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# The On-Line Encyclopedia of Integer Sequences

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**Abstract**

We all recognize $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ but what about $1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, \ldots$? If you come across a number sequence and want to know if it has been studied before, there is only one place to look, the *On-Line Encyclopedia of Integer Sequences* (or *OEIS*). Now in its 49th year, the OEIS contains over 220,000 sequences and 20,000 new entries are added each year. This article will briefly describe the OEIS and its history. It will also discuss some sequences generated by recurrences that are less familiar than Fibonacci's, due to Greg Back and Mihai Caragiu, Reed Kelly, Jonathan Ayres, Dion Gijswijt, and Jan Ritsema van Eck.

*Keywords:* Fibonacci, sequences, recurrences.

*MSC:* Primary 11B

## 1. The Fibonacci numbers

The Fibonacci numbers have been in the On-Line Encyclopedia of Integer Sequences® (or OEIS®) right from the beginning. When I started collecting sequences as a graduate student in 1964, the Fibonacci numbers became sequence A000045 (incidentally, 49 years later, sequences being added have A-numbers around A222000[1]). Over 3000 sequences in the OEIS mention Fibonacci's name in their definition.

Some especially noteworthy variations on the Fibonacci numbers were recently defined by Back and Caragiu [2] in the *Fibonacci Quarterly*. The simplest of their

[1] As of February 2013. Throughout this article, six-digit numbers prefixed by A refer to entries in the OEIS [15]. As in the OEIS, we adopt the convention that $a(n)$ denotes the $n$th term of the sequence being discussed.

examples replaces the Fibonacci recurrence by

$$a(n) = \text{gpf}(a(n-1) + a(n-2)), \tag{1.1}$$

where gpf stands for *greatest prime factor* (A006530). If we start with $1, 1$ we get

$$1, 1, 2, 3, 5, 2, 7, 3, 5, 2, 7, \ldots \tag{1.2}$$

(A175723), and the cycle $3, 5, 2, 7$ repeats for ever. Back and Caragiu show that no matter what the initial values are, (1.1) always becomes periodic and that $3, 5, 2, 7$ is the only nontrivial cycle. On the other hand, consider

$$a(n) = \text{gpf}(a(n-1) + a(n-2) + a(n-3)). \tag{1.3}$$

If we start with $1, 1, 1$ we get

$$1, 1, 1, 3, 5, 3, 11, 19, 11, 41, 71, 41, 17, 43, 101, 23, \ldots \tag{1.4}$$

(A177904), which after 86 steps enters a cycle of length 212. Now it is only a conjecture that (1.3) always becomes periodic, for any initial values.

Another interesting variant of the Fibonacci sequence[2] was very recently introduced into the OEIS by Reed Kelly [12]. Kelly's recurrence is

$$a(n) = \frac{a(n-1) + a(n-3)}{\text{gcd}\{a(n-1), a(n-3)\}}, \tag{1.5}$$

with initial values $1, 1, 1$:

$$1, 1, 1, 2, 3, 4, 3, 2, 3, 2, 2, 5, 7, 9, 14, 3, 4, 9, 4, 2, \ldots \tag{1.6}$$

(A214551). This sequence appears to grow exponentially ($a(n) \approx \text{const.} \cdot e^{0.123\ldots n}$?), but essentially nothing has been proved about it.

The OEIS is an endless source of lovely problems!

## 2. How the OEIS is used

However, the main use for the OEIS is as a reference work for identifying sequences and telling you what is known about them. If you come across a sequence of numbers, and you want to know if it has been studied before, there is only one place to look, the OEIS [15] (`http://oeis.org`).

You enter the first few terms[3], and click "Submit". If you are lucky, the OEIS will return one or more sequences that match what you entered, and, for each one, it will tell you such things as:

[2] Or, more precisely, of another medieval sequence, the Narayana cows sequence, A000930.

[3] When looking up a sequence, it is recommended that you omit the first term or two, since different people may start a sequence in different ways.

- The definition of the sequence
- The first 10, or 10,000, or sometimes 500,000 terms
- Comments explaining further properties of the sequence
- Formulas for generating the sequence
- Computer programs for producing the sequence
- References to books and articles where the sequence is mentioned
- Links to web pages on the Internet where the sequence has appeared
- The name of the person who submitted the sequence to the OEIS
- Examples illustrating some of the terms of the sequence (for example, sequence A000124, which gives the maximal number of pieces that can be obtained when cutting a circular pancake with $n$ cuts, is illustrated with pictures showing the pieces obtained with 1, 2, 3, 4 and 5 cuts)
- The history of each entry in the OEIS as it has evolved over time

You can also view graphs or plots of the sequence, or listen to it when it is converted to sounds.

