The Lie augmentation terminals of groups*

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Abstract. In this paper we give necessery and sufficient conditions for groups which have finite Lie terminals with respect to commutative ring of characteristic p^s where p is a prime and s is a natural number.

1. Introduction. Let R be a commutative ring with identity, G a group and RG its group ring and let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the ring homomorphism $\phi : RG \to R$ which maps the group elements to 1. It is easy to see that as R-module $A(RG)$ is a free module with the elements $g - 1\,(g \in G)$ as a basis. It is clear that $A(RG)$ is the ideal generated by all elements of the form $g - 1, g \in G$.

The Lie powers $A^{[\lambda]}(RG)$ of $A(RG)$ are defined inductively: $A(RG) = A^{[1]}(RG)$, $A^{[\lambda+1]}(RG) = [A^{[\lambda]}(RG), A(RG)]\cdot RG$, if λ is not a limit ordinal, and $A^{[\lambda]}(RG) = \bigcap_{\nu<\lambda} A^{[\nu]}(RG)$ otherwise, where $[K, M]$ denote the $R-$submodule of RG generated by $[k,m] = km - mk, k \in K, m \in M$, and for $K \subseteq RG, K \cdot RG$ denotes the right ideal generated by K in RG (similary $RG \cdot K$ will denote the left ideal generated by K). It is easy to see that the right ideal $A^{[\lambda]}(RG)$ is a two-sided ideal of RG for all ordinals $\lambda \geq 1$.

Evidently there exists a least ordinal $\tau = \tau_R[G]$ such that $A^{[\tau]}(RG) = A^{[\tau+1]}(RG)$ which is called the *Lie augmentation terminal* (or *Lie terminal* for simple when it is obvious from the context what ring R we are working with) of G with respect to R. If $G = \langle 1 \rangle$ we put $\tau_R[G] = 1$.

In general, the question of the classification of groups in regarding to values of the Lie terminals and also of the computation of these terminals, is far from being simple.

We are primarily concerned with finding all groups whose the Lie terminals with respect to commutative ring of characteristic p^s are finite.

In this paper we give necessery and sufficient conditions for groups which have finite Lie terminals with respect to commutative ring of characteristic p^s where p is a prime and s is a natural number (Theorem 3.1).

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2. Notations and some known facts. If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short when it is obvious from the context what ring R we are working with) denotes the ideal of RG generated by all elements of the form $h-1, (h \in H)$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi} : RG \to RG/H$ induced by the group homomorphism ϕ of G onto G/H. It is clear that $I(RG) = A(RG)$.

Let F be a free group on the free generators $x_i (i \in I)$, say, and ZF be its integral group ring (Z denotes the ring of rational integers). Then every homomorphism $\phi : F \to G$ induces a ring homomorphism $\overline{\phi} : ZF \to RG$ by letting $\overline{\phi}(\sum n_y y) = \sum n_y \phi(y)$, where $y \in F$ and the sum rungs over the finite set of $n_y y \in ZF$. If $f \in ZF$, we denote by $A_f(RG)$ the two-sided ideal of RG generated by the elements $\overline{\phi}(f)$, $\phi \in Hom(F,G)$, the set of homomorphism from F to G. In other words $A_f(RG)$ is the ideal generated by the values of f in RG as the elements of G are substituted for the free generators x_i-s.

An ideal J of RG is called a *polynomial ideal* if $J = A_f(RG)$ for some $f \in ZF$, F a free group.

It is easy to see that the augmentation ideal $A(RG)$ is a polynomial ideal. Really, $A(RG)$ is generated as an $R-$module by the elements $g-1 (g \in G)$, i.e. by the values of the polynomial $x-1$.

From [3] (see also [2], Corollary 1.9, page 6) it follows the

Lemma 2.1. ([2]) *The Lie powers* $A^{[n]}(RG), n \geq 1$, *are polynomial ideals in* RG.

We use also the following

Lemma 2.2. ([2] *Proposition 1.4, page 2) Let* $f \in ZF$. *Then* f *defines a polynomial ideal* $A_f(RG)$ *in every group ring* RG. *Furter, if* $\theta : RG \to KH$ *is a ring homomorphism induced by a group homomorphism* $\phi : G \to H$ *and a ring homomorphism* $\psi : R \to K$, *then*

$$\theta(A_f(RG)) \subseteq A_f(KH).$$

(It is assumed here that $\psi(1_R) = 1_K$, where 1_R and 1_K are identity of the rings R and K respectively.)

