

# On the sum of divisors function of linearly recurrent sequences

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**Abstract.** Let  $k \geq 2$  be an integer and  $\sigma_k(n)$  be the sum of the  $k$ th powers of the positive divisors of  $n$ . In this paper, we prove that if  $(U_n)_{n \geq 1}$  is any nondegenerate linearly recurrent sequence of integers whose general term is up to sign not a polynomial in  $n$ , then the inequality  $\sigma_k(|U_n|) \leq |U_{\sigma_k(n)}|$  holds for all  $n$  sufficiently large, and the inequality  $\sigma(|U_{n^k}|) \leq |U_{\sigma_k(n)}|$  holds for almost all natural numbers  $n$ , where  $\sigma(n) = \sigma_1(n)$ . In fact, the second inequality fails on a set of  $n \leq x$  of counting function  $O(x/\log x)$ .

*Keywords:* linear recurrence sequences, sum of divisors function

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## 1. Introduction

Let  $\mathbf{U} := (U_n)_{n \geq 1}$  be a linearly recurrent sequence of integers. Such a sequence satisfies a recurrence of the form

$$U_{n+d} = a_1 U_{n+d-1} + \cdots + a_d U_n \quad \text{for all } n \geq 1$$

with integers  $a_1, \dots, a_d$ , where  $U_1, \dots, U_d$  are integers. Assuming  $d$  is minimal,  $U_n$  can be represented as

$$U_n = \sum_{i=1}^s P_i(n) \alpha_i^n, \tag{1.1}$$

where

$$\Psi(X) := X^d - a_1 X^{d-1} - \cdots - a_d = \prod_{i=1}^s (X - \alpha_i)^{\nu_i}$$

is the characteristic polynomial of  $\mathbf{U}$ ,  $\alpha_1, \dots, \alpha_s$  are the distinct roots of  $\Psi(X)$  with multiplicities  $\nu_1, \dots, \nu_s$ , respectively, and  $P_i(X)$  is a polynomial of degree  $\nu_i - 1$  with coefficients in  $\mathbb{Q}(\alpha_i)$ . The sequence is nondegenerate if  $\alpha_i/\alpha_j$  is not a root of 1 for any  $i \neq j$  in  $\{1, \dots, s\}$ .

Recently (see [4]), we proved that if  $(U_n)_{n \geq 1}$  is any nondegenerate linearly recurrent sequence of integers such that  $|U_n|$  is not a polynomial in  $n$  for large values of  $n$ , then the inequalities

$$\phi(|U_n|) \geq |U_{\phi(n)}| \quad \text{and} \quad \sigma(|U_n|) \leq |U_{\sigma(n)}|$$

hold for almost all natural numbers  $n$ . In fact, we showed that the set of positive integers  $n \leq x$  for which the above inequalities fail has counting function  $O(x/\log x)$ .

Let  $\sigma_k(n)$  be the sum of the  $k$ th powers of the positive divisors of  $n$ . We denote  $\sigma_1(n)$  as  $\sigma(n)$ . In this paper, we continue where we left off in the last paper [4] by considering the inequalities of the arithmetic function  $\sigma_k(n)$  for  $k \geq 2$ .

In this paper, we prove the following theorems. Recall that if  $f(x)$  and  $g(x)$  are functions defined on  $\mathbb{R}_+$  with values in  $\mathbb{R}_+$  we write  $f(x) = O(g(x))$  and  $f(x) = o(g(x))$  if the inequality  $f(x) < Kg(x)$  holds with some constant  $K > 0$  and all  $x > x_0$ , and  $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$ , respectively. Further, the notations  $f(x) \ll g(x)$  and  $g(x) \gg f(x)$  are equivalent to  $f(x) = O(g(x))$ .

**Theorem 1.1.** *Let  $k \geq 2$  be an integer. Let  $\mathbf{U} := (U_n)_{n \geq 1}$  be a nondegenerate linearly recurrent sequence of integers such that  $|U_n|$  is not a polynomial in  $n$  for all large  $n$  and let  $x$  be a large real number. Then the inequality*

$$\sigma_k(|U_n|) \leq |U_{\sigma_k(n)}|$$

*holds for all  $n$  sufficiently large.*

**Theorem 1.2.** *Let  $k \geq 1$  be an integer. Let  $\mathbf{U} := (U_n)_{n \geq 1}$  be a nondegenerate linearly recurrent sequence of integers such that  $|U_n|$  is not a polynomial in  $n$  for all large  $n$  and let  $x$  be a large real number. Then the inequality*

$$\sigma(|U_{n^k}|) \leq |U_{\sigma_k(n)}|$$

*holds for almost all positive integers  $n$ . In fact, the set of positive integers  $n \leq x$  for which it fails is of cardinality  $O(x/\log x)$ .*

## 2. Preliminary results

The following lemma follows from Theorem 323 in [3].