If your sequence is *not* found, you will be encouraged to submit it. This will establish your priority over the sequence, and will help the next person who comes across it. Only sequences of general interest should be submitted. The sequence of primes whose decimal expansion begins with 2012 is an example of a sequence that would not be of general interest. Published sequences are almost always acceptable.

If your sequence was not in the OEIS, you should also try sending it to our email server *Superseeker* (see `http://oeis.org/ol.html`), which will try hard to find an explanation for your sequence. For example, Superseeker might suggest a recurrence or generating function for your sequence, or tell you that it can be obtained by applying one of over a hundred different transformations to one of the over 200,000 sequences in the OEIS. Superseeker is a very powerful tool for analyzing sequences.

Accuracy has always been one of the top priorities in the OEIS. Its standards are those of a mathematics reference work. Ideally, every number, formula, computer program, etc., should be absolutely correct. Formulas that are stated unconditionally should be capable of being proved, and otherwise should be labeled as conjectures. Of course, as the database has grown, these goals have become harder and harder to achieve. Many non-mathematicians have difficulty in understanding the difference between a theorem and a conjecture. ("My formula fits the first 30 terms, so obviously it *must* be correct.")

The OEIS has often been called one of the most useful mathematical sites on the Internet. There is a web page (`http://oeis.org/wiki/Works\_Citing\_OEIS`) that lists over 3000 articles and books that reference it.

# 3. History of the OEIS

I started collecting sequences in 1964, entering them on punched cards (the original motivation was to find an explanation for various sequences that had arisen in my dissertation, the simplest of which was the sequence that became A000435). Eventually two books were published ([16] in 1973, with 2372 entries, and [17], written with Simon Plouffe, in 1995, with 5847 sequences).

In 1996, when the number of entries had risen to 10,000, I put the database on the Internet, calling it the *The On-Line Encyclopedia of Integer Sequences* or *OEIS*. By 2009, the database had grown to over 150,000 entries, and was becoming too big for one person to manage, so I set up a foundation, *The OEIS Foundation Inc* (`http://oeisf.org`), whose goals are to own the intellectual property of the OEIS, to maintain it, and to raise funds to support it.

With major help from Russ Cox (of Google) and my colleague David Applegate (at AT&T), I moved the OEIS off my home page at AT&T to a commercial hosting service, and attempted to set it up as a "wiki." However, this proved to be extremely difficult, and it required a tremendous amount of work by Russ Cox before it started working properly. It was not until November 11, 2010 that the OEIS was officially launched in its new home at `http://oeis.org`. This would not have been possible without the help that Russ Cox and David Applegate provided.

The fact that the OEIS is now a wiki means that I no longer have to process all the updates myself. Once a user has registered[4], he or she can propose new sequences or updates to existing sequences. All submissions are reviewed by a panel of about 80 editors. Nearly two years after it was launched, the wiki system is working well. Since November 2010 the database has grown from 180,000 sequences to its current number of around 220,000. From 1996 to the present, the database has grown at between 10,000 and 20,000 new sequences per year, with about an equal number of entries that are updated.

More about the history of the OEIS can be found on the OEIS Foundation web site, `http://oeisf.org`.[5]

# 4. The poster and the OEIS movie

To celebrate the creation of the the OEIS Foundation, David Applegate and I made a poster that shows 25 especially interesting sequences (several of which will be mentioned in this article). It can be downloaded (along with a key) from the Foundation web site.

Also, Tony Noe made a movie that shows graphs of the first thousand terms of a thousand sequences from the OEIS: it is quite spectacular. It runs for 8.5 minutes,

[4] All readers are encouraged to register: go to `http://oeis.org/wiki` and click "Register."

[5] As President, it would be remiss of me not to mention that the OEIS Foundation is a charitable organization and donations are tax-deductible in the USA. The web site is free, and none of the trustees receive a salary. To make a donation, please go to `http://oeisf.org`.

and it too can be found on the Foundation web site. It is also on YouTube (search for “OEIS movie”).

# 5. Puzzles

One of the goals of the OEIS has always been to help people get higher scores on IQ tests, and the database includes many sequences that have appeared as puzzles. The following are a few examples. If you can’t solve them, you know where to find the answers!

- 61, 21, 82, 43, ...
- 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, 50, 52, 54, 56, 60, 62, 64, 66, 2000, ...
- 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, 55, ...
- 5, 8, 12, 18, 24, 30, 36, 42, 52, 60, ...
- 1, 2, 6, 21, 85, 430, 2586, 18109, 144880, ...

