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Robust estimation in time series with long and short memory properties

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

This paper reviews recent developments of robust estimation in linear time series models, with short and long memory correlation structures, in the presence of additive outliers. Based on the manuscripts Fajardo, Reisen & Cribari-Neto 2009 [7] and Lévy-Leduc, Boistard, Moulines, Taqqu & Reisen 2011 [19], the emphasis in this paper is given in the following directions; the influence of additive outliers in the estimation of a time series, the asymptotic properties of a robust autocovariance function and a robust semiparametric estimation method of the fractional parameter d in ARFIMA(p, d, q) models. Some simulations are used to support the use of the robust method when a time series has additive outliers. The invariance property of the estimators for the first difference in ARFIMA model with outliers is also discussed. In general, the robust long-memory estimator leads to be outlier resistent and is invariant to first differencing.

Keywords: Additive outliers, ARFIMA model, long-memory, robustness.

1. Introduction

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary time series with spectral density that behaves like

$$f_X(\omega) \sim h(\omega) \mid \omega \mid^{-2d}, \text{ as } \omega \to 0 \tag{1.1}$$

where the spectral density $h(\omega)$ is a nonvanishing and continuously differentiable function with bounded derivative for $-\pi \leq \omega \leq \pi$, and $d < 0.5$.

A well-known stationary parametric model with the above spectral density is the ARFIMA(p, d, q) process, which is the solution of the equation

$$X_t - \mu = (1 - B)^{-d}\eta_t, \quad t \in \mathbb{Z}, \tag{1.2}$$

where $\eta_t = \frac{\Theta(B)}{\Phi(B)}\epsilon_t$ is an ARMA(p,q) process, μ is the mean (here it is assumed that $\mu = 0$), $\Phi(B) \equiv 1 - \sum_{j=1}^{p} \phi_j B^j$, $\Theta(B) \equiv 1 - \sum_{i=1}^{q} \theta_i B^i$ and p and q are positive integers (Hosking 1981 [11]). $\Phi(z)$ and $\Theta(z)$, with a scalar z, are the autoregressive and moving average polynomials with all roots outside the unit circle and share no common factors. d is the parameter that holds the memory of the process, that is, when $d \in (-0.5, 0.5)$ the ARFIMA(p,d,q) process is said to be invertible and stationary. Besides, for $d \neq 0$, its autocovariance decays at a hyperbolic rate $(\gamma(j) = O(j^{-1+2d}))$. For $d = 0$, $d \in (-0.5, 0)$ or $d \in (0, 0.5)$, the process is said to be short-memory, intermediate-memory or long-memory, respectively. The long-memory property is related to the behavior of the autocovariances, which are not absolutely summable and the spectral density becomes unbounded at zero frequency. In the intermediate-memory region, the autocovariances are absolutely summable and, consequently, the spectral density is bounded.

The spectral density function of $\{X_t\}_{t\in\mathbb{Z}}$ is given by

$$f_X(\omega) = f_\eta(\omega)\left[2\sin\left(\frac{\omega}{2}\right)\right]^{-2d}, \ \omega \in [-\pi, \pi]. \tag{1.3}$$

$f_X(\omega)$ is continuous except for $\omega = 0$ where it has a pole when $d > 0$. A recent review of the ARFIMA model and its properties can be found in Palma 2007 [23] and Doukhan, Oppenheim & Taqqu 2003 [6].

Many estimators for the fractional parameter d in long-memory time series have already been proposed in the literature. Among them are the semiparametric procedures, a group which includes a wide variety of estimators based on the Ordinary Least Square (OLS) method. These procedures require the use of the spectral density parameterized within a neighborhood of zero frequency. Some references on this subject include the works of Geweke & Porter-Hudak 1983 [10], Reisen 1994 [26] and Robinson 1995 [27], among others. An overview of long-range dependence processes can be found in Beran 1994 [1] and Doukhan et al. 2003 [6].

Time series with outliers or atypical observations is quite common in any area of application. In the case where the data is time-dependent, several authors such as Ledolter 1989 [17], Chang, Tiao & Chen 1988 [4] and Chen & Liu 1993 [5] have studied the effect of outliers in a time series that follows ARIMA models. In general, they have concluded that the parameter estimates of ARMA models become more biased when the data contains outliers. Similar conclusion is also observed when estimating the fractional parameter in ARFIMA models. The outliers cause a substantial bias in the differencing parameter (Fajardo et al. 2009 [7]).

An autocovariance robust function was proposed by Ma & Genton 2000 [20]. The asymptotical properties of this function are studied by Lévy-Leduc et al. 2011 [19]. The results presented in Fajardo et al. 2009 [7], Lévy-Leduc et al. 2011 [19] and Lévy-Leduc, Boistard, Moulines, Taqqu & Reisen 2011 [18] are the motivations of this paper. The impact of outliers in the estimation of ARFIMA models under different context is here studied. The asymptotical properties of a robust autocovariance function is discussed and some empirical examples are used to illustrate the usefulness of a robust fractional parameter estimator. The invariance property of the estimator to the first difference is also empirically studied. The outline of this papers is as follows: Section 2 discusses the model and the impact of the outliers in time series. Section 3 summarizes the main results related to the robust auto-covariance estimator given in Lévy-Leduc et al. 2011 [19] and discusses the robust estimation of the fractional parameter in the ARFIMA model. Section 4 presents some empirical studies and an application is discussed in Section 5. Concluding remarks and future directions are given in Section 6.

