ANNALES MATHEMATICAE ET INFORMATICAE

VOLUME 55. (2022)

INSTITUTE OF MATHEMATICS AND INFORMATICS
ESZTERHÁZY KÁROLY CATHOLIC UNIVERSITY
HUNGARY, EGER

HU ISSN 1787-6117 (Online)

A kiadásért felelős az
Eszterházy Károly Katolikus Egyetem rektora
Megjelent a Líceum Kiadó gondozásában
Kiadóvezető: Dr. Nagy Andor
Műszaki szerkesztő: Dr. Tómács Tibor
Megjelent: 2022. december

Contents

Research papers

Methodological papers

Annales Mathematicae et Informaticae
55 (2022) pp. 1–11
DOI: https://doi.org/10.33039/ami.2022.10.002
URL: https://ami.uni-eszterhazy.hu

Differential geometry properties by using the perturbation methods

Nadhir Bendrici[a], **Malika Boumedien-Zidani**[b]

[a]Laboratory of dynamical systems, University of Science and technology Houari Boumediene, Bab Ezzouar, Algeria
nbendrici@usthb.dz

[b]Laboratory of Decision supported by operational research, computer and statistical technics, University of science and technology Houari Boumediene, Bab Ezzouar, Algeria
mboumedien@usthb.dz

Abstract. In this paper, we present a new method to evaluate the moving Frenet frame along the intersection curves of two parametric surfaces in the tangential intersection situations. To resolve such situation, we have combined a perturbation method with a classical method that works in transversal intersection situation. Unlike the existent methods, our method works even if the order contact of the point is more than one.

Keywords: intersection of surfaces, parametric curve, perturbation methods, differential geometry properties, tangential intersection.

AMS Subject Classification: 65D17

1. Introduction

Tracing the intersection curves of two parametric surfaces is a very important subject in computer-aided geometric design applications, manufacturing as well as in many other industrial fields [4, 5, 7, 8]. The intersection between two parametric surfaces can be either an empty set, several isolated points, a set of curves, a set of patches or a mixture of the aforementioned items. We focused on the case when the two surfaces meet at a collection of curves, and we shall assume that the two parametric surfaces are both so smooth enough that all the derivatives and partial derivatives are given in this paper existent and continuous. There are different ways to resolve this problem. Nevertheless, the most widely used one is the marching method because its algorithm is relatively simple and yet robust, efficient and

Submitted: December 24, 2021
Accepted: October 24, 2022
Published online: November 3, 2022

the required accuracy can be realized by choosing an appropriate step size. The marching methods consist generally of two basic steps:

1. Finding a starting point for each intersection curve.

2. Tracing the intersection curves from these points by using the differential geometry properties (tangent vector, normal, and binormal vectors).

In this manuscript, we will care about the computation of the tangent vector, normal vector and the binormal vectors along an intersection curve of two parametric surfaces. Many papers study this problem in the transversal intersection situations like [1, 12], but only few among them tackled the tangential intersection situations. An intersection point is considered a tangential intersection point if normal vectors of the surfaces are parallel at this point. Among the papers that suggest a solution for the tangential intersection problem is the one proposed by B. U. Düldül and M. Düldül in [2]. They combined the Rodrigues' rotation formula that used in [11] with a new proposed operator so as to determine the differential geometry properties. Their algorithm does not work if the order contact of the point is more than one. Unlike the classical marching method, another approach to trace the intersection curves without using the differential geometry properties was proposed by Grandine and Klein in [3]. This method consists of two steps. Firstly, we have to determine the topology of each intersection curve (starting and ending point). Secondly, we use the first step to trace the intersection curves(boundary value problem). However, it doesn't work in many transversal and tangential situations. This method was improved later by Oh et al. in [6]. They used the perturbation methods to determine the topology of the curves in the tangential intersection situations.

We will suggest in this paper a new method to find the moving Frenet frame at a tangential intersection point. Our method combines the perturbations methods used in [6, 9, 10] with the result proposed by X. Ye and T. Maekawa in [12]. Our solution works even if order contact of the point is at least two.

The paper is organized as follows: in the second section, we will present the definitions and the existent results we need in our work. The proposed method will be presented in section 3. The fourth section is the implementation of our method. The conclusion with future works contained in the fifth section.

2. Review of differential geometry

This section will describe the differential geometry properties along a parametric curve and a parametric surface. Then, we will briefly depict a classical method suggested to evaluate the moving Frenet frame along the intersection curves of two parametric surfaces in transversal intersection situations.

2.1. Differential geometry properties along parametric curves and parametric surfaces

This subsection will mention the differential geometry properties along a parametric curve and also for a parametric surface. For more details, we refer to [2, 12].

Let us consider $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $s, t, u, v \in \mathbb{R}$ two regular parametric surfaces in $\mathbb{R}^3$. A surface is regular if its first partial derivatives are not parallel at each point on this surface. Let us consider also $\mathbf{c}(\tau)$, where $\tau \in \mathbb{R}$ an intersection curve between $\mathbf{f}$ and $\mathbf{g}$ with arc-length parametrization. From the elementary differential geometry, $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ is the moving Frenet frame along the intersection curve $\mathbf{c}$, then, we have

$$\begin{aligned} \mathbf{T} &= \mathbf{c}'(\tau). \\ \mathbf{K} &= \mathbf{c}''(\tau) = \kappa \cdot \mathbf{n}. \end{aligned} \tag{2.1}$$

The binormal vector of $\mathbf{c}$ defined as:

$$\mathbf{b} = \mathbf{T} \times \mathbf{n}, \tag{2.2}$$

where τ is the curve parameter; $\mathbf{T}$ is the unit tangent vector, $\mathbf{n}$ is the unit normal vector and $\mathbf{b}$ is the unit binormal vector along the curve $\mathbf{c}$; κ is the curvature of the curve $\mathbf{c}$ and $\mathbf{K}$ is the curvature vector.

On the other hand, since $\mathbf{c}(\tau)$ lies on both $\mathbf{f}$ and $\mathbf{g}$, we may write:

$$\mathbf{c}(\tau) = \mathbf{f}(u(\tau), v(\tau)) = \mathbf{g}(s(\tau), t(\tau)).$$

Then

$$\begin{aligned} \mathbf{c}'(\tau) &= u' \cdot \mathbf{f}_u + v' \cdot \mathbf{f}_v = s' \cdot \mathbf{g}_s + t' \cdot \mathbf{g}_t. \\ \mathbf{c}''(\tau) &= u'' \cdot \mathbf{f}_u + v'' \cdot \mathbf{f}_v + (v')^2 \cdot \mathbf{f}_{vv} + (u')^2 \cdot \mathbf{f}_{uu} + 2u' \cdot v' \cdot \mathbf{f}_{uv}, \end{aligned} \tag{2.3}$$

where

$$\mathbf{f}_u = \frac{\partial \mathbf{f}(u, v)}{\partial u}, \quad \mathbf{f}_v = \frac{\partial \mathbf{f}(u, v)}{\partial v}, \quad \mathbf{g}_s = \frac{\partial \mathbf{g}(s, t)}{\partial s}, \quad \mathbf{g}_t = \frac{\partial \mathbf{g}(s, t)}{\partial t},$$
$$\mathbf{f}_{uu} = \frac{\partial^2 \mathbf{f}(u, v)}{(\partial u)^2}, \quad \mathbf{f}_{vv} = \frac{\partial^2 \mathbf{f}(u, v)}{(\partial v)^2}, \quad \mathbf{f}_{uv} = \frac{\partial^2 \mathbf{f}(u, v)}{\partial u \partial v}.$$

Since $\mathbf{f}$ and $\mathbf{g}$ are both regular, we can define their normal vectors at a point P as:

$$\mathbf{N_f} = \frac{\mathbf{f_u} \times \mathbf{f_v}}{\|\mathbf{f_u} \times \mathbf{f_v}\|} \quad \text{and} \quad \mathbf{N_g} = \frac{\mathbf{g_s} \times \mathbf{g_t}}{\|\mathbf{g_s} \times \mathbf{g_t}\|}.$$

2.2. Differential geometry properties along an intersection curve of two parametric surfaces

In this subsection, we are going to elaborate the method suggested by X. Ye and T. Maekawa in [12] to evaluate $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ along an intersection curve $\mathbf{c}$ between

f and **g** in the transversal intersection situations. Two parametric surfaces meet transversally at a point P if $\mathbf{N_f}$ and $\mathbf{N_g}$ are not parallel at P.

Since $\mathbf{N_f}$ and $\mathbf{N_g}$ are linearly independent, we may find the tangent vector $\mathbf{T}$ at P by using the formula:

$$\mathbf{T} = \frac{\mathbf{N_f} \times \mathbf{N_g}}{\|\mathbf{N_f} \times \mathbf{N_g}\|}. \tag{2.4}$$

In order to evaluate $\mathbf{n}$ the normal vector at P, we need to compute u', v', s' and t' at P, by (2.3), we get:

$$u' = \frac{g_1 \cdot (\mathbf{T} \cdot \mathbf{f}_u) - f_1 \cdot (\mathbf{T} \cdot \mathbf{f}_v)}{e_1 \cdot g_1 - (f_1)^2}, \quad v' = \frac{e_1 \cdot (\mathbf{T} \cdot \mathbf{f}_v) - f_1 \cdot (\mathbf{T} \cdot \mathbf{f}_u)}{e_1 \cdot g_1 - (f_1)^2}, \tag{2.5}$$

$$s' = \frac{g_2 \cdot (\mathbf{T} \cdot \mathbf{g_s}) - f_2 \cdot (\mathbf{T} \cdot \mathbf{g_t})}{e_2 \cdot g_2 - (f_2)^2}, \quad t' = \frac{e_2 \cdot (\mathbf{T} \cdot \mathbf{g_t}) - f_2 \cdot (\mathbf{T} \cdot G_s)}{e_2 \cdot g_2 - (f_2)^2}, \tag{2.6}$$

where

$$e_1 = \mathbf{f_u} \cdot \mathbf{f_u}, f_1 = \mathbf{f}_v \cdot \mathbf{f}_u, g_1 = \mathbf{f}_v \cdot \mathbf{f}_v, e_2 = \mathbf{g}_s \cdot \mathbf{g}_s, f_2 = \mathbf{g}_s \cdot \mathbf{g}_t \quad \text{and} \quad g_2 = \mathbf{g}_t \cdot \mathbf{g}_t.$$

The curvature $\mathbf{K}$ lies in the normal plane of the intersection point. Hence, there are two reals a and b such that:

$$\mathbf{K} = a \cdot \mathbf{N_f} + b \cdot \mathbf{N_g}.$$

Then, if we denote by $\kappa_\mathbf{n}^\mathbf{f} = \mathbf{K} \cdot \mathbf{N_f}$ and $\kappa_\mathbf{n}^\mathbf{g} = \mathbf{K} \cdot \mathbf{N_g}$, we obtain:

$$\kappa_n^\mathbf{f} = a + b \cdot \cos(\theta). \tag{2.7}$$

$$\kappa_n^\mathbf{g} = a \cdot \cos(\theta) + b, \tag{2.8}$$

where $\cos(\theta) = \mathbf{N_f} \cdot \mathbf{N_g}$.

Now, we can evaluate a and b by (2.7) and (2.8). Hence, we may obtain:

$$\mathbf{K} = \frac{1}{\sin^2(\theta)} \left[\left(\kappa_n^\mathbf{f} - \kappa_n^\mathbf{g} \cos(\theta) \right) \mathbf{N_f} + \left(\kappa_n^\mathbf{g} - \kappa_n^\mathbf{f} \cos(\theta) \right) \mathbf{N_g} \right].$$

We note that $\kappa_n^\mathbf{f}$ and $\kappa_n^\mathbf{g}$ can be evaluated by (2.1), (2.5) and (2.6).

If $\kappa \neq 0$, we can evaluate now $\mathbf{n}$ by the formula:

$$\mathbf{n} = \frac{\mathbf{K}}{\|\mathbf{K}\|}. \tag{2.9}$$

To evaluate the binormal vector $\mathbf{b}$ at P, we use the formula (2.2).

2.3. Tangential intersection situations

In this subsection, we are going to display the weakness of the previous expressions in tangential intersection situations. See [2].

Let us consider P a tangential intersection point between **f** and **g**, then the two vectors $\mathbf{N_f}(P)$ and $\mathbf{N_g}(P)$ are parallel. Hence, we can not use (2.4), (2.2) and (2.9) to compute the moving Frenet frame. We have to suggest a new method to evaluate $\mathbf{T}$, $\mathbf{n}$ and $\mathbf{b}$ in the tangential intersection points.

3. The presentation of our method

In this section, we are going to present a more general method to find $\mathbf{T}$, $\mathbf{n}$ and $\mathbf{b}$ in the tangential intersection situations. We will use an analytic perturbation method that transforms tangential intersection situations into the transversal intersection situation.

Let $P = \mathbf{f}(u_0, v_0) = \mathbf{g}(s_0, t_0)$ be a tangential intersection point lying on an intersection curve $\mathbf{l}$ between $\mathbf{f}$ and $\mathbf{g}$. Then, we have two possibilities:

1. P is a singular point ($\mathbf{f}$ and $\mathbf{g}$ meet transversally in a small neighborhood of P).

2. P lies on a tangential intersection curve $\mathbf{l}' \subset \mathbf{l}$ ($\mathbf{f}$ and $\mathbf{g}$ meet tangentially along $\mathbf{l}'$).

We aim at suggesting an idea that allows us to evaluate the moving Frenet frame $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ using the ideas proposed to resolve the transversal intersection situation.

Firstly, we are going to suggest a method to evaluate $(\mathbf{T}, \mathbf{n}, \mathbf{b})$ when P is a singular point.

As the transversality propriety is both stable and generic on an intersection, we can perturb analytically the point P by considering a new point $P(\varsigma) \in \mathbf{l}$ defined as:

$$P(\varsigma) = \mathbf{f}(u_0 + \varsigma, v_0(\varsigma)) = \mathbf{g}(s_0(\varsigma), t_0(\varsigma)), \tag{3.1}$$

where $\varsigma \neq 0$ is appropriate and small so that the two following conditions are satisfied:

1. For every $\varsigma' \in [0, |\varsigma|]$, the point $P(\varsigma')$ is a transversal intersection point.

2. $\lim_{\varsigma' \to 0} P(\varsigma') = P$.

The former condition allows us to compute the tangent vector at $P(\varsigma')$. The latter condition allows us to compute the tangent vector at P as the limit of the tangent at $P(\varsigma')$ when $\varsigma' \to 0$. For more information about the choice of ς see [6].