The last one is a bit tricky, but it did appear on a quiz.

# 6. Two sequences that agree for a long time

People often ask if it is possible for two sequences to agree for many terms yet not be the same. Here is an extreme example. The sequences

$$\left\lfloor \frac{2n}{\log 2} \right\rfloor \quad \text{and} \quad \left\lceil \frac{2}{2^{1/n}-1} \right\rceil$$

both begin

$$2, 5, 8, 11, 14, 17, 20, 23, 25, 28, 31, 34, 37, \ldots$$

(A078608). In fact they agree for the first 777451915729367 terms! There are infinitely many disagreements, the positions of which form sequence A129935:

$$777451915729368, 140894092055857794, 1526223088619171207, 3052446177238342414, \ldots$$

# 7. Theorems resulting from the OEIS

Another question that is often asked is if there are any theorems that have resulted from the OEIS. The answer is that there are many such examples. In the list of papers that cite the OEIS (`http://oeis.org/wiki/Works\_Citing\_OEIS`) one will find numerous acknowledgments that say things like “This result was discovered with the help of the OEIS.”

I will give three concrete examples of theorems that were discovered with the help of the OEIS. The first concerns the remainder term in Gregory's series for $\pi/2$,

$$\frac{\pi}{2} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{7.1}$$

which is famous for converging very slowly. In 1987, Joseph North observed that if one truncates the series after 50,000 terms, the answer is of course wrong. There is an error in the fifth decimal place. Surprisingly, he noticed that the next nine digits are correct, then there is an error, then there are nine more correct digits, another error, and so on. Here is the decimal expansion of the truncated sum followed by the true value of $\pi/2$ (the sequences of digits form A013706 and A019669). The digits that differ are in bold-face.

1.5707**9**632679489**6**61923132**16**9163975**144**2098584**699**687... (truncated)
1.5707**8**632679489**7**61923132**11**9163975**205**2098583**3314**687... (true value)

The differences between the corresponding bold-faced terms are

$$1, -1, 5, -61, 1385, \ldots.$$

Jonathan Borwein looked up this sequence in [16], and found that (apart from signs) it appeared to be the Euler numbers, A000364. The end result of this investigation was a new theorem.

**Theorem 7.1** (Borwein, Borwein and Dilcher [4]; see also [3, pp. 28–29], [5])**.**

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k+1} \sim \sum_{m=0}^{\infty} \frac{E_m}{N^{2m+1}}, \tag{7.2}$$

*where the $E_m$ are the Euler numbers (A000364):*

$$1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 19391512145, \ldots.$$

The second example is one that I was involved with personally. It began when Eric W. Weisstein (at Wolfram Research, and creator of *MathWorld*) wrote to me about a discovery he had made. He had been classifying real matrices of 0's and 1's according to various properties, and he found that the numbers of such matrices all of whose eigenvalues were positive were $1, 3, 25, 543, 29281$ for matrices of orders $1, 2, \ldots, 5$. He observed that these numbers coincided with the beginning of sequence A003024 (whose definition on the surface seemed to have nothing to do with eigenvalues), and he conjectured that the sequences should in fact be identical. He was right, and this led to the following theorem.

**Theorem 7.2** ([14])**.** *The number of acyclic directed graphs with $n$ labeled vertices is equal to the number of $n \times n$ matrices of $0$'s and $1$'s all of whose eigenvalues are real and positive.*

The third example is a result of Deutsch and Sagan [8]. It is well-known that the famous Catalan numbers

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

(A000108) are odd if and only if $n = 2^k - 1$ for some $k$. Deutsch and Sagan proved (among other things) an analogous result for the almost equally-famous Motzkin numbers (A001006),

$$M_n := \sum_{k=1}^{n} \binom{n}{2k} C_k.$$

**Theorem 7.3** ([8]). *$M_n$ is even if and only if $n \in 4S - 2$ or $4S - 1$, where*

$$S := (1, 3, 4, 5, 7, 9, 11, 12, 13, 15, \ldots)$$

*lists the numbers whose binary expansion ends with an even number of 0's (A003159).*

# 8. Three unusual recurrences

The Fibonacci recurrence is very nice, but it is 800 years old. In the last section of this article I will discuss some *modern* recurrences that I find very appealing.