2. The impact of outliers in time series

Suppose $x_1, ..., x_n$ is a partial realization of $\{X_t\}_{t\in\mathbb{Z}}$. Hence, the periodogram function is defined as $I_x(\omega) = (2\pi n)^{-1}|\sum_{t=1}^{n} x_t e^{i\omega t}|^2$. It follows that, when $d = 0$ in the ARFIMA model,

$$I_x(\omega) = 2\pi f_X(\omega)\frac{I_\epsilon(\omega)}{\sigma_\epsilon^2} + H(\omega) \tag{2.1}$$

where $\mathbb{E}[|H(\omega)|^2] = O(\frac{1}{n^{2\xi}})$ ($\xi > 0$) is uniformly in $\omega \in [-\pi, \pi]$ (Theorem 6.2.2 in Priestley 1981 [25]) and $I_\epsilon(\cdot)$ is the periodogram of the residuals. From (1.2) and Theorem 6.1.1 in Priestley 1981 [25], asymptotic sample properties of $\frac{I_x(\omega)}{f_X(\omega)}$ are derived and they are summarized as follows. If $\{\epsilon_t\}_{t\in\mathbb{Z}}$ are normally distributed, for a fixed set of values of the Fourier frequencies $\omega_j = \frac{2\pi j}{n}$, $j = 1, ..., [n/2]$, where $[\cdot]$ means the integer part, asymptotically the set of variables $\frac{I_x(\omega_j)}{f_X(\omega_j)}$ is independently distributed, each distributed as $\frac{\chi_2^2}{2}$. At $\omega = 0$ and π, the distributions are χ_1^2 (for details see Priestley 1981 [25]). These asymptotic results for the periodogram lead to $\mathbb{E}\left[\frac{I_x(\omega_j)}{f_X(\omega_j)}\right] \to 1$ and $\mathrm{var}\left[\frac{I_x(\omega_j)}{f_X(\omega_j)}\right] \to (1 + \delta(\omega_j))$ as $n \to \infty$, where

$$\delta(\omega_j) = 1 \text{ if } \omega_j = 0, \pi \text{ and } 0 \text{ otherwise.} \tag{2.2}$$

The above results establish the unbiasedness and inconsistency properties of $I_x(\omega_j)$.

Due to the singularity of $f_X(\omega)$ when $d > 0$, the standard results of the asymptotic distribution of the periodogram discussed previously can not be applied to $I_x(\omega_j)$ for small and fixed j. Hurvich & Beltrão 1993 [13] showed that $\lim_{n\to\infty} \mathbb{E}\left[\frac{I_x(\omega_j)}{f_X(\omega_j)}\right]$ depends on j and d, and exceeds unity for most $d \neq 0$ (Künsch 1986 [16]; Robinson 1995 [28]). For $j \neq k$, $\frac{I_x(\omega_j)}{f_X(\omega_j)}$ and $\frac{I_x(\omega_k)}{f_X(\omega_k)}$ are correlated, and for a fixed value j and Gaussian processes, the limiting distribution of $\frac{I_x(\omega_j)}{f_X(\omega_j)}$ is not exponential (Robinson 1995 [28]). That is, under the Gaussian assumption, Hurvich & Beltrão 1993 [13] show that the normalized periodogram $\frac{I(\omega)}{f_X(\omega)}$ is asymptotically distributed as the quadratic form

$$\frac{\alpha_1}{2}\chi_1 + \frac{\alpha_2}{2}\chi_2$$

where χ_1 and χ_2 are variables with Chi-squared distribution with one degree of freedom, $\alpha_1 = L_j(d) - 2L_j^*(d)$, $\alpha_2 = L_j(d) + 2L_j^*(d)$,

$$L_j(d) = \lim_{n\to\infty} \mathbb{E}\left\{\frac{I_x(\omega_j)}{f_X(\omega_j)}\right\} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(\omega/2)}{(2\pi j - \omega)^2}\left|\frac{\omega}{2\pi j}\right|^{-2d} d\omega$$

and

$$L_j^*(d) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(\omega/2)}{(2\pi j - \omega)(2\pi j + \omega)}\left|\frac{\omega}{2\pi j}\right|^{-2d} d\omega.$$

Let $\{Z_t\}_{t\in\mathbb{Z}}$ be a process contaminated by additive outliers, which is described by

$$Z_t = X_t + \sum_{j=1}^{m} \varpi_j Y_{j,t}, \tag{2.3}$$

where m is the maximum number of outliers; the unknown parameter ω_j represents the magnitude of the jth outlier, and $Y_{j,t}$ ($\equiv Y_j$) is a random variable ($r.v.$) with probability distribution $\Pr(Y_j = -1) = \Pr(Y_j = 1) = \frac{p_j}{2}$ and $\Pr(Y_j = 0) = 1 - p_j$, where $\mathbb{E}[Y_j] = 0$ and $\mathbb{E}[Y_j^2] = \mathrm{var}(Y_j) = p_j$. Model 2.3 is based on the parametric models proposed by Fox 1972 [8]. Y_j is the product of *Bernoulli*(p_j) and *Rademacher* random variables; the latter equals 1 or -1, both with probability $\frac{1}{2}$. X_t and Y_j are independent random variables.

Some results related to the effects of outliers on the spectral density and on the autocorrelation functions of $\{Z_t\}_{t\in\mathbb{Z}}$ are presented as follows.

Proposition 2.1. *Suppose that $\{Z_t\}_{t\in\mathbb{Z}}$ follows Model 2.3.*

i. The autocovariance function (ACOVF) of $\{Z_t\}_{t\in\mathbb{Z}}$ is given by

$$\gamma_z(h) = \begin{cases} \gamma_X(0) + \sum\limits_{j=1}^{m} \varpi_j^2 p_j, & \text{if } h = 0, \\ \gamma_X(h), & \text{if } h \neq 0, \end{cases}$$

where $\gamma_X(h) = \mathbb{E}[X_t X_{t+h}] - \mathbb{E}[X_t]\mathbb{E}[X_{t+h}]$ with $h \in \mathbb{Z}$.

ii. The spectral density function of $\{Z_t\}$ is given by

$$f_Z(\omega) = f_X(\omega) + \frac{1}{2\pi}\sum_{j=1}^{m} \varpi_j^2 p_j, \quad \omega \in (-\pi, \pi],$$

where $f_X(\omega) = \frac{1}{2\pi}\sum\limits_{h=-\infty}^{\infty} \gamma_X(h) e^{-ih\omega}$.