The point $P(\varsigma)$ exists because P is a singular point.

As $P(\varsigma')$ is a transversal intersection point, we can evaluate the tangent vector $\mathbf{T}(\varsigma') = \mathbf{T}(P(\varsigma'))$ in term of ς' by the formula:

$$\mathbf{T}(\varsigma') = \frac{\mathbf{N_f}(P(\varsigma')) \times \mathbf{N_g}(P(\varsigma'))}{\|\mathbf{N_f}(P(\varsigma')) \times \mathbf{N_g}(P(\varsigma'))\|}.$$

As the perturbation is analytic with respect ς', also the vector $\mathbf{T}(\varsigma')$ vary continuously in a small neighborhood of $\varsigma = 0$, then, $\mathbf{T}$ is given by:

$$\mathbf{T} = \lim_{\varsigma' \to 0} \mathbf{T}(\varsigma'). \tag{3.2}$$

Now, we can evaluate $u'(\varsigma'), v'(\varsigma'), s'(\varsigma')$ and $t'(\varsigma')$ at $P(\varsigma')$ in term of ς' by (2.5) and (2.6). Hence we may evaluate $\mathbf{K}(\varsigma') = \mathbf{K}(P(\varsigma'))$ in term of ς'. Therefore,

$$\mathbf{n} = \lim_{\varsigma' \to 0} \frac{\mathbf{K}(\varsigma')}{\|\mathbf{K}(\varsigma')\|}, \tag{3.3}$$

because the perturbation is analytic with respect ς' and $\mathbf{K}(\varsigma')$ vary continuously in a small neighborhood of $\varsigma = 0$.

Finally, we can evaluate $\mathbf{b}$ directly by the formula (2.2).

Remark 3.1. If P is a branch point, we have to find all the points P_i which verify the conditions above.

Secondly, we are going to present a method to evaluate $\mathbf{T}$, $\mathbf{n}$ and $\mathbf{b}$ when P belongs to a tangential intersection curve.

Consider $P = \mathbf{f}(u_0, v_0) = \mathbf{g}(s_0, t_0)$ a tangential intersection point lying on a tangential intersection curve $\mathbf{l}'$, then, P is not a singular point. Therefore, we can not use the same idea proposed to resolve the singular point situation, because in a small neighborhood of P, all the points are tangential. Since the transversality is both stable and generic on an intersection, we can perturb $\mathbf{f}$ analytically by a small and appropriate $\varsigma \neq 0$. Then, the curve $\mathbf{l}'$ will be replaced by two new curves $\mathbf{l}_1(\varsigma)$ and $\mathbf{l}_2(\varsigma)$. In order to evaluate the moving frame at P, we have to find thz points $P(\varsigma) = P(u_0, v(\varsigma), s(\varsigma), t(\varsigma))$ such that:

$$\mathbf{f}(u_0, v(\varsigma), \varsigma) = \mathbf{g}(s(\varsigma), t(\varsigma)). \tag{3.4}$$

Suppose that we get two points $P_1(\varsigma) \in \mathbf{l}_1(\varsigma)$ and $P_2(\varsigma) \in \mathbf{l}_2(\varsigma)$ which verify:

1. $P_1(\varsigma')$ and $P_2(\varsigma')$ are transversal intersection points for every $\varsigma' \in [0, |\varsigma|]$.

2. $\lim_{\varsigma \to 0} P_1(\varsigma) = \lim_{\varsigma \to 0} P_2(\varsigma) = P$.

We will take one of them ($P_1(\varsigma)$ for example) and evaluate $\mathbf{T}$, $\mathbf{n}$ and $\mathbf{b}$ by (3.2), (3.3) and (2.2) like the former situation.

Remark 3.2. The perturbation that we used must conserve all the intersection properties in a small neighborhood of P.

Remark 3.3. If (3.4) has no solution to verify the aforementioned conditions, we have to replace the equation $u = u_0$ by $v = v_0$.

Now, we will give some illustrative examples.

4. Examples

In this section, we will implement our method on some examples.

4.1. Example 1

In the first example, we will evaluate the moving Frenet frame at a singular intersection point.

Consider the intersection between the two parametric surfaces $\mathbf{f}(u,v)$ and $\mathbf{g}(s,t)$, where $u,v,s,t \in \mathbb{R}$ that are defined as:

$$\mathbf{f}(u,v) = [\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)].$$
$$\mathbf{g}(s,t) = [0.5\cos(s) + 0.5, 0.5\sin(s), t].$$

(See the Figure 1.)

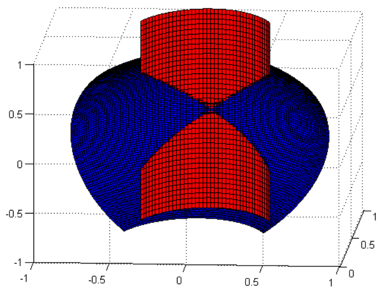


Figure 1. Branch point.

We want to determine the tangent $\mathbf{T}$, the normal $\mathbf{n}$ and the binormal $\mathbf{b}$ at the point $P = \mathbf{f}(0,0) = \mathbf{g}(0,0)$. P is tangential intersection point because

$$\mathbf{N}_f(P) = \mathbf{N}_g(P) = (1,0,0).$$

If we select a small and appropriate ς and by (3.1), we get two points $P_1(\varsigma)$ and $P_2(\varsigma)$ where:

$$P_1(\varsigma) = \mathbf{f}\left(2\arctan\left(\frac{\sqrt{1-\sqrt{\cos(\varsigma)}}}{\cos(\varsigma)+1}\right), \varsigma\right).$$
$$P_2(\varsigma) = \mathbf{f}\left(-2\arctan\left(\frac{\sqrt{1-\sqrt{\cos(\varsigma)}}}{\cos(\varsigma)+1}\right), \varsigma\right).$$

For every ς' $(0 < |\varsigma'| \leq |\varsigma|)$, the two surfaces $\mathbf{f}$ and $\mathbf{g}$ meet transversally at $P_1(\varsigma')$ and $P_1(\varsigma')$. We have $\lim\limits_{\varsigma' \to 0} P_1(\varsigma') = \lim\limits_{\varsigma' \to 0} P_2(\varsigma') = P$. Thus, by (3.2) we get:

$$\mathbf{T}_1 = (0, 0.7071, 0, 7071) \quad \text{and} \quad \mathbf{T}_2 = (0, -0.7071, 0, 7071).$$

Hence,

$$\mathbf{n} = (-1, 0, 0).$$

Thus, by (2.2) we obtain:

$$\mathbf{b}_1 = (0, -0.7071, 0, 7071) \text{ and } \mathbf{b}_2 = (0, 0.7071, 0, 7071).$$

Remark 4.1. We note that we have two tangent vectors at P, then, this point is a branch point.

4.2. Example 2

In the second example, we will implement our method on a second order contact point (cusp point).

Consider the intersection between the two parametric surfaces $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $u, v, s, t \in \mathbb{R}$ that are defined as:

$$\mathbf{f}(u, v) = \left[u, v, u^3 - v^2\right].$$
$$\mathbf{g}(u, v) = [s, t, 0].$$

(See the Figure 2.)

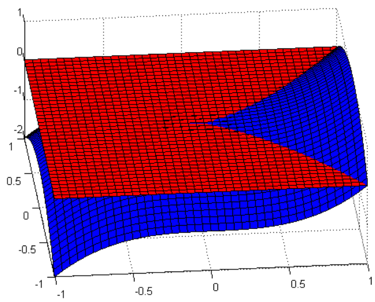


Figure 2. Cusp point.

The point $P = \mathbf{f}(0, 0) = \mathbf{g}(0, 0)$ is a cusp point, then, it is a second order contact point. Since P is a singular point, we use the point

$$P(\varsigma) = \mathbf{f}\left(\sqrt[3]{\varsigma^2}, \varsigma\right) = \mathbf{g}\left(\sqrt[3]{\varsigma^2}, \varsigma\right),$$

where ς is small and appropriate.

For every ς' verifies $0 < |\varsigma'| \leq |\varsigma|$ the two surfaces $\mathbf{f}$ and $\mathbf{g}$ meet transversally at $P(\varsigma')$ and $\lim_{\varsigma' \to 0} P(\varsigma') = P$.

By (3.2), we get:

$$\mathbf{T} = (1, 0, 0).$$

Hence, by (3.3) we obtain:

$$\mathbf{n} = (0, 1, 0).$$

Thus, by (2.2) we get:

$$\mathbf{b} = (0, 0, 1).$$

Now, we are going to implement our method at a point lying on a tangential intersection curve.

4.3. Example 3

Consider the intersection between $\mathbf{f}(u, v)$ and $\mathbf{g}(s, t)$, where $u, v, s, t \in \mathbb{R}$ that are defined as:

$$\mathbf{f}(u, v) = \left[8u - 1, 15v - \frac{15}{2}, 0\right].$$
$$\mathbf{g}(u, v) = \left[\cos(2\pi s)(2 + \cos(2\pi t) - \frac{1}{5}, \sin(2\pi s)(2 + \cos(2\pi t)), 1 + \sin(2\pi t)\right].$$

(See the Figure 3.)

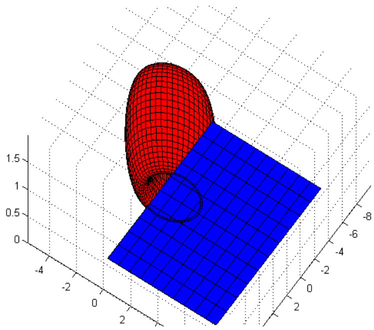


Figure 3. Tangential intersection curve situation.

The surfaces **f** and **g** meet at a tangential intersection curve **l** which contain the point $P = \mathbf{f}(0, 0.3778) = \mathbf{g}(0.6845, 0.75)$.

To obtain the transversal intersection, we have to perturb **g** analytically as

$$\mathbf{g}(\varsigma) = \left[\cos(2\pi s)\left(2 + \cos(2\pi t) - \frac{1}{5}, \sin(2\pi s)(2 + \cos(2\pi t)), 1 + \sin(2\pi t) - \varsigma\right)\right],$$

where ς is small and appropriate. (See the Figure 4.)

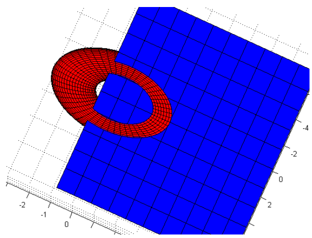


Figure 4. After the perturbation.

By (3.4), we find the point $P(\varsigma) = (u(\varsigma), v(\varsigma), s(\varsigma), t(\varsigma))$ where:

$$u(\varsigma) = 0, \quad v(\varsigma) = \frac{1}{2} - \frac{1}{15}\left[\frac{1}{2} - \sqrt{1 - \frac{16}{25\sqrt{-\varsigma(\varsigma - 2}} + 2)^2}\sqrt{1 - (\varsigma - 2)} + 2\right],$$

$$s(\varsigma) = \frac{1}{2} + \frac{1}{2\pi}\left[\arcsin\left(\frac{4}{5\sqrt{-\varsigma(\varsigma-2)}} + 10\right)\right], \quad t(\varsigma) = 1 + \frac{\arcsin(\varsigma - 1)}{2\pi}.$$

For every ς' $(0 < |\varsigma'| \leq |\varsigma|)$, the point $P(\varsigma')$ is a transversal intersection point and $\lim_{\varsigma' \to 0} P(\varsigma') = P$. Then, we can evaluate $\mathbf{T}$ by (3.2), such that:

$$\mathbf{T} = (-0.9169, 0.4, 0).$$

By (3.3) we get:

$$\mathbf{n} = (0.4, 0.9169, 0).$$

Thus, by (2.2) we obtain:

$$\mathbf{b} = (0, 0, 1).$$

5. Conclusion

We have developed a method to evaluate the tangent vector, normal vector and the binormal vector of a tangential intersection point between two parametric surfaces. We use the perturbation methods that allow us to use the expressions suggested by T. Maekawa and X. Ye to resolve the transversal intersection situations in the tangential intersection situations. If the curvature $\mathbf{K}$ vanishes, we can not evaluate the normal vector and the binormal vector by using our method. This case will be a future work.

References

[1] R. A. M. Alsaidi, A. Musleh: *Two methods for surface/surface intersection problem comparative study*, International Journal of Computer Applications 975 (2014), p. 8887.

[2] B. U. Düldül, M. Düldül: *Can we find Willmore-like method for the tangential intersection problems?*, Journal of Computational and Applied Mathematics 302 (2016), pp. 301–311, doi: https://doi.org/10.1016/j.cam.2016.02.016.

[3] T. A. Grandine, F. W. Klein IV: *A new approach to the surface intersection problem*, Computer Aided Geometric Design 14.2 (1997), pp. 111–134, doi: https://doi.org/10.1016/S0167-8396(96)00024-6.

[4] S. A. Hassan, S. A.-N. Badr: *Differential geometry of self-intersection curves of a parametric surface in R3*, J. Math. Comput. Sci. 6.6 (2016), pp. 1108–1132.

[5] C. M. Hoffmann: *Geometric and solid modeling* (1989).

[6] S. Hur, M.-j. Oh, T.-w. Kim: *Classification and resolution of critical cases in Grandine and Klein's topology determination using a perturbation method*, Computer aided geometric design 26.2 (2009), pp. 243–258, doi: https://doi.org/10.1016/j.cagd.2008.03.003.

[7] J. Masterjohn, V. Milenkovic, E. Sacks: *Finding Intersections of Algebraic curves in a Convex Region using Encasement.* In: CCCG, 2018, pp. 91–97.

[8] M. Mauri: *Intersection cohomology of rank two character varieties of surface groups*, arXiv preprint arXiv:2101.04628 (2021), doi: https://doi.org/10.1017/S1474748021000487.

[9] M.-J. Oh, S. Hur, T.-w. Kim: *Poster: Finding Direction of Intersection Curve in Critical Cases of Surface-surface Intersection*, WSCG'2007 (2007).

[10] M.-J. Oh, S. Hur, T.-w. Kim: *Topology Determination of Critical Cases in Surface-Surface Intersection*, 한국산업응용수학회 학술대회 논문집 1.2 (2006), pp. 149–151.

[11] T. Willmore: *An Introduction to Differential Geometry, Clarendon*, 1959.