**Lemma 2.1.** *Let  $n \geq 3$ . We then have*

$$\sigma(n) \ll n \log \log n.$$

The following lemma follows from a result on page 116 in [6].

**Lemma 2.2.** *Let  $k \geq 2$ . For all  $n \geq 1$ , we have that*

$$\sigma_k(n) \leq 2n^k.$$

For a positive integer  $n$  put  $p(n)$  to be the smallest prime factor of  $n$  with the convention that  $p(1) = 1$ . For  $x \geq y \geq 2$  put

$$\Phi(x, y) := \#\{n \leq x : p(n) > y\}.$$

The following inequality is a consequence of the Brun sieve and appears, for example, as an Exercise on page 11 in [2].

**Lemma 2.3.** *We have uniformly for  $x \geq y \geq 2$ ,*

$$\Phi(x, y) \ll \frac{x}{\log y}.$$

Let  $\Omega(n)$  be the total number of prime factors of  $n$  counting multiplicities.

**Lemma 2.4.** *Let  $x \geq 10$ . The number of positive integers  $n \leq x$  such that  $\Omega(n) \geq 10 \log \log x$  is  $O(x/(\log x)^2)$ .*

**Proof.** See Lemma 3 in [4]. □

Let  $\tau(n)$  be the total number of divisors of  $n$ .

**Lemma 2.5.** *Let  $x \geq 10$  and  $k \geq 1$  be an integer. The number of positive integers  $n \leq x$  such that  $\tau(\sigma_k(n)) > \exp(\sqrt{\log x})$  is  $O(x/(\log x)^2)$ .*

**Proof.** The remarks on page 128 in [5] show that

$$\sum_{n \leq x} \tau(\sigma_k(n)) = x \exp \left( c(x) \left( \frac{\log x}{\log \log x} \right)^{1/2} \left( 1 + O \left( \frac{\log \log \log x}{\log \log x} \right) \right) \right),$$

holds with  $c(x) \in [\alpha, \beta]$ , where  $\alpha$  and  $\beta$  are constants. Then following the proof of Lemma 4 in [4] concludes the proof. □

**Lemma 2.6.** *Let  $\mathbf{U} := (U_n)_{n \geq 1}$  be a nondegenerate linearly recurrent sequence of integers whose general term is given by (1.1) with  $s \geq 2$  and assume that  $|\alpha_1| = \max\{|\alpha_i| : i = 1, \dots, s\} > 1$ . Then for every  $\epsilon \in (0, 1)$ , there exist constants  $x_0$  and  $c := c(\mathbf{U}) \in (0, \epsilon)$  such that for all  $x \geq x_0$ , the number of  $n \leq x$  such that*

$$|U_n| \leq |\alpha_1|^{n(1-\delta)},$$

*with  $\delta := x^{-c}$  is of cardinality  $O(x^\epsilon)$ .*

The proof of the above lemma is similar to the proof of Lemma 5 in [4], by considering the  $n \in (x^\epsilon, x]$  because there are only  $O(x^\epsilon)$  positive integers  $n \leq x^\epsilon$  and by taking  $c := \epsilon/7sr \in (0, \epsilon)$ , where  $r$  is as defined in [4].

### 3. The proof of Theorem 1.1

Let  $k \geq 2$  be an integer. Assume that  $|\alpha_1| \geq \dots \geq |\alpha_s|$  and  $|U_n|$  is not a polynomial in  $n$  for large  $n$ . In particular,  $|\alpha_1| > 1$ . By an application of the subspace theorem (see page 229 in [1]), we have that

$$|U_{\sigma_k(n)}| \geq |\alpha_1|^{\sigma_k(n)/2}$$

holds for all  $n \geq n_0$ , for some constant  $n_0$ . By the fact that  $\sigma_k(n) \geq n^k$ , the above inequality becomes

$$|U_{\sigma_k(n)}| \geq |\alpha_1|^{n^k/2}$$

for all  $n \geq n_0$ . On the other hand, using (1.1) and the triangle inequality, we have that