Let $\overline{\phi} : RG \to RG/L$ be a natural epimorphism induced by the group homomorphism ϕ of G onto G/L. By Lemma 2.1 $A^{[n]}(RG)(n \geq 1)$ are polynomial ideal and from Lemma 2.2 it follows that

$$\overline{\phi}(A^{[n]}(RG)) = A^{[n]}(RG/L). \tag{1}$$

Consequently

$$A^{[n]}(RG/L) \cong (A^{[n]}(RG) + I(RL))/I(RL) \tag{2}$$

for all $n \geq 1$.

If $\mathcal{K}$ denotes a class of groups (by which we understand that $\mathcal{K}$ contains all groups of order 1 and, with each $H \in \mathcal{K}$, all isomorphic copies of H) we define the class $\mathbf{R}\mathcal{K}$ of residually-$\mathcal{K}$ groups by letting $G \in \mathbf{R}\mathcal{K}$ if and only if: whenever $1 \neq g \in G$, there exists a normal subgroup H_g of the group G such that $G/H_g \in \mathcal{K}$ and $g \notin H_g$.

We use the following notations for standard group classes: $\mathcal{D}$: *nilpotent groups whose derived groups are torsion-free nilpotent groups* and $\mathcal{D}_p$: *nilpotent groups whose derived groups are p-groups of finite exponent.*

Let p be a prime and n a natural number. Then we shall denote by G^{p^n} the subgroup generated by all elements of the form $g^{p^n}, g \in G$.

If K, L are two subgroups of G, then we shall denote by (K, L) the subgroup generated by all commutators $(g,h) = g^{-1}h^{-1}gh$, $g \in K$, $h \in L$.

The nth term of the lower central series of G is defined inductively: $\gamma_1(G) = G, \gamma_2(G) = G'$ is the commutator subgroup (G,G) of G, and $\gamma_n(G) = (\gamma_{n-1}(G), G)$.

In this paper we shall use also the following theorems:

Theorem 2.1. ([1]) *Let G be a non-Abelian group, R a commutative ring with identity. Then $A^{[n]}(RG) = 0$ for some $n \geq 2$ if and only if G is nilpotent, G' is a finite p-group and p is nilpotent in R.*

The ideal $J_p(R)$ of a ring R is defined by $J_p(R) = \bigcap_{n=1}^{\infty} p^n R$.

Theorem 2.2. ([2], Theorem 2.13, page 85) *Let G be a residually $\mathcal{D}_p$-group and $J_p(R) = 0$, then $A^{[\omega]}(RG) = 0$.*

We shall use the following lemma, which gives some elementary properties of the Lie powers $A^{[n]}(RG)$ of $A(RG)$.

Lemma 2.3. ([2], Proposition 1.7, page 4) *For an arbitrary natural numbers n and m are true:*

1) $I(\gamma_n(G)) \subseteq A^{[n]}(RG)$

2) $[A^{[n]}(RG), A^{[m]}(RG)] \subseteq A^{[n+m]}(RG)$

3) $A^{[n]}(RG) \cdot A^{[m]}(RG) \subseteq A^{[n+m-1]}(RG)$.

3. The Lie augmentation terminals. *Throughout this section R will denote a commutative ring with identity of characteristic p^s.*

The normal subgroups $G_{p,k}$ is defined by

$$G_{p,k} = \bigcap_{n=1}^{\infty} (G')^{p^n} \gamma_k(G),$$

where $\gamma_k(G)$ is the kth term of the lower central series of G and G' is the commutator subgroup of G. It is clear, that the factor-group $G/G_{p,k}$ is a residually-$\mathcal{D}_p$ group for every k. We have the following sequence

$$G = G_{p,1} \supseteq G_{p,2} \supseteq \ldots \supseteq G_p \tag{3}$$

of normal subgroups $G_{p,k}$ of a group G, where $G_p = \bigcap_{k=1}^{\infty} G_{p,k}$.