[12] X. Ye, T. Maekawa: *Differential geometry of intersection curves of two surfaces*, Computer Aided Geometric Design 16.8 (1999), pp. 767–788, doi: https://doi.org/10.1016/S0167-8396(99)00018-7.

Annales Mathematicae et Informaticae
55 (2022) pp. 12–27
DOI: https://doi.org/10.33039/ami.2022.05.001
URL: https://ami.uni-eszterhazy.hu

Weather forecasting using DBSCAN clustering algorithm

Aida Chefrour

Computer Science Department, Mohamed Cherif Messaadia University,
Souk Ahras,41000, Algeria
LISCO Laboratory, Computer Science Department, Badji Mokhtar-Annaba University,
P.O. Box 12, Annaba, 23000, Algeria
aida.chefrour@univ-soukahras.dz

Abstract. The main objective of this study is the clustering of meteorological parameters and forecasting weather in the region of Annaba (Algeria) using clustering techniques. The proposed two-stage clustering approach is based on the first stage, on the proposition of ANN-DBSCAN, a combination of the DBSCAN algorithm and an Artificial Neural Network (ANN) for grouping the clusters. Internal indices of validation were used to compare and verify the correctness and efficiency of the results. Our experiments identified five groups, each of which was associated with the area's usual weather parameters. Our proposed incremental DBSCAN is employed in the second stage to determine the data pattern that can predict the future atmosphere. The natural molecules of the measured pollutants (nitrogen dioxide (NO2), ozone (O3), carbon dioxide (CO2), and sulfur dioxide (SO2)) are directly dependent on weather forecasting. The focus of this research is on a section of the Samasafia database. The proposed algorithm is used to determine the weather trend in that database. Advanced numerical analysis was applied to a few prediction tasks.

Keywords: Clustering, Forecasting weather, DBSCAN, Artificial Neural Network, SAMASAFIA

1. Introduction

Concern about decreasing air quality and its local and global consequences has increased significantly in recent years [10]. Rapid urbanization, population growth, and industrialization have resulted in alarming levels of air pollution. Scientific planning of analysis methodologies and pollution control are essential to prevent

Submitted: September 21, 2021
Accepted: May 9, 2022
Published online: May 11, 2022

continued declines in air quality. Within this framework, it is required to analyze and specify all pollution sources and their contributions to air quality; research the various elements that generate pollution; and develop instruments to minimize pollution by implementing control measures and alternative practices [18].

Algeria is a beautiful country with a rich and diversified geography. It does, however, have its own set of environmental challenges, as does every country in the world. This is particularly evident in highly industrialized and rapidly rising urban regions such as Annaba [8].

The city of Annaba is one of the most polluted cities in Algeria because of the existence of big industrial complexes such as the El Hadjar steel complex (Arcelor Mittal) and the fertilizer complex (Fertile). In addition, it is known for its dense roundabout traffic and overcrowding.

The main pollutants monitored in the Annaba air are chemicals such as ozone (O3), nitrogen oxide (Nox), carbon monoxide (Co2), and sulfur dioxide (SO2).

In our work, we have used the SAMASAFIA database, it will be described in section 2. Several research has been devoted to identifying the links between air pollution and meteorological variables exist in the literature. In the following, we outline the best known and most recent ones:

Using clustering algorithms, Khedairia and al [18] define meteorological conditions in the Annaba (Algeria) region. The Self-Organizing Maps (SOMs) and the well-known K-means clustering method are used in the suggested two-stage clustering strategy. To compare and validate the accuracy of the results, quantitative (using two kinds of validity indices) and qualitative criteria were used. Five classes emerged from the many experiments, all of which were related to typical weather circumstances in the area. The meteorological clusters obtained are then utilized to better understand the relationship between meteorology and air quality in the presence of seven pollutants. For modeling air contaminants and simulating their reactivity to meteorological parameters of interest, they used Artificial Neural Networks (ANNs), and more specifically, Multi-Layered Perceptron (MLP). This behavior is also examined using the correlation coefficient, where the results are displayed for comparison, and numerous relationships and conclusions are drawn.

Ghazi et al [6] report the construction of air pollution concentration prediction models for five major pollutants (O3, PM10, SO2, NOx, COx) utilizing two neurocomputing paradigms: Radial Basis Function and Elman Networks. As a result, each Artificial Neural Network (ANN) forecasts the concentrations of the five contaminants. These models were created to provide a 12-hour forecast for the Annaba region in northeast Algeria (north of Africa). The models are designed to predict air pollutant concentration at $t + 12$ hours after receiving measurements of air pollutant concentration and meteorological parameters (wind speed, temperature, and humidity) at time t. The performance of both ANN models is fully compared and assessed once anticipated pollutant concentrations are attained and the validity of each ANN model is verified. In light of the acquired results, the usage of one ANN network over another is justified.

Alioua et al [1], present the characterization of air pollution in the region of Annaba. The survey has been conducted using different complementary approaches. On one hand, results were recorded by the monitors operating in the air quality and control network in the region of Annaba (called Sama Safia), and on the other hand, results were provided by a bio-indicator, a lichen species called Xanthoria parietina. A relevant sampling strategy, space and time follow-up of measurements of certain physiological parameters (chlorophyll, proline, breathing), and the proportioning of NO2 have permitted us to characterize the impact of pollution resulting, on the one hand, from intense road traffic and, on the other hand, from the proximity of an iron and steel complex and a phosphate fertilizer complex. The results from the two monitoring techniques used, on one hand, the physico-chemical sensors and, on the other hand, bioindication, have shown a significant correlation not only between the analyzed pollutant (NO2) and the physiological parameters measured (chlorophyll, proline, respiration), but also between the bioindicator and the physical-chemical sensors. This work has allowed a better characterization of air pollution in this region.

The remainder of the paper is organized as follows: the study region and user data are introduced in the following section. The essential concepts of the DBSCAN clustering method, as well as how the suggested clustering strategy ANN-DBSCAN and validity indices are utilized to identify and compare clustering results, are discussed in Sections 3 and 4. In Section 5, we look at the correlations between air pollution and meteorological characteristics, as well as how we applied our novel approach to a portion of the SAMASAFIA database and analyzed the results. Section 6 concludes with a conclusion and recommendations for future research.

2. Studies area and dataset

2.1. Studied area

The city of Annaba is located in the east of Algeria (600 km from Algiers) between the latitudes of 36°30’ Nord and 37°30’ North and the longitudes of 07°20’ East and 08°40’ East, with 12 communes with a total area of 1411.98 km^2. It is situated on a wide plain separated on the northwest by a mountain range that gradually decreases in height towards the southwest, and on the east by the Mediterranean Sea.

These characteristics allow pollutants to accumulate and, as a result, their concentration to develop. I addition, the movement of polluted air is aided by sea and land breezes. Pollutants are transported out to sea by the land wind and then returned to the city by westerly winds along the Seraidi mountain. In the shape of a circle, the clouds stare down on the city. Because the pollutants are deposited slowly by gravity, pollution affects all three receivers (sea, land, and air). Depending on Annaba’s industrial operations, contaminant air pollutants are spread variably. The industry in Annaba is both a source of growth and a source of environmental deterioration, with the majority of industrial sites (complexes) located near the city, such as the Asmidal phosphate and nitrogen fertilizer complex and the El Hadjar metal steel complex. These industrial operations are the primary source of particulate matter and sulfur oxides, whereas the transportation sector is the primary source of carbon monoxide, nitrogen, and lead emissions. Air pollution has risen due to an increase in the number of vehicles (a 5% annual increase in Algeria) and a lack of emission controls. In the open air, waste incineration (domestic, industrial, hospital, toxic) is also a source of pollution.

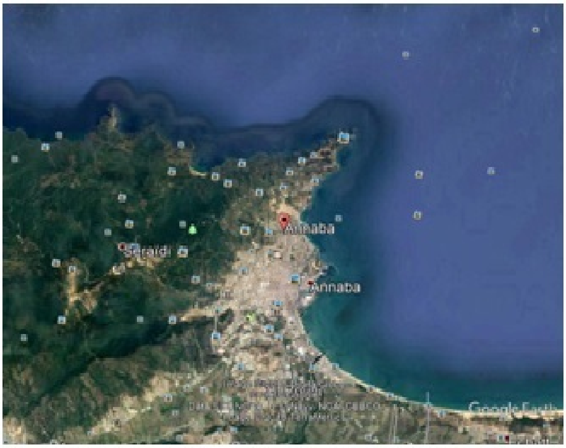


Figure 1. Geographic of Annaba (Google Earth).

2.2. Dataset and pre-processing

The dataset used in this study was collected from the SAMASAFIA network center (www.samasafia.dz/journaux) over a 24-hour period from 2003 to 2004. Nitric oxide (NO), carbon monoxide (CO), ozone (O3), particulate matter (PM10), nitrogen oxides (NOx), nitrogen dioxide (NO2), and sulfur dioxide (SO2) are among the pollutants that are regularly monitored in the air (SO2).The dataset includes Wind Speed, Temperature, and Relative Humidity.

Outliers are introduced by faulty measuring equipment operation or erroneous data collection and processing, and their detection is dependent on a number of criteria, such as the median value, the mode, and the mean,... Outliers are carefully scrutinized since they can skew the prediction model's calibration. Measurement instrument failures result in missing data. Because of the experimental nature of the measurement stations, this is typically caused by power outages and various analyzer problems (Samasafia, 2004). We must follow the same process as in [18], in which we estimate the model's parameters by analyzing observed data without accounting for missing data. Nevertheless, the results may be incorrect because considerable information is lost [7]. The missing data percentages for each year are: 2003 (08.31%), and 2004 (23.67%).

3. The proposed ANN-DBSCAN clustering algorithm

We will recall the principle of classic DBSCAN and Artificial Neural Network:

The DBSCAN algorithm was first proposed by [14], and it is based on the density-based cluster concept. The density of points can be used to identify clusters. Clusters are visible in regions with a high density of points, whereas clusters of noise or outliers are seen in regions with a low density of points. This technique is well suited to deal with huge datasets and noise, as well as identifying clusters of various sizes and forms.

The DBSCAN algorithm's main idea is that for each cluster point, the neighborhood of a specified radius must contain at least a certain number of points, i.e., the density in the neighborhood must surpass a certain threshold. Three input parameters (Eps and MinPts) are required for this algorithm [16]:

- k, the neighbour list size;
- Eps, the radius that delimitates the neighbourhood area of a point (Eps-neighbourhood);
- MinPts, the minimum number of points that must exist in the Eps-neighbourhood.

The clustering technique uses density relations between points (directly density-reachable, density-reachable, density-connected) to construct clusters after classifying the points in the dataset as core points, border points, and noise points.

- **Core Object:** object with at least MinPts objects within a radius 'Eps-neighborhood';
- **Border Object:** object that is on the border of a cluster of NEps(p): q belongs to $D \mid \text{dist}(p, q) \leq \text{Eps}$;
- **Directly Density-Reachable:** a point p is directly density-reachable from a point q w.r.t Eps, MinPts if p belongs to NEps(q) and $|\text{NEps}(q)| \geq \text{MinPts}$;
- **Density-Reachable:** a point p is density-reachable from a point q w.r.t Eps, MinPts if there is a chain of points $p_1, \ldots, p_n$, $p_1 = q$, $p_n = p$ such that p_{i+1} is directly density-reachable from p_i;
- **Density-Connected:** a point p is density-connected to a point q w.r.t Eps, MinPts if there is a point o such that both, p and q are density-reachable from o w.r.t Eps and MinPts.

The algorithm of DBSCAN is as follows [14]:

- Choose a point p at random;
- A cluster is formed if p is a core point;

- If p is a border point, DBSCAN visits the database's next point since no points are density-reachable from p;
- Repeat the process until you've processed all of the points.

Artificial Neural Networks (ANNs) have grown in popularity as a useful approach for modelling environmental systems in recent years. They've already been used to model algal development and transportation in rivers, anticipate salinity and ozone levels to predict air pollution and the functional aspects of ecosystems, and simulate the export of nutrients from river basins [9].

Now, let us recall the principle of ANN, which is a mathematical model that simulates the structure and functionalities of biological neural networks. The artificial neuron builds the basic block of every artificial neural network, that is, a simple mathematical model (function).

It is a set of neurons arranged and placed with each other under a structure of synaptic connections.

An ANN can be divided into three layers:

- **Input layer:** this layer is in charge of receiving information (data) from the outside environment. These inputs are normally normalized within the activation function's limit values;
- **Hidden, intermediate layers:** these layers are composed of neurons that are responsible for extracting data associated with the processor system being analyzed;
- **The output layer:** this layer, like the previous levels, is made up of neurons and is in charge of producing and displaying the final network outputs, which are the consequence of the processing done by the neurons in the previous layers.

3.1. Proposed algorithm

We have used neural networks in our algorithm because they encourage the principle of evolution and make it easy to estimate problems that are real and complex. The following are the steps of our proposed clustering algorithm, ANN-DBSCAN:

1. Apply the DBSCAN algorithm;
2. Create neural networks:

   For every cluster:

   - The centroid is the neuron of the first layer;
   - Other points from neurons constitute the second layer;
   - Every noise point forms a neural network of a single layer and a single neuron.

3. When a new data X arrives:
   - X is considered a neuron.

   For $n \leftarrow 1$ to m do (m is the number of neural networks):
   - Calculate the Euclidian distance d_m between this neuron and the neurons of n neural networks;
   - The neuron which has the minimal distance d_m between this neuron and the new neuron is called a winning neuron and the opposite is a losing neuron.

   We compared d_m with a threshold T:
   - if $d_m \leq T$ then the new neuron will be inserted into the cluster which has $d_m \leq T$ and the winning neuron will form the new centroid of this new cluster; otherwise, X will be considered as a noise point.
4. The most lost neurons will be deleted;
5. Apply the same algorithm if the new data is a cluster.

Remark: The threshold T is the minimal distance between all the centroids and the noise points.

4. Results and discussion

4.1. The cluster validity measures

Despite the fact that clustering methods strive to optimize a criterion, finding the global optimum is not guaranteed [19]. It is necessary to evaluate the quality and validity of results because there is no knowledge of priority in the process of clustering, there are no predefined classes, and there are no examples of what types of acceptable associations should be legitimate among the data. Two different approaches to validity indices are used for comparing the results: External criteria compare the resulting clustering to a ground-based truth available externally, either by previous research or by subjective knowledge from field experts [18]. Internal criteria use quantities and features available within the dataset. In this study, two internal validity indices were applied to determine the appropriate number of clusters within the data set.