$$|U_n| = \left| \sum_{i=1}^s P_i(n) \alpha_i^n \right| \ll n^\beta |\alpha_1|^n,$$

where  $\beta := \max\{\nu_i - 1 : i = 1, \dots, s\}$  and  $\nu_i - 1$  is the degree of the polynomial  $P_i(X)$ . Using Lemma 2.2 and the above inequality, we have that

$$\sigma_k(|U_n|) \leq 2|U_n|^k \ll n^{k\beta} |\alpha_1|^{kn}.$$

Let  $c_1$  be the constant implied by the  $\ll$  symbol in the above inequality. We claim that

$$c_1 n^{k\beta} |\alpha_1|^{kn} \leq |\alpha_1|^{n^k/2} \tag{3.1}$$

holds for all  $n \geq n_0$ , where  $n_0$  is some constant. Taking the logarithm on both sides, the above inequality is equivalent to

$$2kn + O(\log n) \leq n^k,$$

which holds for all  $n \geq n_0(k, \mathbf{U})$  sufficiently large because  $k \geq 2$ . Thus, (3.1) holds and hence,

$$\sigma_k(|U_n|) \leq |U_{\sigma_k(n)}|$$

holds for all  $n \geq n_0$ , where  $n_0$  is some constant depending both on  $k$  and on the sequence  $\mathbf{U}$ .

### 4. The proof of Theorem 1.2

Let us assume that  $|\alpha_1| \geq \dots \geq |\alpha_s|$  and  $|U_n|$  is not a polynomial in  $n$  for large  $n$ . In particular,  $|\alpha_1| > 1$ . We assume that  $k$  is an integer and  $k \geq 2$  because for the case  $k = 1$ , it is already proved (see the proof of Theorem 1 in [4]). For a positive integer  $n$  put  $p(n)$  be the smallest prime factor of  $n$  with the convention that  $p(1) = 1$ . Let

$$\mathcal{A}(x) = \{n \leq x : p(n)^k > x^{c_1}\},$$

where  $c_1 \in (0, 1/4k)$  is a constant to be determined later. Let

$$\mathcal{C}(x) := \{n \leq x : \sigma(|U_{n^k}|) > |U_{\sigma_k(n)}|\}.$$

We need to prove that the set  $\mathcal{C}(x)$  has counting function of  $O(x/\log x)$ . Let us split the set  $\mathcal{C}(x)$  into the following three subsets

$$\begin{aligned}\mathcal{C}_1(x) &:= \{n \leq x : n \in \mathcal{A}(x) \text{ and } n \in \mathcal{C}(x)\}, \\ \mathcal{C}_2(x) &:= \{n \leq x^{1/2k} : n \notin \mathcal{A}(x) \text{ and } n \in \mathcal{C}(x)\}, \\ \mathcal{C}_3(x) &:= \{n \in (x^{1/2k}, x] : n \notin \mathcal{A}(x) \text{ and } n \in \mathcal{C}(x)\}.\end{aligned}$$

The set  $\mathcal{C}_1(x)$  is a subset of  $\mathcal{A}(x)$ . By Lemma 2.3 with  $y := x^{c_1/k}$ , we have that

$$\#\mathcal{A}(x) = \Phi(x, y) \ll \frac{x}{\log y} \ll \frac{x}{\log x}.$$

Thus, the subset  $\mathcal{C}_1(x)$  has counting function of  $O(x/\log x)$ . The second subset has counting function of  $O(x^{1/2k}) = o(x/\log x)$  because the set  $\mathcal{C}_2(x)$  has at most  $x^{1/2k}$  positive integers  $n \leq x^{1/2k}$ . From now on, let  $n \in \mathcal{C}_3(x)$ .

Then  $n \in (x^{1/2k}, x]$  and  $p(n)^k \leq x^{c_1}$ . Since  $p(n)$  divides  $n$ , then  $(n/p(n))$  divides  $n$  as well. Thus, we have that

$$\sigma_k(n) \geq n^k + \frac{n^k}{p(n)^k} \geq n^k + \frac{n^k}{x^{c_1}} = n^k(1 + \delta),$$

where  $\delta := 1/x^{c_1}$ . The above inequality gives

$$n^k \leq \frac{\sigma_k(n)}{1 + \delta} \leq \sigma_k(n)(1 - \delta_1),$$

where  $\delta_1 := \delta/2$ . Let

$$U_n = \sum_{i=1}^s P_i(n) \alpha_i^n.$$

The case  $s = 1$  needs to be treated separately as there are only finitely many  $n$ . Assume that  $s = 1$ . In this case  $\Psi(X) = (X - \alpha_1)^d$  and  $\alpha_1$  is an integer with  $|\alpha_1| \geq 2$ . Thus,