## 8.1. The EKG sequence

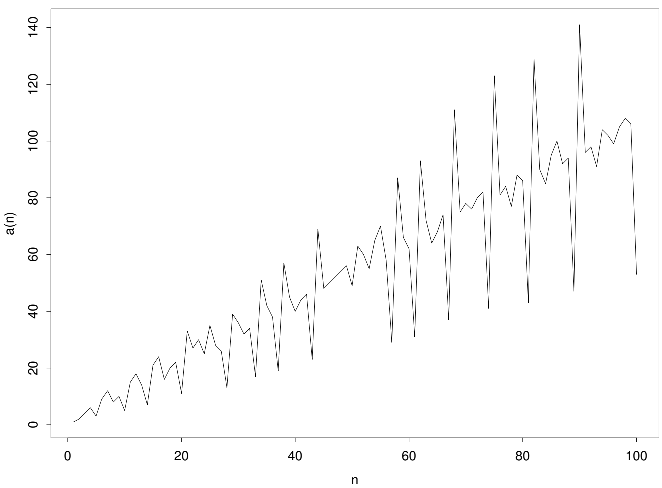


Figure 1: The first 100 terms of the EKG sequence, with successive points joined by lines

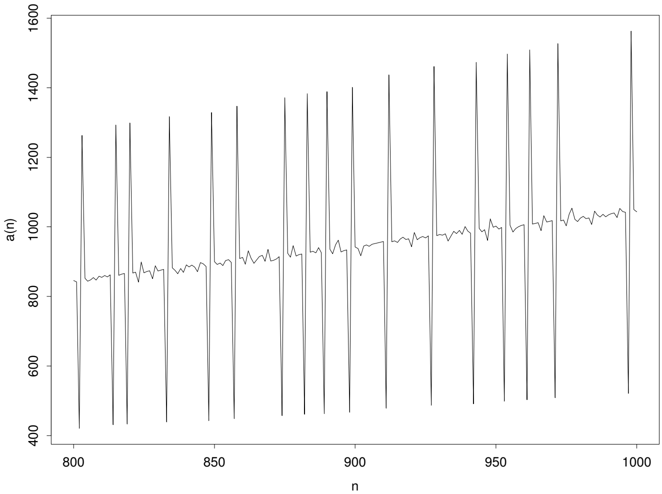


Figure 2: Terms 800 to 1000 of the EKG sequence

Jonathan Ayres contributed this to the OEIS in 2001 [1]. The first 18 terms are

$$1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, 18, 14, 7, 21, 24, 16, 20, \ldots$$

(A064413), and the defining recurrence is $a(1) = 1$, $a(2) = 2$, and, for $n \geq 3$,

> $a(n)$ is the smallest natural number not yet in the sequence which has a common factor $> 1$ with the previous term.

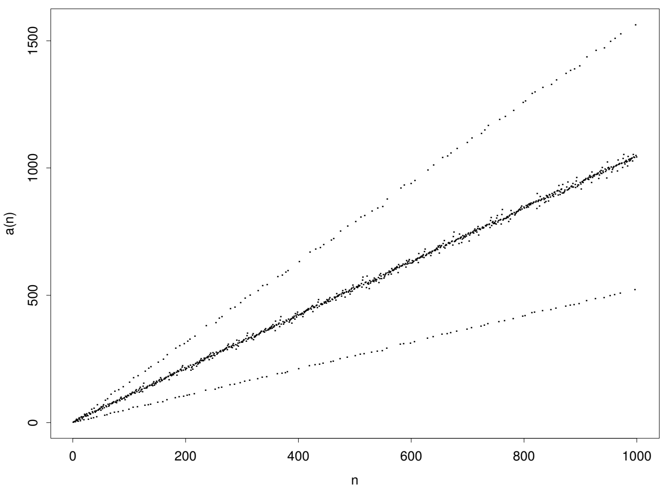


Figure 3: Scatter-plot of the first 1000 terms of the EKG sequence. They lie roughly on three almost-straight lines.

Thus $a(3)$ must have a common factor with 2, i.e. it must be even, and 4 is the smallest candidate, so $a(3) = 4$. The next term must also be even, so $a(4) = 6$. The smallest number not yet in the sequence which has a common factor with 6 is 3, so $a(5) = 3$. Similarly, $a(6) = 9$, $a(7) = 12$, $a(8) = 8$, $a(9) = 10$, $a(11) = 5$, $a(12) = 15$, and so on. Jeffrey Lagarias, Eric Rains and I studied this sequence in [13]. We called it the EKG sequence, since it looks like an electrocardiogram when plotted (Figs. 1, 2).