Proposition 2.1 states that $\gamma_z(h)$, for $h = 0$, depends on $\mathrm{var}(Y_j)$. $\gamma_Z(0)$ increases with $\mathrm{var}(Y_j)$ (see the proof in Fajardo et al. 2009 [7]). This relation between $R_Z(0)$

and $\mathrm{var}(Y_j)$ will certainly affect the model parameter estimates because it reduces the magnitude of the autocorrelations and introduces loss of information on the pattern of serial correlation (see also Chan 1992, 1995 [2, 3]). The spectral form of $\{Z_t\}_{t\in\mathbb{Z}}$ (Model 2.3) when $\{X_t\}_{t\in\mathbb{Z}}$ follows an ARFIMA(p,d,q) model is given in the next lemma.

Lemma 2.2. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary and invertible ARFIMA(p,d,q) process. Also, let $\{Z_t\}_{t\in\mathbb{Z}}$ be such that $Z_t = X_t + \sum_{j=1}^{m} \varpi_j Y_j$, where m is the maximum number of outliers, the unknown parameter ϖ_j is the magnitude of the jth outlier and Y_j is a r.v. with probability distribution $\Pr(Y_j=-1) = \Pr(Y_j = 1) = \frac{p_j}{2}$ and $\Pr(Y_j=0) = 1-p_j$. The spectral density of $\{Z_t\}_{t\in\mathbb{Z}}$ is*

$$f_Z(\omega) = \frac{\sigma_\epsilon^2}{2\pi}\frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2}\left\{2\sin\left(\frac{\omega}{2}\right)\right\}^{-2d} + \frac{1}{2\pi}\sum_{j=1}^{m}\varpi_j^2 p_j.$$

The proof of Lemma 2.2 follows directly from Proposition 2.1.

The effects of an outlier on the sample autocovariance function and on the periodogram are given below.

Proposition 2.3. *Let $z_1, z_2, \ldots, z_n$ be generated from Model 2.3 with one outlier, and let the outlier occur at time $t = T$ with $h < T < n-h$. It follows that:*

i. The sample ACOVF is given by

$$\widehat{\gamma}_z(h) = \widehat{\gamma}_x(h) + \frac{\varpi}{n}(x_{T-h} + x_{T+h} - 2\bar{x}) + \frac{\omega^2}{n}\delta'(h) + o_p(n^{-1}), \tag{2.4}$$

where $\widehat{\gamma}_x(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_t-\bar{x})(x_{t+h}-\bar{x})$ *and* $\delta'(h) = \begin{cases} 1, & \text{when } h = 0,\\ 0, & \text{otherwise.}\end{cases}$

ii. The periodogram is given by

$$I_z(\omega) = I_x(\omega) + \Delta(\varpi), \quad \omega \in (-\pi, \pi],$$

where $I_x(\omega) = \frac{1}{2\pi}\sum_{h=-(n-1)}^{n-1}\widehat{\gamma}_x(h)e^{-ih\omega}$, *and*

$$\Delta(\varpi) = \frac{\varpi^2}{2\pi n} \pm \frac{\varpi}{\pi n}\left\{(x_T - \bar{x}) + \sum_{h=1}^{n-1}(x_{T-h} + x_{T+h} - 2\bar{x})\cos(h\omega)\right\} + o_p(n^{-1}).$$

These results show that outliers may substantially affect the inference performed on stationary models by revealing that there is information loss in the serial correlation dynamics of the process, which is translated into the parameter estimation process.

3. The autocovariance and spectral density robust functions

3.1. The autovariance function

Ma & Genton 2000 [20] proposed a scale covariance estimator which is based on $Q_n(\cdot)$, defined in the sequel, and on the following covariance identity

$$\operatorname{cov}(X,Y) = \frac{1}{4ab}[\operatorname{var}(aX+bY) - \operatorname{var}(aX-bY)], \tag{3.1}$$

where X and Y are random variables, $a = \frac{1}{\sqrt{\operatorname{var}(X)}}$ and $b = \frac{1}{\sqrt{\operatorname{var}(Y)}}$ (Huber 2004 [12]).

Rousseeuw & Croux 1993 [29] proposed a robust scale estimator function $Q_n(\cdot)$ which is based on the τth order statistic of $\binom{n}{2}$ distances $\{|\eta_j - \eta_k|, j < k\}$, and can be written as

$$Q_n(\eta) = c \times \{|\eta_j - \eta_k|; j < k\}_{(\tau)}, \tag{3.2}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)'$, c is a constant used to guarantee consistency ($c = 2.2191$ for the normal distribution) and $\tau = \left\lfloor \frac{\binom{n}{2}+2}{4} \right\rfloor + 1$.

Based on identity (3.1) and on $Q_n(\cdot)$, Ma & Genton 2000 [20] proposed a highly robust estimator for the ACOVF:

$$\widehat{\gamma}_Q(h) = \frac{1}{4}\left[Q^2_{n-h}(\mathbf{u}+\mathbf{v}) - Q^2_{n-h}(\mathbf{u}-\mathbf{v})\right], \tag{3.3}$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors containing the initial $n-h$ and the final $n-h$ observations, respectively. The robust estimator for the autocorrelation function (ACF) is

$$\widehat{\rho}_Q(h) = \frac{Q^2_{n-h}(\mathbf{u}+\mathbf{v}) - Q^2_{n-h}(\mathbf{u}-\mathbf{v})}{Q^2_{n-h}(\mathbf{u}+\mathbf{v}) + Q^2_{n-h}(\mathbf{u}-\mathbf{v})}.$$

It can be shown that $|\widehat{\rho}_Q(h)| \leq 1$ for all h.