$$U_n = P_1(n) \alpha_1^n,$$

where  $P_1(X) \in \mathbb{Z}[X]$ . Let  $n$  be large (say larger than the maximal real root of  $P_1(X)$ ). By Lemma 2.1, we have that

$$\begin{aligned}\sigma(|U_{n^k}|) &\ll |U_{n^k}| \log \log |U_{n^k}| \\ &\ll |P_1(n^k)| |\alpha_1|^{n^k} \log \log |P_1(n^k)| |\alpha_1|^{n^k} \\ &\ll n^{k(d-1)} |\alpha_1|^{\sigma_k(n)(1-\delta_1)} \log n.\end{aligned}\tag{4.1}$$

On the other hand, we have that

$$|U_{\sigma_k(n)}| = |P_1(\sigma_k(n))| |\alpha_1|^{\sigma_k(n)} \gg \sigma_k(n)^{d-1} |\alpha_1|^{\sigma_k(n)} \gg n^{k(d-1)} |\alpha_1|^{\sigma_k(n)}. \quad (4.2)$$

Since  $n \in \mathcal{C}_3(x)$ , then

$$|U_{\sigma_k(n)}| < \sigma(|U_{n^k}|).$$

Using the above inequality together with (4.1) and (4.2), we have that

$$n^{k(d-1)} |\alpha_1|^{\sigma_k(n)} \ll |U_{\sigma_k(n)}| < \sigma(|U_{n^k}|) \ll n^{k(d-1)} |\alpha_1|^{\sigma_k(n)(1-\delta_1)} \log n.$$

The above inequality leads to the following inequality

$$|\alpha_1|^{\delta_1 \sigma_k(n)} \ll \log n.$$

Let  $c_2$  be the constant implied by the  $\ll$  symbol in the above inequality. Then the above inequality is equivalent to

$$|\alpha_1|^{\delta_1 \sigma_k(n)} \leq c_2 \log n.$$

Taking the logarithm on both sides, this is equivalent to

$$\delta_1 \sigma_k(n) \log |\alpha_1| \leq \log c_2 + \log \log n.$$

Since  $n \in (x^{1/2k}, x]$ , then the right-hand side of the above inequality is  $O(\log \log x)$ . Since  $\delta_1 = 0.5\delta$ , then  $\delta_1 \sigma_k(n) \geq 0.5\delta n^k > 0.5x^{-c_1} x^{1/2} = 0.5x^\gamma$ , where  $\gamma := 0.5 - c_1 > 0.5 - 1/4k > 0$ . Thus, the left-hand side is  $\gg x^\gamma$ . Then above inequality becomes

$$x^\gamma \ll \delta_1 \sigma_k(n) \log |\alpha_1| \leq \log c_2 + \log \log n \ll \log \log x,$$

which does not hold for large  $n$  because  $\log \log x = o(x^\gamma)$ . Thus,  $x$  is bounded and so is  $n$ .

From now on, we assume that  $s \geq 2$ . Then the inequality

$$|\alpha_1|^{n^k/2} \leq |U_{n^k}| \ll n^{kd} |\alpha_1|^{n^k}$$

holds for all  $n \geq n_0$ , where  $n_0$  is some constant. The left-hand side is by an application of the subspace theorem (see page 229 in [1]) and the right-hand side is by triangle inequality. Thus, for  $n \geq n_0$ , we have that

$$\log \log |U_{n^k}| = \log n^k + O(1).$$

Using the above inequality, we have that

$$\begin{aligned} \sigma(|U_{n^k}|) &\ll |U_{n^k}| \log \log |U_{n^k}| \ll n^{kd} |\alpha_1|^{n^k} (\log n^k + O(1)) \\ &\leq \sigma_k(n)^d |\alpha_1|^{\sigma_k(n)(1-\delta_1)} (\log \sigma_k(n) + O(1)). \end{aligned}$$