It is not difficult to show that the primes appear in increasing order, and that each odd prime $p$ is either preceded by $2p$ and followed by $3p$, or is preceded by $3p$ and followed by $2p$ (as we just saw, 3 was preceded by 6 and followed by 9, 5 is preceded by 10 and followed by 15).[6]

By definition, no number can can be repeated. But does every number appear? The answer is Yes.

**Theorem 8.1.** *The EKG sequence is a permutation of the natural numbers.*

*Sketch of Proof.* (i) If infinitely many multiples of some prime $p$ occur in the sequence, then every multiple of $p$ must occur. (For if not, let $kp$ be the smallest missing multiple of $p$. Every number below $kp$ either appears or it doesn't, but once we get to a multiple of $p$ beyond all those terms, the next term must be $kp$, which is a contradiction.) (ii) If every multiple of a prime $p$ appears, then every number appears. (The proof is similar.) (iii) Every number appears. (For if there are only finitely many different primes among the prime factors of all the terms, then some prime must divide infinitely many terms, and the result follows from (i) and (ii). On the other hand, if infinitely many different primes $p$ appear, then there are infinitely many terms $2p$, as noted above, so 2 appears infinitely often, and again the result follows from (i) and (ii).) □

Although the initial terms of the sequence jump around, when we look at the big picture we find that the points lie very close to three almost-straight lines (Fig. 3). This is somewhat similar to the behavior of the prime numbers, which are initially erratic, but lie close to a smooth curve (since the $n$th prime is roughly $n \log n$) when we look at the big picture – see Don Zagier's lecture on "The first 50 million prime numbers" [18].

In fact, we have a precise conjecture about the three lines on which the points lie. We believe – but are unable to prove – that almost all $a(n)$ satisfy the asymptotic formula $a(n) \sim n(1 + 1/(3 \log n))$ (the central line in Fig. 3), and that the exceptional values $a(n) = p$ and $a(n) = 3p$, for $p$ a prime, produce the points on the lower and upper lines. We were able to show that the sequence has essentially linear growth (there are constants $c_1$ and $c_2$ such that $c_1 n < a(n) < c_2 n$ for all $n$), but the proof of even this relatively weak result was quite difficult. It would be nice to have better bounds.

[6] We conjectured that $p$ was always preceded by $2p$ rather than $3p$. This was later proved by Hofman and Pilipczuk [11].

## 8.2. Gijswijt's sequence and the Curling Number Conjecture

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | | | | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | 2 | | | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | | | | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 3 | 2 | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | | | | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | 2 | | | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | | | | | | | | | | | | | | |
| 1 | 1 | 2 | | | | | | | | | | | | | | | | | |
| 1 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 3 | 2 | 2 | 2 | 3 | 2 |

Table 1: The first 98 terms of Gijswijt's sequence (A090822)

We start by defining the curling number of a sequence. Let $S$ be a finite nonempty sequence of integers. By grouping consecutive terms, it is always possible to write it as $S = X\,Y\,Y \cdots Y = X\,Y^k$, where $X$ and $Y$ are sequences of integers and $Y$ is nonempty. There may be several ways to do this: choose the one that maximizes the value of $k$: this $k$ is the *curling number* of $S$.

For example, if $S = 0\,1\,2\,2\,1\,2\,2\,1\,2\,2$, we could write it as $X\,Y^2$, where $X = 0\,1\,2\,2\,1\,2\,2\,1$ and $Y = 2$, or as $X\,Y^3$, where $X = 0$ and $Y = 1\,2\,2$. The latter representation is to be preferred, since it has $k = 3$, and as $k = 4$ is impossible, the curling number of this $S$ is 3.

In 2004, Dion Gijswijt, then a graduate student at the University of Amsterdam and also the puzzle editor for the Dutch magazine *Pythagoras*, contributed the following sequence to the OEIS. Start with $a(1) = 1$, and, for $n \geq 2$, use the recurrence

$$\boxed{a(n) = \text{curling number of } a(1), \ldots, a(n-1).}$$

The beginning of the sequence is shown in Table 1 (it has been broken up into sections to show where the curling number drops back to 1):

This sequence was analyzed by Gijswijt, Fokko van de Bult, John Linderman, Allan Wilks and myself [6]. The first time a 4 appears is at $a(220)$. We computed several million terms without finding a 5, and for a while we wondered if perhaps no term greater than 4 was ever going to appear. However, we were able to show that a 5 does eventually appear, although the universe would grow cold before a direct search would find it. The first 5 appears at about term

$$10^{10^{23}}.$$

We also showed that the sequence is actually unbounded, and we conjecture that the first time that a number $m$ $(= 5, 6, 7, \dots)$ appears is at about term number

$$2^{2^{3^{4^{\cdot^{\cdot^{\cdot^{m-1}}}}}}},$$

a tower of height $m - 1$.