Influence Function and Breakdown Point

Influence Function (IF) is an important tool to understand the effect of the contamination of an outlier in any estimator. To define IF supposes that the empirical c.d.f. F_n of $x_1, ..x_n$, adequately normalized, converges. Following Huber 2004 [12], the influence function $x \to IF(x,T,F)$ is defined for a functional T at a distribution F and at point x as the limit

$$IF(x,T,F) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}\{T(F + \varepsilon(\delta_x - F)) - T(F)\},$$

where δ_x is the Dirac distribution at x.

Breakdown Point (BP) indicates the largest proportion of outliers that the data may contain such that the estimator still gives some information about the distribution of the outlier-free data (Maronna, Martin & Yohai 2006 [21]). Rousseeuw & Croux 1993 [29] showed that the asymptotic BP of $Q_n(\cdot)$ is 50%, which means that the data can be contaminated by up to half of the observations with outliers and $Q_n(\cdot)$ will still yield sensible estimates.
The classical notion of sample BP of a scale estimator $S_n(\cdot)$ is given in Definition 3.1.

Definition 3.1. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_n)'$ be a sample of size n. Let $\widetilde{\eta}$ be obtained by replacing any m observations of η by arbitrary values. The sample breakdown point of a scale estimator $S_n(\eta)$ is given by

$$\varepsilon_n^*(S_n(\eta)) = \max\left\{\frac{m}{n} : \sup_{\widetilde{\eta}} S_n(\widetilde{\eta}) < \infty \text{ and } \inf_{\widetilde{\eta}} S_n(\widetilde{\eta}) > 0\right\}.$$

The above BP definition holds for a scale estimator function of a time invariant random sample. As noted by Ma & Genton 2000 [20], in time series, the estimators are based on differences between observations apart by various time lag distances and usually have a BP with respect to these differences. Then, the time location of the outlier becomes important (see also, for example, Ledolter 1989 [17]). Therefore, the authors introduced the following definition of a temporal sample breakdown point of an autocovariance estimator $\hat{\gamma}_\eta(h)$ based on (3.1).

Definition 3.2. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_n)'$ be a sample of size n and let $\widetilde{\eta}$ be obtained by replacing any m observations of η by arbitrary values. Denote by $\mathbb{I}_m$ a subset of size m of $\{1, 2, \ldots, n\}$. The temporal sample breakdown point of an autocovariance estimator $\hat{\gamma}_\eta(h)$ is given by

$$\varepsilon_n^{temp}(\widehat{\gamma}_\eta(h)) = \max\left\{\frac{m}{n} : \sup_{\mathbb{I}_m}\sup_{\widetilde{\eta}} S_{n-h}(\widetilde{\mathbf{u}} + \widetilde{\mathbf{v}}) < \infty, \inf_{\mathbb{I}_m}\inf_{\widetilde{\eta}} S_{n-h}(\widetilde{\mathbf{u}} + \widetilde{\mathbf{v}}) > 0,\right.$$
$$\left.\sup_{\mathbb{I}_m}\sup_{\widetilde{\eta}} S_{n-h}(\widetilde{\mathbf{u}} - \widetilde{\mathbf{v}}) < \infty \text{ and } \inf_{\mathbb{I}_m}\inf_{\widetilde{\eta}} S_{n-h}(\widetilde{\mathbf{u}} - \widetilde{\mathbf{v}}) > 0\right\},$$

where $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{v}}$ are derived from $\widetilde{\eta}$ as in (3.3).

Remark 3.3. The relation between the classical sample and the temporal sample breakdown points can be expressed by the following inequality (Ma & Genton 2000 [20]):

$$\frac{n-h}{2n}\varepsilon_n^*(\widehat{\gamma}_\eta(h)) \leq \varepsilon_n^{temp}(\widehat{\gamma}_\eta(h)) \leq \frac{1}{2}\varepsilon_n^*(\widehat{\gamma}_\eta(h)).$$

It then follows that since the sample breakdown point of the classical autocovariance estimator is zero, the temporal breakdown point of this estimator is also zero. This means that only one single outlier is enough to 'break' the estimator.

Ma & Genton 2000 [20] showed that the maximum temporal breakdown point of the highly robust autocovariance estimator is 25%, which is the highest possible breakdown point for an autocovariance estimator.

Results of the asymptotic properties of the robust aucovariance function for a Gaussian ARFIMA model are summarized as follows (see Lévy-Leduc et al. 2011 [19]).

Short-memory case

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary mean-zero Gaussian process given by Model 1.2 with $d = 0$, that is, the autocovariance function ($\gamma(h) = E(X_1 X_{h+1})$) of $\{X_t\}_{t\in\mathbb{Z}}$ satisfies

$$\sum_{h\geq 1} |\gamma(h)| < \infty.$$

The following theorems present the asymptotic behavior of the robust autocovariance estimator.

Theorem 3.4. *Let h be a non-negative integer. Under the assumption that the autocovariances are absolutely summable, the autocovariance estimator $\widehat{\gamma}_Q(h, X_{1:n}, \Phi)$ satisfies the following Central Limit Theorem:*

$$\sqrt{n}\left(\widehat{\gamma}_Q(h, X_{1:n}, \Phi) - \gamma(h)\right) \xrightarrow{d} \mathcal{N}(0, \check{\sigma}_h^2),$$

where

$$\check{\sigma}^2(h) = E[\psi^2(X_1, X_{1+h})] + 2\sum_{k\geq 1} E[\psi(X_1, X_{1+h})\psi(X_{k+1}, X_{k+1+h})] \tag{3.4}$$

where ψ is a function of $\gamma(h)$ and of IF (see, Theorem 4 in Lévy-Leduc et al. 2011 [19]).