4.1.1. Silhouette index (SC)

In contrast to the above-mentioned indices, the Silhouette value of a data object represents the degree of confidence in its clustering assignment [15]:

$$S_i = (b_i - a_i)/\max\{a_i, b_i\}'$$

where a_i is the average distance between point I and other points in the same cluster, and b_i is the average distance between itself and the "nearest" neighbouring cluster:

$$\begin{aligned} a_i &= \frac{1}{|C_i|} \sum_{j \in C_i} d(i,j) \\ b_i &= \min_{C_k, k \neq i} \sum_{j \in C_k} d(i,j)/|C_k| \end{aligned} \tag{4.1}$$

Let us set as the Silhouette index of a clustering scheme, the average Silhouette score of its objects:

$$\mathrm{SC} = \frac{1}{N} \sum_{i=1}^{N} S_i.$$

The values of the index lie in $[-1;1]$ with unity denoting a perfect assignment. Thus, in practice, values around 0.5 and higher are considered acceptable.

4.1.2. Davies–Bouldin index

The ratio of within-cluster scatter to between-cluster separation (see (4.1)) determines this index:

$$\mathrm{DBI} = \frac{1}{n} \sum_{i=1}^{n} \max_{i \neq j}((S_n(Q_i) + S_n(Q_j))/S(Q_i, Q_j)$$

where n is the number of clusters, S_n is the average distance between cluster centers, and $S(Q_i, Q_j)$ is the distance between cluster centers. As a result, if the clusters are compact and far apart, the ratio is low. As a result, for a decent cluster, the Davies–Bouldin index will have a tiny value [15].

We used the SAMASAFIA database, which is described in Section 2 [18] must deal with two types of data issues: missing data and outliers. Outliers are primarily caused by faulty measuring instrument operation or incorrect data collection and analysis methods. Outlier detection in our situation is based on median values, with the standard deviation of the meteorological factors taken into account. Outliers are carefully evaluated since they can generate bias in the prediction model's calibration. The failure of measurement instruments is the most common cause of missing data. Due to the experimental nature of the measurement stations, this is primarily caused by power outages and other faults in various analyzers (Samasafia, 2004). Missing data might throw off a statistical study by introducing systematic components of mistakes in parameter estimation in the prediction model.

Table 1 shows an example of an outlier detected in attributes: temperature, humidity, and wind speed that appeared on the second day of our database.

Table 2 shows examples of missing data for the different attributes: temperature, and humidity.

The results of clustering were obtained using DBSCAN to group 300 data lines of the database "Meteorological" by changing the input parameters: MinPts = 5

Table 1. Example of an outlier detected in attributes.

Hours	Temperature (C°)	Humidity (%)	Wind speed (m/s)
4	8.5	74	1.1
5	**37989,20833**	**37990,20833**	**37991,20833**
6	8.1	75	1.3
7	8.0	74	2.4

Table 2. Example of missing data detected in attributes.

Hours	Temperature (C°)	Humidity (%)	Wind speed (m/s)
9	9.4	79	5.0
10	**No data**	**No data**	**10**
11	**No data**	**No data**	**11**
12	8.5	85	3.7

and $\varepsilon = 5$, for each run are presented in Table 3, using the following criteria: result, ε, MinPts, number of data lines (NDL), number of clusters (NCL), and evaluation by Silhouette Index (SI) and Davies-Bouldin (DB).

Table 3. DBSCAN algorithm results.

Result	ε	MinPts	NDL	NCL	Evaluation SI	Evaluation DB
1	5	5	300	4	0.021645	0.874169
2	3	5	300	2	0.036589	0.325648
3	7	5	300	3	0.239746	0.345873
4	4.7	3	300	17	0.099475	0.187456

We have chosen partitioning number 1, which gives the best values of Davies Bouldin and Silhouette Index MinPts = 5 and $\varepsilon = 5$. We take the results of the DBSCAN algorithm (Table 3, line1), and we apply the first proposed incremental DBSCAN algorithm to group the 64 new data lines with the same input parameters MinPts = 5 and $\varepsilon = 5$. We get the following results (see Table 4):

Table 4. ANN-DBSCAN algorithm results with MinPts = 5 and $\varepsilon = 5$.

NDL	NCL	Evaluation SI	Evaluation DB
364	5	0.015846	0.896542

Note that the number of clusters in the proposed ANN-DBSCAN algorithm (= 5) is great compared with the static DBSCAN algorithm result (= 4). The optimal number of clusters proposed by this index is 5, according to the Davies–Bouldin index and the Silhouette index. Figure 2 shows the results of these indices.

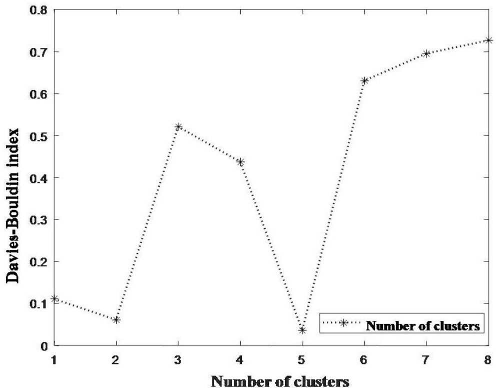


Figure 2. The Davies–Bouldin index minimum value indicates the optimal number of clusters as determined by the proposed ANN-DBSCAN.

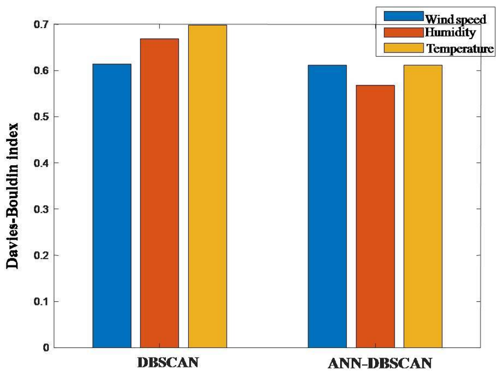


Figure 3. Davies–Bouldin index for the static algorithm and the ANN-DBSCAN algorithm.

From Figure 3, we see the following remarks: For the proposed ANN-DBSCAN algorithm, the Davies–Bouldin index values are low for the three attributes temperature, humidity and wind speed compared to the static DBSCAN algorithm. This explain that the clustering by ANN-DBSCAN is better than the static DBSCAN.

5. The influence of meteorological parameters on air pollution

5.1. Effects of air pollution data on the weather

- **CO2:** carbon dioxide is a significant heat-trapping (greenhouse) gas that is released by human activities like deforestation and fossil fuel combustion, as well as natural processes like respiration and volcanic eruptions [13]. During the past several hundred thousand years of glacial cycles, there has been a substantial correlation between temperature and carbon dioxide levels in the atmosphere. When the concentration of carbon dioxide in the atmosphere rises, so does the temperature. When the amount of carbon dioxide in the atmosphere decreases, the temperature decreases [12];

- **SO2:** sulfur dioxide has a strong stench and is colorless, thick, poisonous, and nonflammable. Temperatures and pressures are typical. Sulfur dioxide is a pollutant that causes respiratory issues, and it irritates the lungs in particular. Sulfur oxides are a primary contributor to acid rain production. Sulfur dioxide oxidizes to sulfur trioxide via many chemical processes. Sulfuric acid is formed when sulfur trioxide reacts with water vapor or droplets (H2SO4). Acid rain contains a variety of acids, including sulfuric acid [2];

- **NOx:** nitrogen dioxide has an unpleasant odor. Some nitrogen dioxide is created by plants, soil, and water, while some is formed naturally in the atmosphere by lightning. However, this method produces just around 1% of the total nitrogen dioxide contained in our cities' air. Nitrogen dioxide is a significant air contaminant because it leads to the creation of photochemical smog, which has serious health implications [17]. It is in charge of raising the temperature;

- **PM:** the most common pollutant that contributes to unhealthy days on the Air Pollution Index is Particulate Matter. Dust, smoke, fumes, mist, fog, aerosols, fly ash, and other pollutants are produced. Increased fine particle levels in the air have been related to health problems such as heart disease, lung problems, and lung cancer [11].

We have used the same database for forecasting weather, as is shown in Table 5. The influence of various air pollution molecules on the weather is seen in Table 6.

Figure 4 shows the average concentration levels of the pollutant CO. This graph represents the primary variances in pollution levels during the four seasons of 2004.

We have applied ANN-DBSCAN algorithm to the air pollutant on the data for the Spring season of 2014, we obtain the following results: Minpts = 5 and eps = 5.

Table 5. Hourly data.

Date	Temp	CO	NO2	O3	PM10	SO2
01/02/2004	14	1.56	82.12	37.6	158.46	26.41
02/02/2004	12	1.95	63.85	22.74	162.45	3.654
03/02/2004	20	1.75	122.64	29.45	228.96	5.96
04/02/2004	11	1.44	156.45	44.14	189.75	9.45
05/02/2004	16	0.96	74.28	30.96	154.85	9.88

Table 6. Parameters table.

Concentration	CO	NO2	O3	PM10	SO2
Low	Temp low	No Effect	No Effect	No Effect	No Effect
Normal	Temp Normal	Dry	No Effect	Dust	Dry
High	Temp High	Fog, dry	Humid high	Smog, dust, fog	Smog, dry
Extreme high	Temp Extreme High	Fog, dry	Humid high	Smog, dust, fog	Smog, dry

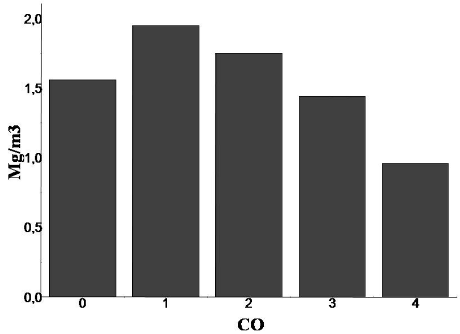


Figure 4. Pollutant concentrations for the four seasons of 2004 are averaged.

5.2. Evaluation measures for forecasting

A diagram identical to the one shown in figure is used to analyze the forecasts. The area "H" represents the intersection of the forecast and observed areas, or the "Hits" area; "M" represents the observed area that was missed by the forecast area, or the "Misses" area; and "F" represents the part of the forecast that did not overlap with an area of observed precipitation, or the "False Alarm" area, in this diagram [20].

We represent this situation by the "contingency table" as shown in Table 7. (Marginal of forecast: MF)

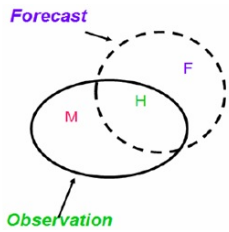


Figure 5. Diagram showing hits, misses, and false alarms for dichotomous forecast/observations.

Table 7. The contingency table for (yes-no) events.

Forecast	Observation				MF
	-	Yes	No	-	
	Yes	H: Hits	F False Alarm	H + F Yes forecast	
	No	M: Misses	N: Correct rejection	M + N 'No forecast'	
		H + M 'Yes observe'	F + N 'No observe'	Total forecast	
	Marginal of Observe				

Equitable Threat Score:

The Threat Score (TS) is a competence score for binary occurrences that necessitates the setting of a threshold. It's calculated as the ratio of hits to the total of hits, false alarms, and misses. The Equitable Threat Score (ETS) is a modified Threat Score that subtracts the number of hits that would be predicted by chance alone from the total number of hits [20].

$$ETS = (H - CH)/(F + M - H - CH)$$

where $CH = (F \times M)/N$.

The number of random hits is CH, and the number of points in the verification domain is N. The ETS is the same as TS, but with a bias adjustment for random hits.

Bias score:

The number of times an event was predicted against the number of times it was observed is referred to as bias [20].

$$B = F/M$$

This score is above (below) 1 when the predicted precipitation rate is higher (lower) than the observed. $ETS = 1$ and $BIAS = 1$ are the results of a perfect forecast.

The number of verification grid-boxes containing observations affects the validity of these indexes.

Accuracy:

The ratio between the number of well-predicted (correctly classified) examples and the total number of examples. It is given by the following formula:

$$\mathrm{Acc} = (F + H)(H + M)/(H + F + M + N)$$

5.3. Prediction evaluation

We evaluate the prediction by using the proposed ANN-DBSCAN algorithm. Using DBSCAN, we were able to determine the number of clusters for each air pollutant data set. Weather conditions are often the same inside a cluster. Now for the forecast, we have used the measures explained earlier (ETS, Bias and accuracy) for the purpose of the given date mentioned in [5]:

PWC: Predict weather conditions;
PT: Predict temp;
AT: Actual temp.

Table 8. The resultant table

Date	PWC	PT	AT	Hit	Miss
01/03/2014	Smog, fogs, Lightning, and cold	11-21	13	+	
02/03/2014	Humid, Smog	11-21	16	+	
03/03/2014	Humid, mist, dust	13-24	20	+	
04/03/2014	Smog, Lightning Thunder, and mist	09-17	11	+	
05/03/2014	Humid, fog	12-19	15	+	

These results are according to the cluster data and their nature, calculating the temperature range. The above table shows the comparison between the current temperature and the forecast temperature.

The forecasts will be evaluated using the Equitable Threat Score (ETS) and the Bias Score ($BIAS$), as well as their accuracy (acc):

$$\mathbf{ETS} = (364 - 311)/(5 + 968 - 364 - 311) = \mathbf{0.117}$$
$$\mathbf{Bias} = 398/968 = \mathbf{0.411}$$
$$\mathbf{Acc} = 364/398 \times 100 = \mathbf{91.45}\%$$

According to the use of the SAMASAFIA database and the result of the accuracy, we conclude that we have obtained the best score and the perfect forecast.

6. Conclusion and perspectives

In this work, two parts have been presented to depict and identify the Annaba region's meteorological day types; we have focused on the database Samasafia, and the following concluding statements can be made:

- We proposed the ANN-DBSCAN algorithm, combining the DBSCAN and Artificial Neural Network, based on density and proximity concepts;
- We evaluated our proposed ANN-DBSCAN algorithm versus static DBSCAN, using internal criteria: Davies Bouldin index and Silhouette index. There are five distinct clusters that have been found. Our results suggest that the proposed methodologies outperform the DBSCAN;
- Each cluster's meteorological parameters are simple to interpret based on previuos work. We have predicted weather using the ANN-DBSCAN clustering algorithm on the Samasafia database (Spring season);
- Verification of the Forecast Metrics like accuracy are calculated using their respective hit and miss times. Our results suggest that the proposed method produces more accurate results.