Our arguments could be considerably simplified if the *Curling Number Conjecture* were known to be true. This states that:

> If one starts with any initial sequence of integers, and extends it by repeatedly calculating the curling number and appending it to the sequence, the sequence will eventually reach 1.

The conjecture is still open. One way to tackle it is to consider starting sequences $S_0$ that contain only 2's and 3's, and to see how far such a sequence will extend (by repeatedly appending the curling number) before reaching a 1.

Let $\mu(n)$ denote the maximal length that can be achieved before a 1 appears, for any starting sequence $S_0$ consisting of $n$ 2's and 3's. For $n = 4$, for example, $S_0 = 2\,3\,2\,3$ produces the sequence

$$2\,3\,2\,3\,2\,2\,2\,3\,1\,\dots,$$

and no other starting string does better, so $\mu(4) = 8$. The Curling Number Conjecture would imply that $\mu(n) < \infty$ for all $n$. Reference [6] gave $\mu(n)$ for $1 \le n \le 30$, and Benjamin Chaffin and I have determined $\mu(n)$ for all $n \le 48$ [7]. By making certain plausible assumptions about $S_0$, we have also computed lower bounds on $\mu(n)$ (which we conjecture to be the true values) for all $n \le 80$. The results are shown in Table 2 and Figure 4. The values of $\mu(n)$ also form sequence A094004 in [15].

As can be seen from Fig. 4, up to $n = 80$, it appears that $\mu(n)$ increases in a piecewise linear manner. At the values $n = 1, 2, 4, 6, 8, 9, 10, 11, 14, 19, 22, 48, 68, 76, 77$ (A160766), assuming that the values in Table 2 are correct, there is a jump, but at the other values of $n$, $\mu(n)$ is simply $\mu(n-1)+1$. Table 3 gives the starting sequences where $\mu(n) > \mu(n-1)+1$ for $n \le 48$.

For example, Table 2 shows that

$$\mu(n) = n + 120 \quad \text{for } 22 \le n \le 47. \tag{8.1}$$

In this range one cannot do any better than taking the starting sequence for $n = 22$ and prefixing it by an irrelevant sequence of $47 - n$ 2's and 3's. However, at $n = 48$ a new record-holder appears, and it seems that

$$\mu(n) = n + 131 \text{ for } 48 \le n \le 67. \tag{8.2}$$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $\mu(n)$ | 1 | 4 | 5 | 8 | 9 | 14 | 15 | 66 | 68 | 70 | 123 | 124 |
| $n$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| $\mu(n)$ | 125 | 132 | 133 | 134 | 135 | 136 | 138 | 139 | 140 | 142 | 143 | 144 |
| $n$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 |
| $\mu(n)$ | 145 | 146 | 147 | 148 | 149 | 150 | 151 | 152 | 153 | 154 | 155 | 156 |
| $n$ | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 |
| $\mu(n)$ | 157 | 158 | 159 | 160 | 161 | 162 | 163 | 164 | 165 | 166 | 167 | 179 |
| $n$ | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| $\mu(n)$ | 180 | 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 | 191 |
| $n$ | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 |
| $\mu(n)$ | 192 | 193 | 194 | 195 | 196 | 197 | 198 | 200 | 201 | 202 | 203 | 204 |
| $n$ | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 |
| $\mu(n)$ | 205 | 206 | 207 | 209 | 250 | 251 | 252 | 253 | ? | ? | ? | ? |

Table 2: Lower bounds on $\mu(n)$, the record for a starting sequence of $n$ 2's and 3's. Entries for $n \leq 48$ are known to be exact (and we conjecture the other entries are exact).

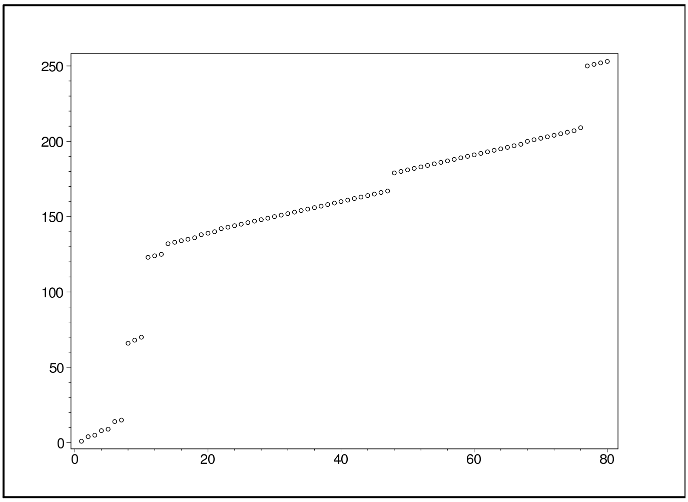


Figure 4: Scatter-plot of lower bounds on $\mu(n)$, the record for a starting sequence of $n$ 2's and 3's. Entries for $n \leq 48$ are known to be exact (and we conjecture the other entries are exact).