Long-memory case

Now, let $d \neq 0$ in Model 1.2 and let $D = 1 - 2d$. The ACF behaves like

$$\gamma(h) = h^{-D} L(h),\ 0 < D < 1\ ,$$

where L is slowly varying at infinity and is positive for large h. Note that, for positive d, as previously stated, the ACF of the process is not absolutely summable.

Theorem 3.5. *Let h be a non negative integer. Then, $\widehat{\gamma}_Q(h, X_{1:n}, \Phi)$ satisfies the following limit theorems as n tends to infinity.*

- *If $D > 1/2$,*

$$\sqrt{n}\left(\widehat{\gamma}_Q(h, X_{1:n}, \Phi) - \gamma(h)\right) \xrightarrow{d} \mathcal{N}(0, \check{\sigma}^2(h))\ ,$$

where

$$\breve{\sigma}^2(h) = \mathbb{E}[\psi^2(X_1, X_{1+h})] + 2\sum_{k\geq 1}\mathbb{E}[\psi(X_1, X_{1+h})\psi(X_{k+1}, X_{k+1+h})] \, ,$$

where ψ *is a function of* $\gamma(h)$ *and of IF (see, Theorems 4 and 5 in Lévy-Leduc et al. 2011 [19]).*

- *If* $D < 1/2$,

$$\beta(D)\frac{n^D}{\widetilde{L}(n)}\left(\widehat{\gamma}_Q(h, X_{1:n}, \Phi) - \gamma(h)\right) \xrightarrow{d} \frac{\gamma(0)+\gamma(h)}{2}(Z_{2,D}(1) - Z^2_{1,D}(1))$$

where $\beta(D) = \mathrm{B}((1-D)/2, D)$, B *denotes the Beta function, the processes* $Z_{1,D}(\cdot)$ *and* $Z_{2,D}(\cdot)$ *are defined by Equations 53 and 54, respectively, in Lévy-Leduc et al. 2011 [19], and*

$$\widetilde{L}(n) = 2L(n) + L(n+h)(1+h/n)^{-D} + L(n-h)(1-h/n)^{-D} \, . \qquad (3.5)$$

Remark 3.6. For Model 1.2 with $1/4 < d < 1/2$, the robust autocovariance estimator $\widehat{\gamma}_Q(h, X_{1:n}, \Phi)$ has the same asymptotic behavior as the classical autocovariance estimator $\widehat{\gamma}_x(h)$.

Theories related to the use of the robust ACF function to obtain an spectral estimate are still opened questions. However, this was first empirically investigated by Fajardo et al. 2009 [7]. The authors considered a robust estimator of the spectral density based on the robust ACF function when the time series follows an ARFIMA Model. Their estimation method is discussed in the next sub-section.

3.2. The sample spectral function

The results discussed in the previous sections and the spectral representation of a stationary process justify the use of the robust ACF function in the calculus of an estimator of a spectral density.

As previously stated, for the stationary process $\{X_t\}_{t\in\mathbb{Z}}$, the spectral density is a real-valued function of the Fourier transform of the autocovariance function, that is,

$$f_X(\omega) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_X(h)e^{-ih\omega} \qquad (3.6)$$

where $\gamma_X(\cdot)$ is the autocovariance of the process.

Equation (3.6) suggests to replace $\gamma_X(\cdot)$ by its estimate to obtain an estimate of $f_X(\omega)$. The periodogram function is the classical tool to estimate the spectral function. Other variants of the periodogram are called smoothed window periodogram (see, for example, Priestley 1981 [25]). In the same direction, Fajardo et al. 2009 [7] suggested to use the robust autocovariance function as an estimator of the classical ACF to obtain a robust spectral function. Although the theoretical justification of this estimator is still an opened question, the authors have empirically shown that the robust spectral estimator can be an alternative method to estimate a time series with outliers. A robust spectral estimator is

$$I_Q(\omega) = \frac{1}{2\pi} \sum_{|h|<n} \kappa(h)\widehat{\gamma}_Q(h)\cos(h\omega), \tag{3.7}$$

where $\widehat{\gamma}_Q(h)$ is the sample autocovariance function given in (3.3) and $\kappa(h)$ is defined as

$$\kappa(h) = \begin{cases} 1, & |h| \le M, \\ 0, & |h| > M. \end{cases}$$

$\kappa(h)$ is a particular case of the *lag window* functions used in classical spectral theory to obtain a consistent spectral estimator, and M is the truncation point which is a function of n, say $M = G(n)$, where $G(n)$ must satisfy $G(n) \to \infty$, $n \to \infty$, with $\frac{G(n)}{n} \to 0$. $G(n)$ is usually chosen to be $G(n) = n^\beta$, where $0 < \beta < 1$ (see, e.g. Priestley 1981 [25, pp. 433–437]). Note that, equivalently to the classical spectral estimation theories, other different *lag window* functions can be used to obtain a robust spectral estimator.

Since (3.7) does not have the same finite-sample properties as the periodogram, it is defined here as *robust truncated pseudo-periodogram*. For large h, the numbers of observations in the calculus of $\widehat{\gamma}_Q(h)$ are very small and, consequently, this function becomes very unstable. Then, to avoid these undesirable covariance estimates in the calculus of the estimator given in (3.7) justify the use of a truncation point M in the calculus of this sample function (see Fajardo et al. 2009 [7]). The authors suggested M that satisfies

$$M \le h' = \min\left\{0 < h < n : \varepsilon_n^{temp}\left(\widehat{\gamma}_Q(h)\right) \le \frac{m}{n}\right\} - 1.$$

4. Semiparametric estimation methods of d and empirical studies

The semiparametric estimation procedure based on the OLS estimator proposed by Geweke & Porter-Hudak 1983 [10](GPH) is considered. Since the GPH estimator is well-discussed in the literature, this method and its asymptotic statistical properties are briefly summarized as follows.