In the future:

- We will apply our proposed algorithm to the same aim as [18];
- We will use our proposed algorithm AMF IDBSCAN for the same objective as in this paper [4];
- We will utilize another incremental algorithm [3] for forecasting.

Acknowledgements. The author would like to thank the DGRSDT (General Directorate of Scientific Research and Technological Development) – MESRS (Ministry of Higher Education and Scientific Research), ALGERIA, for the financial support of LISCO Laboratory.

References

[1] A. Amel, M. Naila, M. Lyllia, T. Ali: *Caractérisation de la pollution par le NO2 à l'aide d'un couplage de technique biologique et physico-chimique dans la région d'Annaba (Algérie)*, Association pour la prévention de la pollution atmosphérique 69 (2008), pp. 2268–3798, doi: https://doi.org/10.4267/pollution-atmospherique.1367.

[2] J. Bachmann, R. Damberg, J. Caldwell, C. Edwards, P. Koman: *Review of the national ambient air quality standards for particulate matter: Policy assessment of scientific and technical information. OAQPS staff paper. Final report*, tech. rep., Environmental Protection Agency, Research Triangle Park, NC (United States ...), 1996.

[3] A. Chefrour: *Incremental supervised learning: algorithms and applications in pattern recognition*, Evolutionary Intelligence (2019), pp. 1–16, doi: https://doi.org/10.1007/s12065-019-00203-y.

[4] A. Chefrour, L. Souici-Meslati: *AMF-IDBSCAN: Incremental Density Based Clustering Algorithm Using Adaptive Median Filtering Technique*, Informatica 43.4 (2019), doi: https://doi.org/10.31449/inf.v43i4.2629.

[5] S. C. Chou, M. Justi da Silva: *Objective evaluation of Eta model precipitation forecasts over South America*, Revista Climanálise 1.1 (1999), pp. 1–17.

[6] S. Ghazi, T. K. Med: *Combination of artificial neural network models for air quality predictions for the region of Annaba, Algeria*, International Journal of Environmental Studies 69.1 (2012), pp. 79–89, doi: https://doi.org/10.1080/00207233.2012.644900.

[7] E. Hamdy, A.-G. Hala: *Estimation of air pollutant concentrations from meteorological parameters using artificial neural network*, Journal of Electrical engineering 57.1-2 (2006), pp. 105–110.

[8] A. Hamza, C. M. Reda, H. Ahmed: *Forecasting PM 10 in Algiers: Efficacy of multilayer perceptron networks*, Environmental Science and Pollution Research 23.2 (2016), pp. 1634–1641, doi: https://doi.org/10.1007/s11356-015-5406-6.

[9] M. Holger, D. Grame: *Neural network based modelling of environmental variables: a systematic approach*, Mathematical and Computer Modelling 33.6-7 (2001), pp. 669–682.

[10] T. M. Kamal, S. Najib: *Arab environment: Future challenges*, in: Proceedings of the 2008 Symposium on Interactive 3D Graphics and Games, Beirut: Arab Forum for Environment and Development, 2008, doi: https://doi.org/10.1145/1342250.1342261.

[11] F. J. Kelly, J. C. Fussell: *Air pollution and public health: emerging hazards and improved understanding of risk*, Environmental geochemistry and health 37.4 (2015), pp. 631–649, doi: https://doi.org/10.1007/s10653-015-9720-1.

[12] S. G. Kulkarni, H. M. Mehendale: *Carbon dioxide* (2005), doi: https://doi.org/10.1130/0091-7613.

[13] S. G. Laws: *Chemistry libretexts*.

[14] E. Martin, K. Hans-Peter, S. Jörg, X. Xiaowei, et al.: *A density-based algorithm for discovering clusters in large spatial databases with noise*, in: kdd, vol. 96, 34, ACM, 1996, pp. 226–231, doi: https://doi.org/10.1023/A:1009745219419.

[15] A. L. Mary, K. S. Kumar: *A density based dynamic data clustering algorithm based on incremental dataset* (2012), doi: https://doi.org/10.3844/jcssp.2012.656.664.

[16] A. Moreira, M. Y. Santos, S. Carneiro: *Density-based clustering algorithms–DBSCAN and SNN*, University of Minho-Portugal (2005), pp. 1–18.

[17] W. H. Organization: *Air quality guidelines: global update 2005: particulate matter, ozone, nitrogen dioxide, and sulfur dioxide*, World Health Organization, 2006.

[18] K. Soufiane, K. M. Tarek: *Impact of clustered meteorological parameters on air pollutants concentrations in the region of Annaba, Algeria*, Atmospheric research 113 (2012), pp. 89–101, doi: https://doi.org/10.1016/j.atmosres.2012.05.002.

[19] I. J. Turias, F. J. Gonzalez, M. L. Martín, P. L. Galindo: *A competitive neural network approach for meteorological situation clustering*, Atmospheric Environment 40.3 (2006), pp. 532–541, doi: https://doi.org/10.1016/j.atmosenv.2005.09.065.

[20] D. S. Wilks: *Statistical methods in the atmospheric sciences*, vol. 100, Academic press, 2011.

Annales Mathematicae et Informaticae
55 (2022) pp. 28–48
DOI: https://doi.org/10.33039/ami.2022.01.002
URL: https://ami.uni-eszterhazy.hu

Fibonacci–Lucas–Pell–Jacobsthal relations

Robert Frontczak[a*], Taras Goy[b], Mark Shattuck[c]

[a]Landesbank Baden-Württemberg, Stuttgart, Germany
robert.frontczak@lbbw.de

[b]Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine
taras.goy@pnu.edu.ua

[c]University of Tennessee, Knoxville, USA
mshattuc@utk.edu

Abstract. In this paper, we prove several identities involving linear combinations of convolutions of the generalized Fibonacci and Lucas sequences. Our results apply more generally to broader classes of second-order linearly recurrent sequences with constant coefficients. As a consequence, we obtain as special cases many identities relating exactly four sequences amongst the Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas number sequences. We make use of algebraic arguments to establish our results, frequently employing the Binet-like formulas and generating functions of the corresponding sequences. Finally, our identities above may be extended so that they include only terms whose subscripts belong to a given arithmetic progression of the non-negative integers.

Keywords: Generalized Fibonacci sequence, generalized Lucas sequence, Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, generating function

AMS Subject Classification: 11B39, 11B37

1. Introduction

Let $U_n = U_n(p,q)$ denote the sequence defined recursively by

$$U_0 = 0,\ U_1 = 1, \quad U_n = pU_{n-1} + qU_{n-2},\ n \geq 2,$$

*Statements and conclusions made in this paper by R. Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

Submitted: September 4, 2021
Accepted: January 4, 2022
Published online: January 26, 2022

and let $V_n = V_n(p,q)$ be given by

$$V_0 = 2,\ V_1 = p, \quad V_n = pV_{n-1} + qV_{n-2},\ n \geq 2.$$

Note that U_n and V_n correspond to special cases of the Horadam sequence and will be referred to here as *generalized* Fibonacci and Lucas sequences, respectively.

We note the special cases $F_n = U_n(1,1)$, $P_n = U_n(2,1)$, and $J_n = U_n(1,2)$ corresponding to the Fibonacci, Pell, and Jacobsthal number sequences, respectively, as well as $L_n = V_n(1,1)$, $Q_n = V_n(2,1)$, and $j_n = V_n(1,2)$ corresponding to the Lucas, Pell–Lucas, and Jacobsthal–Lucas numbers. In addition, we note that the balancing numbers $B_n = U_n(6,-1)$ also belong to the class of generalized Fibonacci sequences, while Lucas-balancing numbers C_n, usually defined by the initial values $C_0 = 1$ and $C_1 = 3$, do not belong to the class V_n.

The sequences F_n, L_n, P_n, Q_n, J_n, j_n, and B_n are indexed in the On-Line Encyclopedia of Integer Sequences [14], the first few terms of which are stated below:

n	0	1	2	3	4	5	6	7	8	9	Sequence in [14]
F_n	0	1	1	2	3	5	8	13	21	34	A000045
L_n	2	1	3	4	7	11	18	29	47	76	A000032
P_n	0	1	2	5	12	29	70	169	408	985	A000129
Q_n	2	2	6	14	34	82	198	478	1154	2786	A002203
J_n	0	1	1	3	5	11	21	43	85	171	A001045
j_n	2	1	5	7	17	31	65	127	257	511	A014551
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	A001109

In this paper, we adopt a unifying approach to identities involving various combinations of these sequences. In this direction, Adegoke [2] derived several identities for arbitrary homogeneous second order recursive sequences with constant coefficients and applied these results to present a unified study of the sequences above. Later in [1], he found binomial and ordinary summation formulas arising from an identity connecting any two second-order linearly recurrent sequences having the same recurrence but whose initial terms may differ. Illustrative examples were drawn from the aforementioned sequences and their generalizations.

Further, some isolated results in this direction have also occurred. For example, in [12], the author asked to express P_n in terms of F_n and L_n. One possible solution is to express this relationship as [9]

$$\sum_{s=0}^{n} F_s P_{n-s} = P_n - F_n.$$

A generalization of this identity was given by Seiffert in [13]:

$$\sum_{s=0}^{n} F_{k(s+1)} P_{k(n+1-s)} = \frac{F_k P_{k(n+2)} - P_k F_{k(n+2)}}{2Q_k - L_k}, \quad k \geq 1.$$

Moreover, similar convolution identities involving Fibonacci, Lucas, and generalized balancing numbers can be found in [5], whereas new convolution relations between Fibonacci, Lucas, tribonacci, and tribonacci-Lucas numbers were derived by the second author in [7]. A short time later, in [6], these results were extended to generalized Fibonacci and tribonacci sequences defined, respectively, by the recurrences $u_n = u_{n-1} + u_{n-2}$ and $v_n = v_{n-1} + v_{n-2} + v_{n-3}$ with arbitrary initial values.

The first and second authors [8] have established connection formulas between the Mersenne numbers $M_n = 2^n - 1$ and Horadam numbers w_n defined by $w_n = pw_{n-1} + qw_{n-2}$ for $n \geq 2$ with $w_0 = a$ and $w_1 = b$ and stated several explicit examples involving Fibonacci, Lucas, Pell, and Jacobsthal numbers to highlight the results. In [3], some special families of finite sums with squared Horadam numbers were found, which yield formulas involving squared Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, Jacobsthal–Lucas, and tribonacci numbers as particular cases. In [11], Koshy and Griffiths developed convolution formulas linking the Fibonacci, Lucas, Jacobsthal, and Jacobsthal–Lucas polynomials, and then deduced the corresponding results for Fibonacci–Jacobsthal–Lucas, Lucas–Jacobsthal, and Lucas–Jacobsthal–Lucas convolutions. Bramham and Griffiths in [4] obtained, using combinatorial arguments, a number of convolution identities involving the Jacobsthal and Jacobsthal–Lucas numbers as well as various generalizations of the Fibonacci numbers. Using generating functions, Koshy [10] developed a number of properties for sums of products of Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers. In [15, 16], Szakács dealt with convolutions of second order recursive sequences and gave some special convolutions involving the Fibonacci, Pell, Jacobsthal, and Mersenne sequences and their associated sequences.

In the next section, we prove several general formulas involving linear combinations of certain convolutions of U_n and V_n. These results in turn are obtained as special cases of more general identities involving second-order linearly recurrent sequences with constant coefficients and arbitrary initial values meeting at times certain auxiliary conditions. As a consequence of our formulas for U_n and V_n, we obtain several identities for F_n, L_n, P_n, Q_n, J_n, and j_n, each involving exactly four of these sequences. In the third section, it is demonstrated that the aforementioned formulas for U_n and V_n may be extended so that the subscript of each summand term belongs to a given arithmetic progression. Finally, some further general results are given in which it is required that the sequences appearing in the convolutions meet certain conditions with regard to their initial values and recurrence coefficients.

2. Main results

Let $T_n = T_n(a, b, p, q)$ denote the sequence defined recursively by

$$T_n = pT_{n-1} + qT_{n-2}, \quad n \geq 2,$$

with $T_0 = a$ and $T_1 = b$, where a, b, p, and q are arbitrary and $p^2 + 4q \neq 0$. Note that T_n reduces to U_n when $a = 0$, $b = 1$ and to V_n when $a = 2$, $b = p$. It can be shown that $T_n = \alpha r_1^n + \beta r_2^n$ for $n \geq 0$, where

$$\alpha = \frac{2b - ap + a\Delta}{2\Delta}, \quad \beta = \frac{ap - 2b + a\Delta}{2\Delta}, \quad r_1 = \frac{p + \Delta}{2}, \quad r_2 = \frac{p - \Delta}{2},$$

and $\Delta = \sqrt{p^2 + 4q}$. Note that

$$\alpha r_2 + \beta r_1 = \frac{(2b - ap + a\Delta)(p - \Delta) + (ap - 2b + a\Delta)(p + \Delta)}{4\Delta} = ap - b. \tag{2.1}$$

Thus, we get

$$\begin{aligned}\sum_{n\geq 0} T_n x^n &= \sum_{n\geq 0} \big(\alpha r_1^n + \beta r_2^n\big) x^n = \frac{\alpha}{1 - r_1 x} + \frac{\beta}{1 - r_2 x} \\ &= \frac{\alpha + \beta - (\alpha r_2 + \beta r_1)x}{(1 - r_1 x)(1 - r_2 x)} = \frac{a - (ap - b)x}{1 - px - qx^2}. \end{aligned} \tag{2.2}$$

Let $T_n^{(i)} = T_n(a_i, b_i, p_i, q_i)$, where i is fixed and (a_i, b_i, p_i, q_i) is arbitrary for each i. We will make frequent use of the following generating function formula for the product $T_n^{(1)} T_n^{(2)}$.