Lemma 3.1. *Let R be a commutative ring of characteristic p^s. Then $I(G_{p,k}) \subseteq A^{[k]}(RG)$ for all $k \geq 1$.*

Proof. Let the element $h - 1$ be in $I(G_{p,k})$. It will be sufficient to show that $h - 1 \in A^{[k]}(RG)$. For an arbitrary n written the element h as $h = h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k$ $(h_i \in G', y_k \in \gamma_k(G))$ and using the identity

$$ab - 1 = (a-1)(b-1) + (a-1) + (b-1) \tag{4}$$

we have that

$$h - 1 = (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k - 1)(y_k - 1) + (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) + (y_k - 1).$$

By Lemma 2.3, $I(\gamma_k(G)) \subseteq A^{[k]}(RG)$ and hence $y_k - 1 \in A^{[k]}(RG)$. Therefore

$$h - 1 \equiv (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) \pmod{A^{[k]}(RG)}.$$

Applying (4) repeatedly to $(h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)$ from the previous expression it follows that

$$h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv \sum_{i=1}^{m} \sum_{j=1}^{p^n} \binom{p^n}{j} (h_i - 1)^j b_i \pmod{A^{[k]}(RG)},$$

where $b_i \in RG$. From Lemma 2.3 (cases 1 and 3) we obtain, that the element $(h_i - 1)^j$ lie in $A^{[j+1]}(RG)$ for every i and j. If $n \geq s + k$, then p^s divides $\binom{p^n}{j}$ for $j = 1, 2, \ldots, k-1$. Therefore

$$h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv p^s \sum_{i=1}^{m} \sum_{j=1}^{k-1} d_j (h_i - 1)^j b_i \equiv p^s F_k(h) \pmod{A^{[k]}(RG)},$$

where $F_k(h) = \sum_{i=1}^{m}\sum_{j=1}^{k-1} d_j(h_i-1)^j b_i$ and $p^s d_j = \binom{p^n}{j}$. Since p^s is zero in R we have that $h-1 \in A^{[k]}(RG)$ which implies the inclusion $I(G_{p,k}) \subseteq A^{[k]}(RG))$ and completes the proof of the lemma.

Lemma 3.2. *Let R be a commutative ring of characteristic p^s. Then*

$$A^{[\omega]}(RG) = I(G_p).$$

Proof. From (3) and from Lemma 3.1 the inclusion $I(G_p) \subseteq A^{[\omega]}(RG)$ follows. We can readily verify that G/G_p is the residually-$\mathcal{D}_p$ group and by Theorem 2.2

$$A^{[\omega]}(RG/G_p) = 0. \tag{5}$$

By (1) $\overline{\phi}(A^{[n]}(RG)) = A^{[n]}(RG/G_p)$ for all $n \geq 1$, where $\overline{\phi} : RG \to RG/G_p$ the natural epimorphism induced by the group homomorphism ϕ of G onto G/G_p. Consequently $\overline{\phi}(A^{[\omega]}(RG)) \subseteq A^{[n]}(RG/G_p)$ for all n and therefore $\overline{\phi}(A^{[\omega]}(RG)) \subseteq A^{[\omega]}(RG/G_p)$. Then from the isomorphism $RG/G_p \cong RG/I(G_p)$ and from (5) we conclude that $A^{[\omega]}(RG) \subseteq I(G_p)$. Therefore $A^{[\omega]}(RG) = I(G_p)$. This completes the proof of the lemma.

If G is a nilpotent group with a finite p-group as the commutator subgroup and R a commutative ring of characteristic p^s then the ideal $A(RG)$ is Lie nilpotent (see Theorem 2.1). Denote $\tau^\circ[A(RG)]$ the *Lie nilpotency index* of $A(RG)$ i.e. the natural number n for which $A^{[n-1]}(RG) \neq A^{[n]}(RG) = 0$ holds. If $G = \langle 1 \rangle$ we put $\tau^\circ[A(RG)] = 1$.

Let $\tau_p[G]$ denote the smallest natural number k (if it exists) such that $G_{p,k-1} \neq G_{p,k} = \cdots = G_p$.

Theorem 3.1. *Let R be a commutative ring of characteristic p^s. Then:*

1) $\tau_R[G] = 1$ *if and only if* $G = G_p$,
2) $\tau_R[G] = 2$ *if and only if* $G \neq G' = G_p$,
3) $\tau_R[G] > 2$ *if and only if* G/G_p *is a nilpotent group whose derived group is a finite p-group.*

Proof. The statement 1) follows from Lemma 3.2.