We have not succeeded in finding any algebraic constructions for good starting sequences. For more about the Curling Number Conjecture see [7].

| $n$ | Starting sequence |
|---|---|
| 1 | 2 |
| 2 | 2 2 |
| 4 | 2 3 2 3 |
| 6 | 2 2 2 3 2 2 |
| 8 | 2 3 2 2 2 3 2 3 |
| 9 | 2 2 3 2 2 2 3 2 3 |
| 10 | 2 3 2 3 2 2 2 3 2 2 |
| 11 | 2 2 3 2 3 2 2 2 3 2 2 |
| 14 | 2 2 3 2 3 2 2 2 3 2 2 3 2 3 |
| 19 | 2 2 3 2 2 3 2 3 2 2 2 3 2 2 3 2 2 3 2 |
| 22 | 2 3 2 2 3 2 2 3 2 3 2 2 2 3 2 3 2 2 3 2 2 3 |
| 48 | 2 2 3 2 2 3 2 3 2 2 2 3 2 2 2 3 2 2 3 2 2 2 3 2 2 3 2 3 2 2 2 3 2 2 2 3 2 2 3 2 2 2 3 2 2 3 2 3 |

Table 3: Starting sequences of $n$ 2's and 3's for which $\mu(n) > \mu(n-1)+1$. This is complete for $n \leq 48$ and is believed to be complete for $n \leq 67$.

## 8.3. Van Eck's sequence

In 2010, Jan Ritsema van Eck [9] contributed a sequence to the OEIS that is defined by yet another unusual recurrence. Again we start with $a(1) = 0$, and then for $n \geq 2$,

> $a(n)$ is the number of steps backwards before the previous appearance of $a(n-1)$, or $a(n) = 0$ if $a(n-1)$ has never appeared before.

Since $a(1) = 0$ has never appeared before, $a(2) = 0$. Now 0 has appeared one step before, at $a(1)$, so $a(3) = 1$. We have not seen a 1 before, so $a(4) = 0$. We had an earlier 0 at $a(2)$, so $a(5) = 4 - 2 = 2$. This is the first 2 we have seen, so $a(6) = 0$. And so on. The first 36 terms are shown in Table 4.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0, | 0, | 1, | 0, | 2, | 0, | 2, | 2, | 1, | 6, | 0, | 5, | |
| 0, | 2, | 6, | 5, | 4, | 0, | 5, | 3, | 0, | 3, | 2, | 9, | |
| 0, | 4, | 9, | 3, | 6, | 14, | 0, | 6, | 3, | 5, | 15, | 0, | |
| 5, | 3, | 5, | 2, | 17, | 0, | 6, | 11, | 0, | 3, | 8, | 0, | ... |

Table 4: The first 48 terms of Van Eck's sequence (A181391)

Figure 5 shows a scatter-plot of the first 800 terms. The plot suggests that after $n$ terms, there are occasionally terms around $n$, or in other words that $\limsup a(n)/n \approx 1$. This is confirmed by looking at the first million terms, and the data also strongly suggests that every number appears in the sequence. However, at present these are merely conjectures.

Van Eck was able to show that there are infinitely many 0's in the sequence, or, equivalently, that the sequence is unbounded.