Lemma 2.1. *We have*

$$\sum_{n\geq 0} T_n^{(1)} T_n^{(2)} x^n = \frac{G_1(x)}{G_2(x)}, \tag{2.3}$$

where

$$\begin{aligned} G_1(x) &= a_1 a_2 + (b_1 b_2 - a_1 a_2 p_1 p_2)x \\ &\quad + \big(a_1 b_2 p_2 q_1 + a_2 b_1 p_1 q_2 - a_1 a_2 (p_1^2 q_2 + p_2^2 q_1 + q_1 q_2)\big)x^2 \\ &\quad - q_1 q_2 (b_1 - a_1 p_1)(b_2 - a_2 p_2) x^3, \\ G_2(x) &= 1 - p_1 p_2 x - (p_1^2 q_2 + p_2^2 q_1 + 2 q_1 q_2)x^2 - p_1 p_2 q_1 q_2 x^3 + q_1^2 q_2^2 x^4. \end{aligned}$$

Proof. For $i = 1, 2$, let

$$\begin{aligned} \Delta_i &= \sqrt{p_i^2 + 4q_i}, \quad r_1^{(i)} = \frac{p_i + \Delta_i}{2}, \quad r_2^{(i)} = \frac{p_i - \Delta_i}{2}, \\ \alpha_i &= \frac{2b_i - a_i p_i + a_i \Delta_i}{2\Delta_i}, \quad \beta_i = \frac{a_i p_i - 2b_i + a_i \Delta_i}{2\Delta_i}. \end{aligned}$$

Then

$$T_n^{(1)} T_n^{(2)} = \big(\alpha_1 (r_1^{(1)})^n + \beta_1 (r_2^{(1)})^n\big)\big(\alpha_2 (r_1^{(2)})^n + \beta_2 (r_2^{(2)})^n\big)$$

implies

$$\sum_{n\geq 0} T_n^{(1)} T_n^{(2)} x^n = \frac{\alpha_1 \alpha_2}{1 - r_1^{(1)} r_1^{(2)} x} + \frac{\alpha_1 \beta_2}{1 - r_1^{(1)} r_2^{(2)} x} + \frac{\alpha_2 \beta_1}{1 - r_1^{(2)} r_2^{(1)} x} + \frac{\beta_1 \beta_2}{1 - r_2^{(1)} r_2^{(2)} x}$$

$$= \alpha_2\left(\frac{\alpha_1}{1-r_1^{(1)}(r_1^{(2)}x)} + \frac{\beta_1}{1-r_2^{(1)}(r_1^{(2)}x)}\right) + \beta_2\left(\frac{\alpha_1}{1-r_1^{(1)}(r_2^{(2)}x)} + \frac{\beta_1}{1-r_2^{(1)}(r_2^{(2)}x)}\right)$$
$$= \alpha_2 \cdot \frac{a_1-(a_1p_1-b_1)(r_1^{(2)}x)}{1-p_1r_1^{(2)}x-q_1(r_1^{(2)})^2x^2} + \beta_2 \cdot \frac{a_1-(a_1p_1-b_1)(r_2^{(2)}x)}{1-p_1r_2^{(2)}x-q_1(r_2^{(2)})^2x^2},$$

where we have used (2.2) (with x replaced by $r_1^{(2)}x$ and by $r_2^{(2)}x$) in the last equality.

Thus we have

$$\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n = \frac{H_1(x)}{H_2(x)},$$

where

$$\begin{aligned}H_1(x) &= \alpha_2\big(a_1-(a_1p_1-b_1)r_1^{(2)}x\big)\big(1-p_1r_2^{(2)}x-q_1(r_2^{(2)})^2x^2\big)\\ &\quad + \beta_2\big(a_1-(a_1p_1-b_1)r_2^{(2)}x\big)\big(1-p_1r_1^{(2)}x-q_1(r_1^{(2)})^2x^2\big),\\ H_2(x) &= \big(1-p_1r_1^{(2)}x-q_1(r_1^{(2)})^2x^2\big)\big(1-p_1r_2^{(2)}x-q_1(r_2^{(2)})^2x^2\big).\end{aligned}$$

We now work separately on the numerator and denominator of the last expression, starting with the numerator. Expanding the numerator, and using the facts $\alpha_2+\beta_2 = a_2$ and $r_1^{(2)}r_2^{(2)} = -q_2$, gives

$$\begin{aligned}H_1(x) &= a_1(\alpha_2+\beta_2)\\ &\quad - \Big(a_1\alpha_2p_1r_2^{(2)} + \alpha_2(a_1p_1-b_1)r_1^{(2)} + a_1\beta_2p_1r_1^{(2)} + \beta_2(a_1p_1-b_1)r_2^{(2)}\Big)x\\ &\quad + \Big(\alpha_2\big(p_1(a_1p_1-b_1)r_1^{(2)}r_2^{(2)} - a_1q_1(r_2^{(2)})^2\big)\\ &\quad + \beta_2\big(p_1(a_1p_1-b_1)r_1^{(2)}r_2^{(2)} - a_1q_1(r_1^{(2)})^2\big)\Big)x^2\\ &\quad + \Big(\alpha_2q_1(a_1p_1-b_1)r_1^{(2)}(r_2^{(2)})^2 + \beta_2q_1(a_1p_1-b_1)(r_1^{(2)})^2r_2^{(2)}\Big)x^3\\ &= a_1a_2 - \Big((a_1a_2p_1-b_1\alpha_2)r_1^{(2)} + (a_1a_2p_1-b_1\beta_2)r_2^{(2)}\Big)x\\ &\quad + \Big(a_2p_1q_2(b_1-a_1p_1) - a_1q_1\big(\alpha_2(r_2^{(2)})^2 + \beta_2(r_1^{(2)})^2\big)\Big)x^2\\ &\quad + q_1q_2(b_1-a_1p_1)\Big(\alpha_2r_2^{(2)} + \beta_2r_1^{(2)}\Big)x^3.\end{aligned}$$

Concerning the coefficient of x in the last expression, note that

$$a_1a_2p_1\big(r_1^{(2)}+r_2^{(2)}\big) - b_1\big(\alpha_2r_1^{(2)} + \beta_2r_2^{(2)}\big) = a_1a_2p_1p_2 - b_1b_2.$$

Also, observe that $\alpha r_2^2 + \beta r_1^2$ (in the notation above) is given by

$$\frac{(2b-ap+a\Delta)(p^2+2q-p\Delta) + (ap-2b+a\Delta)(p^2+2q+p\Delta)}{4\Delta} = ap^2+aq-bp.$$

Thus, the coefficient of x^2 in the numerator equals

$$a_2p_1q_2(b_1-a_1p_1) + a_1q_1(b_2p_2-a_2q_2-a_2p_2^2).$$

Finally, note that $\alpha_2 r_2^{(2)} + \beta_2 r_1^{(2)} = a_2p_2 - b_2$, by (2.1) (with a_2 and b_2 in place of a and b and p_2 and q_2 in place of p and q), which implies that the coefficient of x^3 is given by $q_1q_2(b_1 - a_1p_1)(a_2p_2 - b_2)$. Thus, the numerator of the generating function works out to

$$a_1a_2 + (b_1b_2 - a_1a_2p_1p_2)x + \big(a_2p_1q_2(b_1 - a_1p_1) + a_1q_1(b_2p_2 - a_2q_2 - a_2p_2^2)\big)x^2 - q_1q_2(b_1 - a_1p_1)(b_2 - a_2p_2)x^3.$$

In the denominator, we have

$$\begin{aligned} H_2(x) &= \Big(1 - p_1r_1^{(2)}x - q_1(r_1^{(2)})^2x^2\Big)\Big(1 - p_1r_2^{(2)}x - q_1(r_2^{(2)})^2x^2\Big) \\ &= 1 - p_1\big(r_1^{(2)} + r_2^{(2)}\big)x - \Big(q_1\big((r_1^{(2)})^2 + (r_2^{(2)})^2\big) - p_1^2r_1^{(2)}r_2^{(2)}\Big)x^2 \\ &\quad + p_1q_1r_1^{(2)}r_2^{(2)}\big(r_1^{(2)} + r_2^{(2)}\big)x^3 + q_1^2(r_1^{(2)}r_2^{(2)})^2x^4 \\ &= 1 - p_1p_2x - \big(q_1(p_2^2 + 2q_2) + p_1^2q_2\big)x^2 - p_1p_2q_1q_2x^3 + q_1^2q_2^2x^4. \end{aligned}$$

Combining this expression with the one above for the numerator yields (2.3). □

We having the following general formula involving certain sums of convolutions of $T_n^{(1)}T_n^{(2)}$ with $T_n^{(3)}T_n^{(4)}$ where there are no restrictions on the parameters of the various $T_n^{(i)}$.

Theorem 2.2. *If $n \geq 4$, then*

$$\begin{aligned} &\sum_{s=0}^{n-4}\Big((p_1p_2 - p_3p_4)T_{n-1-s}^{(3)}T_{n-1-s}^{(4)} \\ &\quad + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)T_{n-2-s}^{(3)}T_{n-2-s}^{(4)} \\ &\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)T_{n-3-s}^{(3)}T_{n-3-s}^{(4)} - (q_1^2q_2^2 - q_3^2q_4^2)T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}\Big)T_s^{(1)}T_s^{(2)} \\ &= -a_1a_2T_n^{(3)}T_n^{(4)} + (a_1a_2p_1p_2 - b_1b_2)T_{n-1}^{(3)}T_{n-1}^{(4)} \\ &\quad + \big(a_1a_2(p_1^2q_2 + p_2^2q_1 + q_1q_2) - a_1b_2p_2q_1 - a_2b_1p_1q_2\big)T_{n-2}^{(3)}T_{n-2}^{(4)} \\ &\quad + q_1q_2(b_1 - a_1p_1)(b_2 - a_2p_2)T_{n-3}^{(3)}T_{n-3}^{(4)} + a_3a_4T_n^{(1)}T_n^{(2)} \\ &\quad - (a_3a_4p_1p_2 - b_3b_4)T_{n-1}^{(1)}T_{n-1}^{(2)} \\ &\quad - \Big(a_3a_4(p_1^2q_2 + p_2^2q_1 + 2q_1q_2 - q_3q_4) + b_3b_4(p_1p_2 - p_3p_4) \\ &\quad - a_3b_4p_4q_3 - a_4b_3p_3q_4\Big)T_{n-2}^{(1)}T_{n-2}^{(2)} \\ &\quad - \Big(a_3a_4p_1p_2q_3q_4 + a_3b_4p_1p_2p_4q_3 + a_4b_3p_1p_2p_3q_4 + b_3b_4p_1p_2p_3p_4 - a_3a_4p_3p_4q_3q_4 \\ &\quad - a_3b_4p_3p_4^2q_3 - a_4b_3p_3^2p_4q_4 - b_3b_4p_3^2p_4^2 + a_3a_4p_1p_2q_1q_2 - a_3b_4p_3q_3q_4 \\ &\quad - a_4b_3p_4q_3q_4 + b_3b_4\big(p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - q_3q_4\big)\Big)T_{n-3}^{(1)}T_{n-3}^{(2)}. \end{aligned} \tag{2.4}$$

Proof. Consider the quantity

$$\begin{aligned}
&a_3a_4T_n^{(1)}T_n^{(2)} + (b_3b_4 - a_3a_4p_3p_4)T_{n-1}^{(1)}T_{n-1}^{(2)} \\
&\quad + \Big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4)\Big)T_{n-2}^{(1)}T_{n-2}^{(2)} \\
&\quad - q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4)T_{n-3}^{(1)}T_{n-3}^{(2)} \\
&\quad - \Big(a_1a_2T_n^{(3)}T_n^{(4)} + (b_1b_2 - a_1a_2p_1p_2)T_{n-1}^{(3)}T_{n-1}^{(4)} \\
&\quad + \big(a_1b_2p_2q_1 + a_2b_1p_1q_2 - a_1a_2(p_1^2q_2 + p_2^2q_1 + q_1q_2)\big)T_{n-2}^{(3)}T_{n-2}^{(4)} \\
&\quad - q_1q_2(b_1 - a_1p_1)(b_2 - a_2p_2)T_{n-3}^{(3)}T_{n-3}^{(4)}\Big), \qquad (2.5)
\end{aligned}$$

where $T_m^{(i)}$ is taken to be zero for all i if $m < 0$. By Lemma 2.1, the generating function of the quantity (2.5) for $n \geq 0$ is given by the product of $\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n$ and $\sum_{n\geq 0} T_n^{(3)}T_n^{(4)}x^n$ with

$$\begin{aligned}
&(p_1p_2 - p_3p_4)x + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)x^2 \\
&\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)x^3 - (q_1^2q_2^2 - q_3^2q_4^2)x^4.
\end{aligned}$$

Extracting the coefficient of x^n of this generating function gives for $n \geq 4$,

$$\begin{aligned}
&(p_1p_2 - p_3p_4)\sum_{s=0}^{n-1} T_{n-1-s}^{(3)}T_{n-1-s}^{(4)}T_s^{(1)}T_s^{(2)} \\
&\quad + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)\sum_{s=0}^{n-2} T_{n-2-s}^{(3)}T_{n-2-s}^{(4)}T_s^{(1)}T_s^{(2)} \\
&\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)\sum_{s=0}^{n-3} T_{n-3-s}^{(3)}T_{n-3-s}^{(4)}T_s^{(1)}T_s^{(2)} \\
&\quad - (q_1^2q_2^2 - q_3^2q_4^2)\sum_{s=0}^{n-4} T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}T_s^{(1)}T_s^{(2)},
\end{aligned}$$

which holds also for $0 \leq n \leq 3$ since empty sums are zero by convention. Equating this last quantity with (2.5) above, and shifting summands to the other side so that each sum has upper index $n-4$, gives

$$\begin{aligned}
&\sum_{s=0}^{n-4}\Big((p_1p_2 - p_3p_4)T_{n-1-s}^{(3)}T_{n-1-s}^{(4)} \\
&\quad + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)T_{n-2-s}^{(3)}T_{n-2-s}^{(4)} \\
&\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)T_{n-3-s}^{(3)}T_{n-3-s}^{(4)} - (q_1^2q_2^2 - q_3^2q_4^2)T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}\Big)T_s^{(1)}T_s^{(2)} \\
&= -a_1a_2T_n^{(3)}T_n^{(4)} + (a_1a_2p_1p_2 - b_1b_2)T_{n-1}^{(3)}T_{n-1}^{(4)}
\end{aligned}$$

$$
\begin{aligned}
&+ \big(a_1a_2(p_1^2q_2 + p_2^2q_1 + q_1q_2) - a_1b_2p_2q_1 - a_2b_1p_1q_2\big)T^{(3)}_{n-2}T^{(4)}_{n-2}\\
&+ q_1q_2(b_1 - a_1p_1)(b_2 - a_2p_2)T^{(3)}_{n-3}T^{(4)}_{n-3} + a_3a_4T^{(1)}_nT^{(2)}_n\\
&+ \big(b_3b_4 - a_3a_4p_3p_4 - a_3a_4(p_1p_2 - p_3p_4)\big)T^{(1)}_{n-1}T^{(2)}_{n-1}\\
&+ \Big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4) - b_3b_4(p_1p_2 - p_3p_4)\\
&- a_3a_4(p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)\Big)T^{(1)}_{n-2}T^{(2)}_{n-2}\\
&- \Big(q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4) + b_3b_4(p_1^2q_1 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)\\
&+ a_3a_4(p_1p_2q_1q_2 - p_3p_4q_3q_4)\\
&+ (p_1p_2 - p_3p_4)(a_3q_3 + b_3p_3)(a_4q_4 + b_4p_4)\Big)T^{(1)}_{n-3}T^{(2)}_{n-3}.
\end{aligned}
$$

Simplifying the right side of the last equality gives (2.4). □

Note that (2.4) also holds for $0 \le n \le 3$, by the convention for empty sums, with this applying comparably to subsequent results.