For a single realization $x_1, ..., x_n$ of $\{X_t\}_{t\in\mathbb{Z}}$, the GPH estimate of d is obtained from the regression equation

$$\log I_x(\omega_j) = a_0 - 2d\log\left[2\sin(\omega_j/2)\right] + \xi_j, j = 1, ..., m' \tag{4.1}$$

where ω_j is the Fourier frequency at j, m' is the bandwidth in the regression equation which has to satisfy $m' \to \infty$, $n \to \infty$, with $\frac{m'}{n} \to 0$ and $\frac{m'\log(m')}{n} \to 0$,

$a_0 = \log f_\eta(0) + \log \frac{f_\eta(\omega_j)}{f_\eta(0)} + \text{C}$, $\xi_j = \log \frac{I_x(\omega_j)}{f_X(\omega_j)} - C$ and $\text{C} = \varphi(1)$ ($\varphi(.)$ is the digamma function).

The GPH estimate of d is given by

$$d_{GPH} = (-0.5)\frac{\sum_{j=1}^{m'}(v_j - \bar{v})\log I_x(\omega_j)}{S_{vv}} \tag{4.2}$$

where $S_{vv} = \sum_{j=1}^{m'}(v_j - \overline{v})^2$, $v_j = \log\left\{4\sin^2(\omega_j/2)\right\}$.

Under some conditions, Hurvich, Deo & Brodsky 1998 [14] proved that the GPH-estimator is consistent for the memory parameter and asymptotically normal for Gaussian time series processes. The authors established that the optimal m' in (4.1) and (4.2) is of order $o(n^{4/5})$ and $(m')^{1/2}(d_{GPH} - d) \xrightarrow{d} N(0, \frac{\pi^2}{24})$.

To obtain a robust estimator of d, Fajardo et al. 2009 [7] proposed to replace in (4.1) the $\log I_x(\omega_j)$ by $\log I_Q(\omega_j)$ which gives the following OLS regression estimator

$$d_{GPHR} = -(0.5)\frac{\sum_{j=1}^{m'}(v_j - \bar{v})\log I_Q(\omega_j)}{S_{vv}}, \tag{4.3}$$

where S_{vv}, m' are defined as before and $I_Q(\omega)$ is the function given in (3.7). As previously mentioned, the asymptotical properties of d_{GPHR} still remains to be established. However, based on the following empirical investigation, the robust method seems to be a reasonable robust alternative method to estimate long-memory time series in the presence of additive outliers.

4.1. Numerical evaluation using the ARFIMA$(0, d, 0)$ model

The finite series were simulated from zero-mean ARFIMA models (Eq. 1.2) with $\{\epsilon_t\}_{t\in\mathbb{Z}}$, $t = 1, ..., n$, i.i.d. $N(0, 1)$. The models, parameters, sample sizes and empirical results are displayed in the following tables. The empirical mean, standard deviation (s.d.), bias and mean squared error (MSE) were obtained as a mean of 10.000 replications. The contaminated data were generated from Model 2.3 with $m = 1$, $p = 0.05$ for magnitude $\varpi = 10$ and bandwidth values for d_{GPH} and d_{GPHR} were computed for $\alpha = 0.7$ and truncation point $M = n^\beta$, $\beta = 0.7$. In the tables d_{GPH_c} and d_{GPHR_c} mean the estimates of d when the series has outliers. The simulations were carried out using the `Ox` matrix programming language (see `http://www.doornik.com`). The empirical study was divided into the following model properties: stationary and non-stationary processes.

Stationary model

Table 1 displays results for $d = 0.3, 0.45$ and $\alpha = \beta = 0.7$. From the table, it can be seen that when the series does not contain outliers, both estimators present similar behavior in the estimation of d, which is not a surprising result. However, the introduction of outliers in the series dramatically changes the performance of the classical estimator (GPH), in particular, it significantly underestimates the true parameter. On the other hand, in this scenario, the robust method (GPHR) seems to be not sensitive to outliers. Other cases were also simulated such as ARFIMA models with AR and MA parts and different values of p and ϖ. All cases indicated similar conclusions to the one given in Table 1. These are available upon request. Table 2 gives the estimates of d when different lag-windows are used to compute the robust periodogram estimator. The lag-windows are Parzen (P), Tukey-Hamming(TH) and Bartlett (B) and the fractional estimators were computed with the same bandwidths as in the previous case. The choice of the lag-window does not appear to be too important in the estimation of d since the estimates obtained from different lag-windows are, in general, numerically very close to each other. In other words, the estimates are not too sensitive to the choice of the lag-window. These lag-windows yield similarly accurate estimates compared to the one given in (3.7).