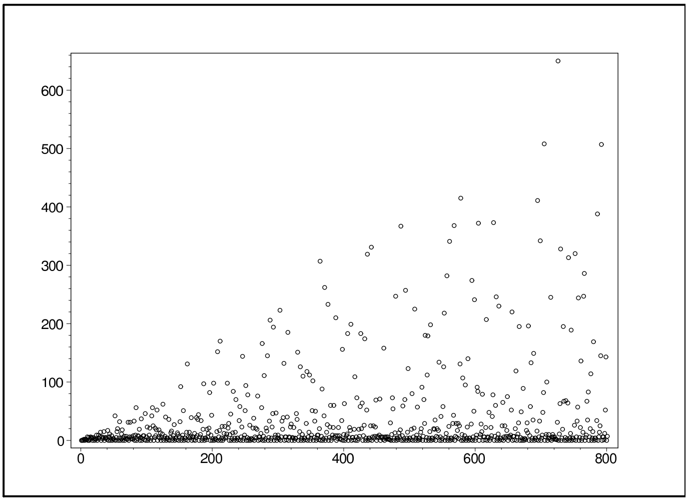


Figure 5: Scatter-plot of the first 800 terms of Van Eck's sequence A181391

**Theorem 8.2** (Van Eck, personal communication)**.** *The sequence contains infinitely many* 0*'s.*

*Proof.* Suppose, seeking a contradiction, that there are only finite number of 0's in the sequence. Then after a certain point no new terms can appear, so the sequence is bounded. Let $M$ be the largest term. This means that any block of $M$ successive terms determines the sequence. But there are only $M^M$ different possible blocks. So a block must repeat and the sequence is eventually periodic. Furthermore, the period cannot contain a 0.

Suppose the period has length $p$, and starts at term $r$, with $a(r) = x, \ldots, a(r + p - 1) = z, a(r + p) = x, \ldots$. There is another $z$ after $q \leq p$ steps, which is immediately followed by $q$. But this $q$ implies that $a(r - 1) = z$. Therefore the periodic part really began at step $r - 1$.

Repeating this argument shows that the periodic part starts at $a(1)$. But $a(1) = 0$, and the periodic part cannot contain a 0. Contradiction. □

It would be nice to know more about this fascinating sequence!

# 9. Conclusion

I will end with a few general remarks.

- The OEIS needs more editors. If you are interested in helping, please write to me or one of the other Editors-in-Chief. There are no formal duties, everything is done on a volunteer basis, and you will get to see a lot of interesting new problems.
- Everyone should register with the OEIS – see Sect. 3.
- If you write a paper that mentions a sequence in the OEIS, please do two things. Add it to the list of papers that cite the OEIS – see Sec. 2, and add a reference pointing to your paper to any entries in the OEIS that it mentions.
- The same thing if you come across a sequence in your work, in the library, or on a web site: send it in to the OEIS if it is missing (it need not be your own sequence – just mention the source) or add a reference to the source if it is already in the OEIS. It is these cross-connections that make the database so valuable.

# References

[1] J. Ayres, Sequence A064413 in [15], Sept. 30, 2001.

[2] G. Back and M. Caragiu, The greatest prime factor and recurrent sequences, *Fib. Quart.*, **48** (2010), 358–362.

[3] J. M. Borwein, D. H. Bailey and R. Girgensohn, *Experimentation in Mathematics*, A K Peters, Natick, MA, 2004.

[4] J. M. Borwein, P. B. Borwein and K. Dilcher, Pi, Euler numbers and asymptotic expansions, *Amer. Math. Monthly*, **96** (1989), 681–687.

[5] J. M. Borwein and R. M. Corless, Review of "An Encyclopedia of Integer Sequences" by N. J. A. Sloane and Simon Plouffe, *SIAM Review*, **38** (1996), 333–337.

[6] F. J. van der Bult, D. C. Gijswijt, J. P. Linderman, N. J. A. Sloane and A. R. Wilks, A slow-growing sequence defined by an unusual recurrence, *J. Integer Sequences*, **10** (2007), Article 07.1.2.

[7] B. Chaffin, J. P. Linderman, N. J. A. Sloane and A. R. Wilks, On curling numbers of integer sequences, Preprint, 2012; arXiv:1212.6102.

[8] E. Deutsch and B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Num. Theory*, **117** (2006), 191–215.

[9] J. R. van Eck, Sequence A181391 in [15], Oct. 17, 2010.

[10] D. Gijswijt, Sequence A090822 in [15], Feb. 27, 2004.

[11] P. Hofman and M. Pilipczuk, A few new facts about the EKG sequence, *J. Integer Sequences*, **11** (2008), Article 08.4.2.

[12] R. Kelly, Sequence A214551 in [15], July 20, 2012.

[13] J. C. Lagarias, E. M. Rains and N. J. A. Sloane, The EKG sequence, *Experimental Math.*, **11** (2002), 437–446.

[14] B. D. McKay, F. E. Oggier, N. J. A. Sloane, G. F. Royle, I. M. Wanless and H. S. Wilf, *J. Integer Sequences*, **7** (2004), Article 04.3.3.

[15] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`, 2012.

[16] N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, NY, 1973.

[17] N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, 1995.

[18] D. Zagier, The first 50 million prime numbers, *Math. Intelligencer*, **0** (1977), 7–19.