We now state some special cases of (2.4) involving the generalized Fibonacci and Lucas sequences. Let $U^{(i)}_n = T_n(0, 1, p_i, q_i)$, $V^{(i)}_n = T_n(2, p_i, p_i, q_i)$ for a fixed i. Equivalently, these are the specializations of $T^{(i)}_n$ when $a_i = 0$, $b_i = 1$ and when $a_i = 2$, $b_i = p_i$, respectively.

Letting (a_i, b_i, p_i, q_i) for $1 \le i \le 4$ be given by $(0, 1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_1, p_1, q_1)$, $(2, p_2, p_2, q_2)$, respectively, in (2.4) yields the following formula.

Corollary 2.3 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n U^{(2)}_n}\big)$ and $\big(\boldsymbol{V^{(1)}_n V^{(2)}_n}\big)$)**.** *For $n \ge 3$,*

$$
\begin{aligned}
V^{(1)}_{n-1}V^{(2)}_{n-1} - q_1q_2V^{(1)}_{n-3}V^{(2)}_{n-3} &= 4U^{(1)}_nU^{(2)}_n - 3p_1p_2U^{(1)}_{n-1}U^{(2)}_{n-1}\\
&\quad - 2(p_1^2q_2 + p_2^2q_1 + 2q_1q_2)U^{(1)}_{n-2}U^{(2)}_{n-2} - p_1p_2q_1q_2U^{(1)}_{n-3}U^{(2)}_{n-3}.
\end{aligned}
$$

Example 2.4.

$$
\begin{aligned}
&L_{n-1}Q_{n-1} - L_{n-3}Q_{n-3}\\
&= 2\big(2F_nP_n - 3F_{n-1}P_{n-1} - 7F_{n-2}P_{n-2} - F_{n-3}P_{n-3}\big), \qquad (2.6)\\
&L_{n-1}j_{n-1} - 2L_{n-3}j_{n-3} = 4F_nJ_n - 3F_{n-1}J_{n-1} - 14F_{n-2}J_{n-2} - 2F_{n-3}J_{n-3},\\
&Q_{n-1}j_{n-1} - 2Q_{n-3}j_{n-3} = 2\big(2P_nJ_n - 3P_{n-1}J_{n-1} - 13P_{n-2}J_{n-2} - 2P_{n-3}J_{n-3}\big).
\end{aligned}
$$

Letting (a_i, b_i, p_i, q_i) for $1 \le i \le 4$ be given by $(0, 1, p_1, q_1)$, $(2, p_2, p_2, q_2)$, $(0, 1, p_2, q_2)$, $(2, p_1, p_1, q_1)$, respectively, in (2.4), and replacing n by $n + 1$, yields the following result.

Corollary 2.5 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n V^{(2)}_n}\big)$ and $\big(\boldsymbol{U^{(2)}_n V^{(1)}_n}\big)$)**.** *For $n \ge 2$,*

$$
\begin{aligned}
&p_1U^{(1)}_nV^{(2)}_n + 2p_2q_1U^{(1)}_{n-1}V^{(2)}_{n-1} + p_1q_1q_2U^{(1)}_{n-2}V^{(2)}_{n-2}\\
&\qquad = p_2U^{(2)}_nV^{(1)}_n + 2p_1q_2U^{(2)}_{n-1}V^{(1)}_{n-1} + p_2q_1q_2U^{(2)}_{n-2}V^{(1)}_{n-2}.
\end{aligned}
$$

Example 2.6.

$$F_nQ_n + 4F_{n-1}Q_{n-1} + F_{n-2}Q_{n-2} = 2(L_nP_n + L_{n-1}P_{n-1} + L_{n-2}P_{n-2}),$$
$$F_nj_n + 2F_{n-1}j_{n-1} + 2F_{n-2}j_{n-2} = L_nJ_n + 4L_{n-1}J_{n-1} + 2L_{n-2}J_{n-2},$$
$$2\big(P_nj_n + P_{n-1}j_{n-1} + 2P_{n-2}j_{n-2}\big) = Q_nJ_n + 8Q_{n-1}J_{n-1} + 2Q_{n-2}J_{n-2}.$$

Taking $(0,1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(2,p_2,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in (2.4) gives the following result.

Corollary 2.7 (Sequence pairs $\big(\boldsymbol{U_n^{(1)}U_n^{(2)}}\big)$ and $\big(\boldsymbol{V_n^{(2)}V_n^{(3)}}\big)$)**.** *For $n \geq 4$,*

$$\begin{aligned}
&\sum_{s=1}^{n-4}\Big(p_2(p_1-p_3)V^{(2)}_{n-1-s}V^{(3)}_{n-1-s}\\
&\quad + \big(p_1^2q_2 + p_2^2q_1 - p_2^2q_3 - p_3^2q_2 + 2q_1q_2 - 2q_2q_3\big)V^{(2)}_{n-2-s}V^{(3)}_{n-2-s}\\
&\quad + p_2q_2(p_1q_1 - p_3q_3)V^{(2)}_{n-3-s}V^{(3)}_{n-3-s} + q_2^2(q_3^2 - q_1^2)V^{(2)}_{n-4-s}V^{(3)}_{n-4-s}\Big)U_s^{(1)}U_s^{(2)}\\
&= -V^{(2)}_{n-1}V^{(3)}_{n-1} + q_1q_2V^{(2)}_{n-3}V^{(3)}_{n-3} + 4U_n^{(1)}U_n^{(2)} + p_2(p_3-4p_1)U^{(1)}_{n-1}U^{(2)}_{n-1}\\
&\quad + \Big(p_2^2p_3^2 - p_1p_2^2p_3 - 4p_1^2q_2 - 4p_2^2q_1 + 2p_2^2q_3 + 2p_3^2q_2 + 4q_2q_3 - 8q_1q_2\Big)U^{(1)}_{n-2}U^{(2)}_{n-2}\\
&\quad + p_2\Big(p_2^2p_3^3 + 3p_2^2p_3q_3 + 3p_3^3q_2 + 9p_3q_2q_3 - p_1p_2^2p_3^2 - 2p_1p_2^2q_3 - 2p_1p_3^2q_2\\
&\quad - 4p_1q_2q_3 - p_1^2p_3q_2 - p_2^2p_3q_1 - 2p_3q_1q_2 - 4p_1q_1q_2\Big)U^{(1)}_{n-3}U^{(2)}_{n-3}.
\end{aligned}$$

Example 2.8.

$$\begin{aligned}
&\sum_{s=1}^{n-4}\big(6Q_{n-2-s}j_{n-2-s} + 2Q_{n-3-s}j_{n-3-s} - 3Q_{n-4-s}j_{n-4-s}\big)F_sP_s\\
&= Q_{n-1}j_{n-1} - Q_{n-3}j_{n-3} - 4F_nP_n + 6F_{n-1}P_{n-1} + 2F_{n-2}P_{n-2} - 16F_{n-3}P_{n-3},\\
&\sum_{s=1}^{n-4}\big(Q_{n-1-s}j_{n-1-s} + 6Q_{n-2-s}j_{n-2-s} + 2Q_{n-3-s}j_{n-3-s}\big)F_sJ_s\\
&= Q_{n-1}j_{n-1} - 2Q_{n-3}j_{n-3} - 4F_nJ_n + 2F_{n-1}J_{n-1} - 46F_{n-3}J_{n-3},\\
&\sum_{s=1}^{n-4}\big(L_{n-1-s}j_{n-1-s} + 6L_{n-2-s}j_{n-2-s} + 2L_{n-3-s}j_{n-3-s}\big)P_sJ_s\\
&= -L_{n-1}j_{n-1} + 2L_{n-3}j_{n-3} + 4P_nJ_n - 7P_{n-1}J_{n-1} - 39P_{n-2}J_{n-2} - 31P_{n-3}J_{n-3}.
\end{aligned}$$

Taking $(0,1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(0,1,p_3,q_3)$, $(2,p_2,p_2,q_2)$ as the four sets of parameters in Theorem 2.2, and replacing n by $n+1$, gives the following.

Corollary 2.9 (Sequence pairs $\big(\boldsymbol{U_n^{(1)}U_n^{(2)}}\big)$ and $\big(\boldsymbol{U_n^{(3)}V_n^{(2)}}\big)$)**.** *For $n \geq 3$,*

$$\sum_{s=1}^{n-3}\Big(p_2(p_1-p_3)U^{(3)}_{n-s}V^{(2)}_{n-s}$$

$$+\big(p_1^2q_2+p_2^2q_1-p_2^2q_3-p_3^2q_2+2q_1q_2-2q_2q_3\big)U^{(3)}_{n-1-s}V^{(2)}_{n-1-s}$$
$$+p_2q_2(p_1q_1-p_3q_3)U^{(3)}_{n-2-s}V^{(2)}_{n-2-s}+q_2^2(q_3^2-q_1^2)U^{(3)}_{n-3-k}V^{(2)}_{n-3-k}\Big)U^{(1)}_sU^{(2)}_s$$
$$=-U^{(3)}_nV^{(2)}_n+q_1q_2U^{(3)}_{n-2}V^{(2)}_{n-2}+p_2U^{(1)}_nU^{(2)}_n+(p_2^2p_3-p_1p_2^2+2p_3q_2)U^{(1)}_{n-1}U^{(2)}_{n-1}$$
$$+p_2\Big(p_2^2p_3^2+3p_3^2q_2-p_1p_2^2p_3-2p_1p_3q_2+3q_2q_3-p_1^2q_2$$
$$-p_2^2q_1-2q_1q_2+p_2^2q_3\Big)U^{(1)}_{n-2}U^{(2)}_{n-2}.$$

Example 2.10.

$$\sum_{s=1}^{n-3}\big(6Q_{n-1-s}J_{n-1-s}+2Q_{n-2-s}J_{n-2-s}-3Q_{n-3-s}J_{n-3-s}\big)F_sP_s$$
$$=Q_nJ_n-Q_{n-2}J_{n-2}-2F_nP_n-2F_{n-1}P_{n-1}-16F_{n-2}P_{n-2},$$
$$\sum_{s=1}^{n-3}\big(P_{n-s}j_{n-s}+6P_{n-1-s}j_{n-1-s}+2P_{n-2-s}j_{n-2-s}\big)F_sJ_s$$
$$=P_nj_n-2P_{n-2}j_{n-2}-F_nJ_n-9F_{n-1}J_{n-1}-18F_{n-2}J_{n-2},$$
$$\sum_{s=1}^{n-3}\big(F_{n-s}j_{n-s}+6F_{n-1-s}j_{n-1-s}+2F_{n-2-s}j_{n-2-s}\big)P_sJ_s$$
$$=-F_nj_n+2F_{n-2}j_{n-2}+P_nJ_n+3P_{n-1}J_{n-1}-9P_{n-2}J_{n-2}.$$

Taking $(0,1,p_2,q_2)$, $(2,p_3,p_3,q_3)$, $(2,p_1,p_1,q_1)$, $(2,p_2,p_2,q_2)$ in Theorem 2.2 gives the following.

Corollary 2.11 (Sequence pairs $\big(\boldsymbol{U_n^{(2)}V_n^{(3)}}\big)$ and $\big(\boldsymbol{V_n^{(1)}V_n^{(2)}}\big)$)**.** *For $n\ge 4$,*

$$\sum_{s=1}^{n-4}\Big(p_2(p_1-p_3)V^{(1)}_{n-1-s}V^{(2)}_{n-1-s}$$
$$+\big(p_1^2q_2+p_2^2q_1-p_2^2q_3-p_3^2q_2+2q_1q_2-2q_2q_3\big)V^{(1)}_{n-2-s}V^{(2)}_{n-2-s}$$
$$+p_2q_2(p_1q_1-p_3q_3)V^{(1)}_{n-3-s}V^{(2)}_{n-3-s}+q_2^2(q_3^2-q_1^2)V^{(1)}_{n-4-s}V^{(2)}_{n-4-s}\Big)U^{(2)}_sV^{(3)}_s$$
$$=p_3V^{(1)}_{n-1}V^{(2)}_{n-1}+2p_2q_3V^{(1)}_{n-2}V^{(2)}_{n-2}+p_3q_2q_3V^{(1)}_{n-3}V^{(2)}_{n-3}-4U^{(2)}_nV^{(3)}_n$$
$$+p_2(4p_3-p_1)U^{(2)}_{n-1}V^{(3)}_{n-1}+\big(p_1p_2^2p_3-p_1^2p_2^2-2p_1^2q_2-2p_2^2q_1+4p_2^2q_3$$
$$+4p_3^2q_2-4q_1q_2+8q_2q_3\big)U^{(2)}_{n-2}V^{(3)}_{n-2}$$
$$-p_2\Big(9p_1q_1q_2-4p_3q_2q_3-p_1p_2^2q_3-2p_1q_2q_3-p_1p_3^2q_2+p_1^3p_2^2-p_1^2p_2^2p_3$$
$$+3p_1^3q_2-2p_1^2p_3q_2+3p_1p_2^2q_1-2p_2^2p_3q_1-4p_3q_1q_2\Big)U^{(2)}_{n-3}V^{(3)}_{n-3}.$$

Example 2.12.

$$\sum_{s=1}^{n-4}\big(6L_{n-2-s}Q_{n-2-s}+2L_{n-3-s}Q_{n-3-s}-3L_{n-4-s}Q_{n-4-s}\big)P_sj_s$$

$$= -L_{n-1}Q_{n-1} - 8L_{n-2}Q_{n-2} - 2L_{n-3}Q_{n-3} + 4P_nj_n - 6P_{n-1}j_{n-1} - 38P_{n-2}j_{n-2} - 22P_{n-3}j_{n-3},$$

$$\sum_{s=1}^{n-4}\big(L_{n-1-s}j_{n-1-s} + 3L_{n-4-s}j_{n-4-s}\big)F_sQ_s$$
$$= -2L_{n-1}j_{n-1} - 2L_{n-2}j_{n-2} - 2L_{n-3}j_{n-3} + 4F_nQ_n - 7F_{n-1}Q_{n-1} - 15F_{n-2}Q_{n-2} - 17F_{n-3}Q_{n-3},$$

$$\sum_{s=1}^{n-4}\big(Q_{n-1-s}j_{n-1-s} + 6Q_{n-2-s}j_{n-2-s} + 2Q_{n-3-s}j_{n-3-s}\big)L_sJ_s$$
$$= Q_{n-1}j_{n-1} + 2Q_{n-2}j_{n-2} + 2Q_{n-3}j_{n-3} - 4L_nJ_n + 2L_{n-1}J_{n-1} - 46L_{n-3}J_{n-3}.$$

Taking $(0,1,p_1,q_1)$, $(2,p_2,p_2,q_2)$, $(0,1,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in Theorem 2.2, and replacing n by $n+1$, gives the following.