d	n		d_{GPH}	d_{GPH_c}	d_{GPHR}	d_{GPHR_c}
0.30	100	mean	0.2988	0.1134	0.2584	0.2449
		s.d.	0.1735	0.1619	0.1558	0.1556
		bias	−0.0012	−0.1866	−0.0416	−0.0551
		MSE	0.0301	0.0610	0.0260	0.0272
	300	mean	0.3062	0.1007	0.2907	0.2837
		s.d.	0.1005	0.0978	0.0926	0.0960
		bias	0.0062	−0.1993	−0.0093	−0.0163
		MSE	0.0101	0.0493	0.0087	0.0095
	800	mean	0.3003	0.1184	0.2949	0.2869
		s.d.	0.0679	0.0715	0.0573	0.0610
		bias	0.0003	−0.1816	−0.0051	−0.0131
		MSE	0.0046	0.0381	0.0033	0.0039
0.45	100	mean	0.4561	0.1923	0.3975	0.3778
		s.d.	0.1722	0.1727	0.1506	0.1433
		bias	0.0061	−0.2577	−0.0525	−0.0722
		MSE	0.0297	0.0962	0.0254	0.0258
	300	mean	0.4594	0.2015	0.4329	0.4233
		s.d.	0.0986	0.0976	0.1041	0.1013
		bias	0.0094	−0.2485	−0.0171	−0.0267
		MSE	0.0098	0.0713	0.0111	0.0110
	800	mean	0.4620	0.2306	0.4457	0.4349
		s.d.	0.0688	0.0809	0.0562	0.0576
		bias	0.0121	−0.2194	−0.0043	−0.0151
		MSE	0.0049	0.0547	0.0032	0.0035

Table 1: Simulation results; ARFIMA$(0, d, 0)$ model with $\alpha = \beta = 0.7$ and $\varpi = 0, 10$.

Non-stationary model

As is well-known, the GPH estimator has been widely used even for ARFIMA models with d in $(0.5, 1.0]$ (see, for example, Franco & Reisen 2007 [9], Hurvich & Ray 1995 [15],Olbermann, Lopes & Reisen 2006 [22], Phillips 2007 [24] among

uncontaminated series					
Parameter	n		d_P	d_{TH}	d_B
	100	mean	0.2699	0.2602	0.2459
		s.d.	0.1497	0.1575	0.1444
		bias	−0.0301	−0.0398	−0.0541
		MSE	0.0233	0.0264	0.0238
	300	mean	0.2880	0.2833	0.2857
$d = 0.3$		s.d.	0.1050	0.1037	0.0976
		bias	−0.0119	−0.0167	−0.0143
		MSE	0.0112	0.0110	0.0097
	800	mean	0.2985	0.2966	0.3001
		s.d.	0.0554	0.0584	0.0561
		bias	−0.0015	−0.0034	0.0001
		MSE	0.0031	0.0034	0.0031
contaminated series					
Parameter	n		d_P	d_{TH}	d_B
	100	mean	0.2504	0.2446	0.2419
		s.d.	0.1552	0.1482	0.1405
		bias	−0.0496	−0.0554	−0.0581
		MSE	0.0266	0.0250	0.0231
	300	mean	0.2806	0.2729	0.2796
$d = 0.3$		s.d.	0.1028	0.0925	0.0964
		bias	−0.0194	−0.0271	−0.0204
		MSE	0.0109	0.0093	0.0097
	800	mean	0.2934	0.2889	0.2928
		s.d.	0.0578	0.0606	0.0553
		bias	−0.0066	−0.0111	−0.0072
		MSE	0.0034	0.0038	0.0031

Table 2: Empirical results of d's estimators in ARFIMA$(0, d, 0)$ model using different lag-windows.

others).

Based on the theory discussed in the previous sections, the robust method can not be applied in a non-stationary time series. However, it may be interesting to verify if GPHR estimator is invariant to the first difference, i.e. estimative of the memory parameter based on the original data is equal to one plus the estimated d based on the differenced data.

Now, let Model 1.2 be defined with parameter $d^* = d + \kappa$, where $d \in (-0.5, 0.5)$, $\kappa > 0$, $\kappa \in \mathbb{Z}$. Then, Model 1.2, with zero-mean, becomes

$$X_t = (1 - B)^{-d^*} \eta_t, \quad t \in \mathbb{Z}. \tag{4.4}$$

Process given in (4.4) is non-stationary when $d^* \geq 0.5$; however, it is still persistent. For $d^* \in [0.5, 1.0)$ it is level-reverting in the sense that there is no long-run impact of an innovation on the value of the process. The level-reversion property no longer holds when $d^* \geq 1$. Note that when $d^* = 1$ the process is a random walk.

From Model 4.4 with $\kappa = 1$ and $p = q = 0$,

$$W_t = (1 - B) X_t, t \in \mathbb{Z},$$

is an $ARFIMA(0, d, 0)$ process. Let $\hat{d}^*$ be the estimator of d^* and let $\hat{d}$ be the fractional estimator obtained from the differenced data. The main goal is to verify the equality $\hat{d}^* = \hat{d} + 1$ for uncontaminated and contaminated series. Based on the same simulation procedure previously described, series from Model 4.4 were generated and some cases are displayed in Table 3 (other cases are available upon request). Similar conclusions to the previous study are observed. Both estimators present equivalent performance when they are applied in the first difference of uncontaminated series. This suggests that both can be used in practical situations when dealing with non-stationary data. However, since the first difference does not eliminate the effect of an outlier, the estimates clearly indicate that caution has to be exercised when there is suspicion of outliers in the data. The GPH estimator presents poor performance in terms of bias (high positive bias) and MSE. In contrast to the GPH estimator, the GPHR method seems to be invariant to the first difference of non-stationary time series with outliers. This empirical study suggests that, in practical situations when dealing with non-stationary data with outliers, one solution is to apply the first difference in the series and then to estimate d with the robust estimator discussed in this paper.