Let  $c_3$  and  $c_4$  be constants implied by the  $\ll$  symbol and  $O(1)$  in the above inequality, respectively. We claim that with  $\delta_2 := 1/x^{2c_1}$  and  $m = \sigma_k(n)$ , the inequality

$$c_3 m^d |\alpha_1|^{m(1-\delta_1)} (\log m + c_4) < |\alpha_1|^{m(1-\delta_2)}$$

holds for all  $n \geq n_0$ , where  $n_0$  is some constant. Taking logarithm on both sides, the above inequality is equivalent to

$$\log c_3 + d \log m + \log(\log m + c_4) < m(\delta_1 - \delta_2) \log |\alpha_1|. \quad (4.3)$$

Since  $n \in (x^{1/2k}, x]$  and  $m = \sigma_k(n) \leq 2x^k$  (by Lemma 2.2), then the left-hand side of (4.3) is  $O(\log x)$ . Since  $\delta_1 = \delta/2$  and  $\delta_2 = \delta^2$ , then it follows that  $\delta_1 - \delta_2 \geq 0.25\delta = 0.25x^{-c_1}$  for  $x \geq x_0$ , for some constant  $x_0$ . Thus, the right-hand side of (4.3) is  $\gg x^\gamma$ , where  $\gamma := 0.5 - c_1 > 0$ . Thus, since  $\log x = o(x^\gamma)$ , then we have that

$$\log c_3 + d \log m + \log(\log m + c_4) \ll \log x \ll x^\gamma \ll m(\delta_1 - \delta_2) \log |\alpha_1|$$

holds for all  $n \geq n_0$ . Thus, inequality (4.3) holds for all  $n \geq n_0$  and hence,

$$|U_m| < |\alpha_1|^{m(1-\delta_2)}$$

holds for all  $x \geq x_0$ , for some constant  $x_0$ . Note that by Lemma 2.2 (since  $k \geq 2$ ),  $m = \sigma_k(n) \leq 2x^k$ . By Lemma 2.5, we can choose  $c_1 := c/2$  with  $\epsilon = 1/2k$  and then the set of  $m \leq 2x^k$  satisfying the above inequality is of cardinality

$$O(\sqrt{x}).$$

But this is only an upper bound on the number of distinct values of  $m = \sigma_k(n)$  and we have to get the upper bound on the number of  $n$ 's themselves. By Lemmas 2.3 and 2.4, we may assume that  $\Omega(n) \leq 10 \log \log x$  and  $\tau(\sigma_k(n)) \leq \exp(\sqrt{\log x})$  since the number of  $n \leq x$  for which one of the above inequalities fails is  $O(x/(\log x)^2) = o(x/\log x)$ . Writing

$$n = p_1^{a_1} \cdots p_\ell^{a_\ell}$$

with distinct primes  $p_1, \dots, p_\ell$  and positive exponents  $a_1, \dots, a_\ell$ , so

$$\sigma_k(n) = \prod_{i=1}^{\ell} \left( \frac{p_i^{k(a_i+1)} - 1}{p_i^k - 1} \right).$$

Given  $m = \sigma_k(n)$ , each of  $d_i := (p_i^{k(a_i+1)} - 1)/(p_i^k - 1)$  is a divisor of  $\sigma_k(n)$ . Additionally, given  $d_i$  and also  $a_i$ ,  $p_i$  is uniquely determined. Thus, since  $d_i$  can be fixed in at most  $\tau(\sigma_k(n))$  ways and  $a_i \leq \Omega(n)$  can be fixed in at most  $\Omega(n)$  ways, it follows that  $p_i^{a_i}$  can be fixed in at most  $\tau(\sigma_k(n))\Omega(n)$  ways. This is so for a fixed  $i$ , but  $i \leq \ell = \omega(n) \leq \Omega(n)$ . Thus, the number of such  $n$  when given  $\sigma_k(n)$  and  $\Omega(n)$  is at most

$$\left( (10 \log \log x) \exp(\sqrt{\log x}) \right)^{10 \log \log x} < \exp \left( 20(\log \log x) \sqrt{\log x} \right)$$

for all  $x \geq x_0$ , for some constant  $x_0$ . Now varying  $\Omega(n)$  up to  $10 \log \log x$  and  $\sigma_k(n)$  up to  $O(\sqrt{x})$ , we get that the number of possible  $n \leq x$  is

$$\ll \sqrt{x}(\log \log x) \exp \left( 20(\log \log x) \sqrt{\log x} \right) = o(x/\log x)$$

as  $x \rightarrow \infty$ . Hence, the set  $\mathcal{C}_3(x)$  has counting function  $o(x/\log x)$  and thus the set  $\mathcal{C}(x)$  has a counting function of  $O(x/\log x)$ . This concludes the proof.

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