This inequality and (3.12) imply

$$
\frac{|S_{\mathbf{t}}(\omega) - B_{\mathbf{t}}|}{|\mathbf{t}|^{1/r}} < \delta \quad \forall \mathbf{n} \geq \mathbf{N}_2, \mathbf{t} \in (\mathbf{k}_{\mathbf{n}+\mathbf{1}}, \mathbf{k}_{\mathbf{n}+\mathbf{2}}]. \tag{3.13}
$$

If $\mathbf{t} \geq \mathbf{k}_{\mathbf{N}_2+\mathbf{1}} + \mathbf{1}$, then there exists $\mathbf{n} \geq \mathbf{N}_2$ such that $\mathbf{t} \in (\mathbf{k}_{\mathbf{n}+\mathbf{1}}, \mathbf{k}_{\mathbf{n}+\mathbf{2}}]$. Hence (3.13) implies

$$
\frac{|S_{\mathbf{t}}(\omega) - B_{\mathbf{t}}|}{|\mathbf{t}|^{1/r}} < \delta \quad \forall \mathbf{t} \geq \mathbf{k}_{\mathbf{N}_2+\mathbf{1}} + \mathbf{1}.
$$

Therefore the statement is proved. □

4. An application for multi-index sequence of non-negative random variables with finite variances

In this section we give an application of Theorem 3.1. In case $d = r = 1$, this result gives Theorem of Petrov [9].

Theorem 4.1. *Let* $X_{\mathbf{n}}$ $(\mathbf{n} \in \mathbb{N}^d)$ *be a sequence of non-negative random variables with finite variances,* $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$, $c > 0$, $K \in \mathbb{N}$ *and* $0 < r \le 1$. *If*

$$\mathrm{E}\, S_{\mathbf{n}} - \mathrm{E}\, S_{\mathbf{m}} \le c(|\mathbf{n}| - |\mathbf{m}|) \quad \forall \mathbf{n}, \mathbf{m} \in \mathbb{N}^d, \mathbf{n} \ge \mathbf{m}, |\mathbf{n}| - |\mathbf{m}| \ge K \tag{4.1}$$

and

$$\sum_{\mathbf{n}} \frac{\operatorname{Var} S_{\mathbf{n}}}{|\mathbf{n}|^{1+2/r}} < \infty, \tag{4.2}$$

then

$$\lim_{\mathbf{n}\to\infty} \frac{S_{\mathbf{n}} - \mathrm{E}\, S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} = 0 \quad \textit{almost surely.}$$

Proof. With notation $b_{\mathbf{k}} = \mathrm{E}\, X_{\mathbf{k}}$ and $B_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} b_{\mathbf{k}} = \mathrm{E}\, S_{\mathbf{n}}$, (4.1) implies (3.1). On the other hand, if $\varepsilon > 0$, then the Chebyshev inequality and (4.2) imply

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathrm{P}\left(|S_{\mathbf{n}} - B_{\mathbf{n}}| > \varepsilon |\mathbf{n}|^{1/r}\right) \le \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \frac{\operatorname{Var} \frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}}}{\varepsilon^2} = \varepsilon^{-2} \sum_{\mathbf{n}} \frac{\operatorname{Var} S_{\mathbf{n}}}{|\mathbf{n}|^{1+2/r}} < \infty.$$

Therefore (3.2) holds. Hence, using Theorem 3.1, we have that the statement is true. □

References

[1] N. Etemadi: *An elementary proof of the strong law of large numbers*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete 55.1 (1981), pp. 119–122, doi: 10.1007/bf01013465.

[2] I. Fazekas: *Convergence rates in the Marcinkiewicz strong law of large numbers for Banach space valued random variables with multidimensional indices*, Publicationes Mathematicae, Debrecen 32 (1985), pp. 203–209.

[3] I. Fazekas, T. Tómács: *Strong laws of large numbers for pairwise independent random variables with multidimensional indices*, Publicationes Mathematicae, Debrecen 53.1-2 (1998), pp. 149–161.

[4] A. Gut: *Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices*, The Annals of Probability 6.3 (1978), pp. 469–482, doi: 10.1214/aop/1176995531.

[5] O. I. Klesov: *Strong law of large numbers for random fields with orthogonal values*, Dokl. Akad. Nauk. Ukr. SSR Ser. A 7 (1982), pp. 9–12.

[6] O. I. Klesov: *The law of large numbers for multiple sums of independent identically distributed random variables*, Theor. Probab. Math. Statist. 50 (1995), pp. 77–87.

[7] V. M. Kruglov: *Strong law of large numbers*, in: Stability Problems for Stochastic Models: Proceedings of the Fifteenth Perm Seminar, Perm, Russia, June 2-6, 1992, Moscow, Utrecht, Tokyo: TVP/VSP, 1994, pp. 139–150, isbn: 90-6764-159-6.

[8] M. Loève: *Probability Theory I.* New York: Springer-Verlag, 1977.

[9] V. V. Petrov: *On the strong law of large numbers for a sequence of nonnegative random variables*, Zapiski Nauchnnykh Seminarov POMI 384 (2010), pp. 182–184, doi: 10.1007/s10958-011-0411-x.