Parameter	n		d_{GPH}	d_{GPH_c}	d_{GPHR}	d_{GPHR_c}
	300	mean	−0.2141	−0.5066	−0.1906	−0.2211
$d_X = 0.8, d_W = -0.2$		bias	0.0141	0.3066	−0.0094	0.0211
		s.d	0.1076	0.1469	0.1127	0.1421
		MSE	0.0118	0.1155	0.0128	0.0206
	800	mean	−0.1906	−0.4283	−0.2062	−0.2250
		bias	−0.0094	0.2283	0.0062	0.0251
		s.d	0.0630	0.0883	0.0851	0.1081
		MSE	0.0041	0.0599	0.0073	0.0123
	100	mean	−0.0048	−0.4166	−0.0449	−0.0871
		bias	0.0048	0.4166	0.0449	0.0871
		s.d	0.1763	0.2215	0.1620	0.1811
		MSE	0.0311	0.2226	0.0283	0.0404
	300	mean	−0.0122	−0.3230	−0.0273	−0.0426
$d_X = 1.0, d_W = 0.0$		bias	0.0122	0.3230	0.0273	0.0426
		s.d	0.1076	0.1296	0.1094	0.1277
		MSE	0.0117	0.1211	0.0127	0.0181
	800	mean	0.0059	−0.2181	−0.0107	−0.0222
		bias	−0.0059	0.2181	0.0107	0.0222
		s.d	0.0648	0.0823	0.0629	0.0909
		MSE	0.0042	0.0544	0.0041	0.0088

Table 3: Empirical results: ARFIMA$(0, d, 0)$ model with differenced data and $\omega = 0, 10$.

5. Application

IGP-DI is the general price index with domestic availability and is calculated by Fundação Getúlio Vargas, Brazil. The series comprises monthly observations from

August 1994 to April 2011 (total of 201 observations). The series and its ACF are displayed in Figure 1. The observations of the months February 1999 (4.44%), October 2002 (4.21%) and November 2002 (5.84%) are possibly outliers. Looking at the plots in Figure 1, these suggest that the series is stationary and possess long-memory behavior. From the data and using the methodologies previously discussed, the parameter d is estimated and the results are displayed in Table 4. For this application, the estimates of d were computed from the original data (OD) and from the modified data (MD), where the observations of February 1999, October 2002 and November 2002 were replaced by the sample mean of the series. This analysis is a simple exercise to verify the robustness of the estimators in a real application and, also, to investigate whether the data contains outliers. The d' estimates of OD and MD series are given, respectively, on the left and right sides of Table 4. These estimates were calculated using different bandwidths in (4.2)$(m' = n^{\alpha})$ and β was fixed as in the simulation study. In both series, for a fixed α, the robust methods present similar results. The estimates maintain the same empirical property across the bandwidth values. In contrast to the robust methods, the classical GPH estimator gives estimates that dramatically change from OD to MD data, showing that the observations replaced by the mean are possible atypical data.

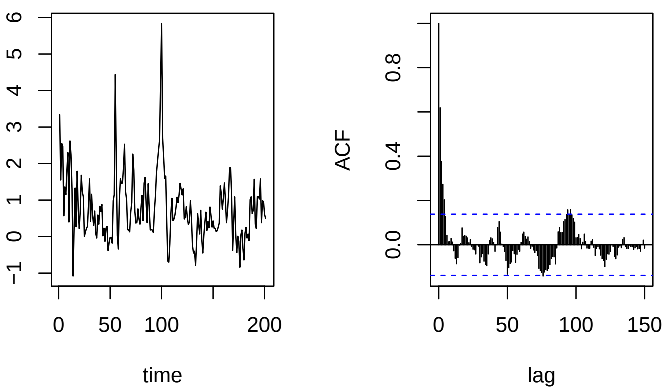

Figure 1: IGP-DI series and its sample autocorrelation function: period from Aug/94 to Apr/11.

6. Concluding remarks and future direction

This paper investigates the effect of outliers in the estimation of the fractional parameter d in the ARFIMA(p, d, q) model and, also, discusses the asymptotical and empirical properties of the robust autocovariance and spectral estimators, previously given in Fajardo et al. 2009 [7] and Lévy-Leduc et al. 2011 [19], for the case of time series with short and long-memory properties. These studies support the use

	Original time series				Modified time series			
Estimator	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
d_{GPH}	0.0757	0.1205	0.3431	0.3759	0.3110	0.3116	0.3713	0.3875
	(0.3417)	(0.1869)	(0.1389)	(0.0888)	(0.1586)	(0.1077)	(0.0909)	(0.0683)
d_{GPHR_P}	0.1802	0.2335	0.2269	0.2397	0.1630	0.2077	0.2078	0.2230
	(0.0857)	(0.0745)	(0.0469)	(0.0331)	(0.0782)	(0.0603)	(0.0385)	(0.0251)
$d_{GPHR_{TH}}$	0.1718	0.1919	0.2125	0.2379	0.1545	0.1782	0.1968	0.2231
	(0.0742)	(0.0508)	(0.0303)	(0.0210)	(0.0673)	(0.0436)	(0.0259)	(0.0170)
d_{GPHR_B}	0.1522	0.1788	0.2047	0.2327	0.1379	0.1667	0.1896	0.2181
	(0.0641)	(0.0433)	(0.0262)	(0.0183)	(0.0586)	(0.0378)	(0.0227)	(0.0151)
d_{GPHR}	0.1662	0.2628	0.2454	0.2285	0.1500	0.2211	0.2215	0.2228
	(0.0862)	(0.0995)	(0.0671)	(0.0436)	(0.0794)	(0.0717)	(0.0511)	(0.0328)

Table 4: Estimates of d: IGP-DI data, period from Aug/94 to Apr/11.

of the robust estimators to estimate the long-memory parameter when Gaussian long-memory time series are contaminated with additive outliers. Non-stationary time series with outliers are also studied and the investigation reveals that the robust method can be used as an alternative estimation procedure in time series with fractional differences. As previously stated, the asymptotical properties of the robust estimator under the study still remain to be investigated. The robust ACF method discussed here has also been used in other contexts such as in the estimation of periodic process (Sarnaglia, Reisen & Lévy-Leduc 2010 [30]) and in seasonal ARFIMA processes (this is one of the current research of the authors).

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