2) Let $\tau_R[G] = 2$, i.e.

$$A(RG) \neq A^{[2]}(RG) = A^{[3]}(RG) = \cdots = A^{[\omega]}(RG).$$

By statement 1) of our theorem $G_p \neq G$ and consequently $G \neq G'$. Because G/G' is an Abelian group, $A^{[2]}(RG/G') = 0$. From the isomorphism

$$A^{[2]}(RG/G') \cong (A^{[2]}(RG) + I(G'))/I(G'),$$

which follows from (2), we conclude that $A^{[2]}(RG) \subseteq I(G')$. By Lemma 2.3 we obtain the inclusion $A^{[2]}(RG) \supseteq I(G')$. Consequently $A^{[2]}(RG) = I(G')$. Since $\tau_R[G] = 2, A^{[2]}(RG) = A^{[\omega]}(RG)$. Then from Lemma 3.2 we have the equality $A^{[2]}(RG) = I(G_p)$. Therefore $I(G_p) = I(G')$ and $G_p = G'$.

Conversely. If $G \neq G' = G_p$, then $A^{[2]}(RG/G_p) = 0$ because G/G_p is an Abelian group. From this equality it follows that $A^{[2]}(RG) \subseteq I(G_p)$ and by Lemma 3.2 $A^{[2]}(RG) = A^{[\omega]}(RG)$. Since $G \neq G_p$, $A(RG) \neq A^{[2]}(RG)$. Consequently $\tau_R[G] = 2$ which prove 2) of our theorem.

3) Suppose that $\tau_R[G] = n > 2$. From the statements 1) and 2) it follows that $G \neq G_p$ and $G' \neq G_p$. It is very simple to see that $G/G_{p,i}$ are residually$-\mathcal{D}_p$ groups and consequently, by Theorem 2.2,

$$A^{[\omega]}(RG/G_{p,i}) = 0$$

for all $i \geq 1$. Because $\tau_R[G]$ is finite then

$$\cdots \supset A^{[n-1]}(RG) \supset A^{[n]}(RG) = A^{[n+1]}(RG) = \cdots = A^{[\omega]}(RG)$$

and hence

$$\begin{aligned} \cdots \supseteq A^{[n-1]}(RG/G_{p,i}) \supseteq A^{[n]}(RG/G_{p,i}) = \\ = A^{[n+1]}(RG/G_{p,i}) = \cdots = A^{[\omega]}(RG/G_{p,i}). \end{aligned}$$

It follows that $\tau_R[G/G_{p,i}]$ are finite and not greater than $\tau_R[G]$ for all $i \geq 1$. Then there exists a natural number $k \leq n$ such that

$$A^{[k]}(RG/G_{p,i}) = 0 \tag{6}$$

for all i. Then from (2) we have that $A^{[k]}(RG) \subseteq I(G_{p,i})$ for all i. If $i = k$, by Lemma 3.1 we obtain that $A^{[k]}(RG) = I(G_{p,k})$. Hence $I(G_{p,k}) \subseteq I(G_{p,i}))$ and therefore, $G_{p,i} \supseteq G_{p,k}$ for all $i \geq 1$. This implies that

$$\cdots \supseteq G_{p,k} = G_{p,k+1} = \cdots = G_p \tag{7}$$

and by (6) we have that

$$A^{[k]}(RG/G_p) = 0. \tag{8}$$

By Theorem 2.1 it follows that G/G_p is a nilpotent group whose commutator subgroup is a finite p-group.

We remind that in the proof of this part we obtained the following inequalitions: from (7) we have that

$$\tau_R[G] \geq \tau_p[G] \tag{9}$$

and from (8) we obtain that

$$\tau_R[G] \geq \tau^{\circ}[A(RG/G_p]. \tag{10}$$

Conversely, let G/G_p is a nilpotent group whose derived group is a finite p-group. Then by Theorem 2.1

$$A^{[k]}(RG/G_p) = 0$$

for the Lie nilpotency index $\tau^{\circ}[A(RG)] = k$ of $A(RG/G_p)$. It follows that $A^{[k]}(RG) \subseteq I(G_p)$. Hence, by Lemma 3.2, we obtain that $A^{[k]}(RG) \subseteq A^{[\omega]}(RG)$. The inverse inclusion, of course, is trivial. Therefore $A^{[k]}(RG) = A^{[\omega]}(RG)$. Consequently, $\tau_R[G]$ is finite and

$$\tau_R[G] \leq \tau^{\circ}[A(RG/G_p)]. \tag{11}$$

The proof of the theorem is complete.

Theorem 3.2. *Let R be a commutative ring of characteristic p^s and the Lie augmentation terminal of G is finite. Then*

$$\tau_R[G] = \tau^{\circ}[A(RG/G_p)] \geq \tau_p[G].$$

The proof of this theorem follows from statements 1) and 2) of Theorem 3.1 and from (9), (10) and (11).

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