Corollary 2.13 (Sequence pairs $\boldsymbol{(U_n^{(1)}V_n^{(2)})}$ and $\boldsymbol{(U_n^{(2)}V_n^{(3)})}$)**.** *For $n \geq 3$,*

$$\sum_{s=1}^{n-3}\Big(p_2(p_1-p_3)U_{n-s}^{(2)}V_{n-s}^{(3)} + \big(p_1^2q_2 + p_2^2q_1 - p_2^2q_3 - p_3^2q_2 + 2q_1q_2 - 2q_2q_3\big)U_{n-1-s}^{(2)}V_{n-1-s}^{(3)} + p_2q_2(p_1q_1 - p_3q_3)U_{n-2-s}^{(2)}V_{n-2-s}^{(3)} + q_2^2(q_3^2 - q_1^2)U_{n-3-s}^{(2)}V_{n-3-s}^{(3)}\Big)U_s^{(1)}V_s^{(2)}$$
$$= -p_2U_n^{(2)}V_n^{(3)} - 2p_1q_2U_{n-1}^{(2)}V_{n-1}^{(3)} - p_2q_1q_2U_{n-2}^{(2)}V_{n-2}^{(3)} + p_3U_n^{(1)}V_n^{(2)} + p_2(p_3^2 - p_1p_3 + 2q_3)U_{n-1}^{(1)}V_{n-1}^{(2)} + \Big(p_2^2p_3^3 + 3p_2^2p_3q_3 - p_1p_2^2p_3^2 - 2p_1p_2^2q_3 - p_1^2p_3q_2 - 2p_3q_1q_2 - p_2^2p_3q_1 + 3p_3q_2q_3 + p_3^3q_2\Big)U_{n-2}^{(1)}V_{n-2}^{(2)}.$$

Example 2.14.

$$\sum_{s=1}^{n-3}\big(6P_{n-1-s}j_{n-1-s} + 2P_{n-2-s}j_{n-2-s} - 3P_{n-3-s}j_{n-3-s}\big)F_sQ_s$$
$$= 2P_nj_n + 2P_{n-1}j_{n-1} + 2P_{n-2}j_{n-2} - F_nQ_n - 8F_{n-1}Q_{n-1} - 8F_{n-2}Q_{n-2},$$

$$\sum_{s=1}^{n-3}\big(L_{n-s}J_{n-s} + 6L_{n-1-s}J_{n-1-s} + 2L_{n-2-s}J_{n-2-s}\big)P_sj_s$$
$$= -L_nJ_n - 8L_{n-1}J_{n-1} - 2L_{n-2}J_{n-2} + P_nj_n + P_{n-1}j_{n-1} - 7P_{n-2}j_{n-2},$$

$$\sum_{s=1}^{n-3}\big(Q_{n-s}J_{n-s} + 6Q_{n-1-s}J_{n-1-s} + 2Q_{n-2-s}J_{n-2-s}\big)F_sj_s$$
$$= Q_nJ_n + 4Q_{n-1}J_{n-1} + 2Q_{n-2}J_{n-2} - 2F_nj_n - 4F_{n-1}j_{n-1} - 22F_{n-2}j_{n-2}.$$

We now prove a general identity of a similar form to Theorem 2.2 above in which $T_n^{(1)}$ and $T_n^{(2)}$ are assumed to share p_i and q_i parameter values meeting a certain restriction.

Theorem 2.15. *Suppose $p_1 = p_2 = p$ and $q_1 = q_2 = q$, with p and q satisfying*

$$2b_1b_2 - 2a_1a_2q = (a_1b_2 + a_2b_1)p.$$

Then for $n \geq 4$,

$$\begin{aligned}
&\sum_{s=0}^{n-4}\Big((p^2+2q-p_3p_4)T^{(3)}_{n-1-s}T^{(4)}_{n-1-s} - (p_3^2q_4 + p_4^2q_3 + 2q_3q_4 + q^2)T^{(3)}_{n-2-s}T^{(4)}_{n-2-s}\\
&\quad - p_3p_4q_3q_4T^{(3)}_{n-3-s}T^{(4)}_{n-3-s} + q_3^2q_4^2T^{(3)}_{n-4-s}T^{(4)}_{n-4-s}\Big)T^{(1)}_sT^{(2)}_s\\
&= -a_1a_2T^{(3)}_nT^{(4)}_n + (b_1-a_1p)(b_2-a_2p)T^{(3)}_{n-1}T^{(4)}_{n-1}\\
&\quad + a_3a_4T^{(1)}_nT^{(2)}_n + \big(b_3b_4 - a_3a_4(p^2+2q)\big)T^{(1)}_{n-1}T^{(1)}_{n-1}\\
&\quad + \big(a_3b_4p_4q_3 + a_4b_3p_3q_4 + a_3a_4q_3q_4 + a_3a_4q^2 - b_3b_4(p^2+2q-p_3p_4)\big)T^{(1)}_{n-2}T^{(2)}_{n-2}\\
&\quad + \Big(b_3b_4(p_3^2q_4 + p_4^2q_3 + q_3q_4 + q^2) + q_3q_4(a_4b_3p_4 + a_3b_4p_3)\\
&\quad - (p^2+2q-p_3p_4)(a_3q_3 + b_3p_3)(a_4q_4 + b_4p_4)\Big)T^{(1)}_{n-3}T^{(2)}_{n-3}.
\end{aligned} \tag{2.7}$$

Proof. Using $2b_1b_2 - 2a_1a_2q = (a_1b_2 + a_2b_1)p$, one can show

$$2a_1a_2q - (b_1-a_1p)(b_2-a_2p) = b_1b_2 - a_1a_2p^2$$

and

$$a_1a_2q - 2(b_1-a_1p)(b_2-a_2p) = a_1b_2p + a_2b_1p - a_1a_2(2p^2+q),$$

from which it follows the factorization

$$\begin{aligned}
&a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2+q)\big)x^2\\
&\quad - q^2(b_1-a_1p)(b_2-a_2p)x^3 = \big(a_1a_2 - (b_1-a_1p)(b_2-a_2p)x\big)(1+2qx+q^2x^2).
\end{aligned}$$

Thus for $T^{(1)}_n$ and $T^{(2)}_n$ whose parameters meet the stated conditions, we have by (2.3) that the generating function $\sum_{n\geq 0} T^{(1)}_nT^{(2)}_nx^n$ is given by

$$\begin{aligned}
&\frac{a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2+q)\big)x^2}{1 - p^2x - 2q(p^2+q)x^2 - p^2q^2x^3 + q^4x^4}\\
&\quad - \frac{q^2(b_1-a_1p)(b_2-a_2p)x^3}{1 - p^2x - 2q(p^2+q)x^2 - p^2q^2x^3 + q^4x^4}\\
&= \frac{\big(a_1a_2 - (b_1-a_1p)(b_2-a_2p)x\big)(1+2qx+q^2x^2)}{(1-(p^2+2q)x+q^2x^2)(1+2qx+q^2x^2)}\\
&= \frac{a_1a_2 - (b_1-a_1p)(b_2-a_2p)x}{1-(p^2+2q)x+q^2x^2}.
\end{aligned}$$

Consider now the quantity

$$a_3a_4T^{(1)}_nT^{(2)}_n + (b_3b_4 - a_3a_4p_3p_4)T^{(1)}_{n-1}T^{(2)}_{n-1}$$

$$+ \big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4)\big)T_{n-2}^{(1)}T_{n-2}^{(2)}$$
$$- q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4)T_{n-3}^{(1)}T_{n-3}^{(2)} - a_1a_2T_n^{(3)}T_n^{(4)}$$
$$+ (b_1 - a_1p)(b_2 - a_2p)T_{n-1}^{(3)}T_{n-1}^{(4)}. \quad (2.8)$$

Then the generating function of (2.8) for $n \geq 0$ is given by the product of

$$\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n \quad \text{and} \quad \sum_{n\geq 0} T_n^{(3)}T_n^{(4)}x^n$$

with

$$(p^2 + 2q - p_3p_4)x - (p_3^2q_4 + p_4^2q_3 + 2q_3q_4 + q^2)x^2 - p_3p_4q_3q_4x^3 + q_3^2q_4^2x^4.$$

Extracting the coefficient of x^n then yields

$$(p^2 + 2q - p_3p_4)\sum_{s=0}^{n-1} T_{n-1-s}^{(3)}T_{n-1-s}^{(4)}T_s^{(1)}T_s^{(2)}$$
$$- (p_3^2q_4 + p_4^2q_3 + 2q_3q_4 + q^2)\sum_{s=0}^{n-2} T_{n-2-s}^{(3)}T_{n-2-s}^{(4)}T_s^{(1)}T_s^{(2)}$$
$$- p_3p_4q_3q_4\sum_{s=0}^{n-3} T_{n-3-s}^{(3)}T_{n-3-s}^{(4)}T_s^{(1)}T_s^{(2)} + q_3^2q_4^2\sum_{s=0}^{n-4} T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}T_s^{(1)}T_s^{(2)}.$$

Equating this last expression with (2.8) above and shifting the appropriate summands gives

$$\sum_{s=0}^{n-4}\Big((p^2 + 2q - p_3p_4)T_{n-1-s}^{(3)}T_{n-1-s}^{(4)} - (p_3^2q_4 + p_4^2q_3 + 2q_3q_4 + q^2)T_{n-2-s}^{(3)}T_{n-2-s}^{(4)}$$
$$- p_3p_4q_3q_4T_{n-3-s}^{(3)}T_{n-3-s}^{(4)} + q_3^2q_4^2T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}\Big)T_s^{(1)}T_s^{(2)}$$
$$= - a_1a_2T_n^{(3)}T_n^{(4)} + (b_1 - a_1p)(b_2 - a_2p)T_{n-1}^{(3)}T_{n-1}^{(4)}$$
$$+ a_3a_4T_n^{(1)}T_n^{(2)} + \big(b_3b_4 - a_3a_4p_3p_4 - a_3a_4(p^2 + 2q - p_4p_4)\big)T_{n-1}^{(1)}T_{n-1}^{(2)}$$
$$+ \Big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4)$$
$$+ a_3a_4(p_3^2q_4 + p_4^2q_3 + 2q_3q_4 + q^2) - b_3b_4(p^2 + 2q - p_3p_4)\Big)T_{n-2}^{(1)}T_{n-2}^{(2)}$$
$$+ \Big(a_3a_4p_3p_4q_3q_4 + b_3b_4(p_3^2q_4 + p_4^2q_3 + 2q_3q_4 + q^2)$$
$$- (p^2 + 2q - p_3p_4)(a_3q_3 + b_3p_3)(a_4q_4 + b_4p_4)$$
$$- q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4)\Big)T_{n-3}^{(1)}T_{n-3}^{(2)},$$

which may be rewritten as (2.7). □

Note that taking $a_1 = 0$ in the condition on p and q in the preceding theorem reduces it to $2b_2 = a_2 p$ since it may be assumed $b_1 \neq 0$ (note $a_1 = b_1 = 0$ would result in a triviality). In particular, the condition is satisfied when $T_n^{(1)} = U_n(p,q)$ and $T_n^{(2)} = V_n(p,q)$, upon taking $a_1 = 0$, $b_1 = 1$ and $a_2 = 2$, $b_2 = p$.

Letting (a_i, b_i, p_i, q_i) be given by $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(0,1,p_2,q_2)$ and $(0,1,p_3,q_3)$ in (2.7), and replacing n by $n+1$, yields the following result.

Corollary 2.16 (Sequence pairs $\big(\boldsymbol{U_n^{(1)} V_n^{(1)}}\big)$ and $\big(\boldsymbol{U_n^{(2)} U_n^{(3)}}\big)$)**.** *For $n \geq 3$,*

$$\begin{aligned}
&p_1 U_n^{(2)} U_n^{(3)} + \sum_{s=1}^{n-3} \Big((p_1^2 - p_2 p_3 + 2q_1) U_{n-s}^{(2)} U_{n-s}^{(3)} \\
&\quad - (p_2^2 q_3 + p_3^2 q_2 + q_1^2 + 2q_2 q_3) U_{n-1-s}^{(2)} U_{n-1-s}^{(3)} \\
&\quad - p_2 p_3 q_2 q_3 U_{n-2-s}^{(2)} U_{n-2-s}^{(3)} + q_2^2 q_3^2 U_{n-3-s}^{(2)} U_{n-3-s}^{(3)} \Big) U_s^{(1)} V_s^{(1)} \\
&= U_n^{(1)} V_n^{(1)} - (p_1^2 - p_2 p_3 + 2q_1) U_{n-1}^{(1)} V_{n-1}^{(1)} \\
&\quad - \big(p_1^2 p_2 p_3 + 2p_2 p_3 q_1 - p_2^2 p_3^2 - q_2 q_3 - p_2^2 q_3 - p_3^2 q_2 - q_1^2\big) U_{n-2}^{(1)} V_{n-2}^{(1)}.
\end{aligned}$$

Example 2.17.

$$\begin{aligned}
&P_n J_n + \sum_{s=1}^{n-3} \big(P_{n-s} J_{n-s} - 14 P_{n-1-s} J_{n-1-s} \\
&\qquad - 4 P_{n-2-s} J_{n-2-s} + 4 P_{n-3-s} J_{n-3-s} \big) F_s L_s \\
&= F_n L_n - F_{n-1} L_{n-1} + 10 F_{n-2} L_{n-2}, \\
&F_n P_n + \sum_{s=1}^{n-3} \big(3F_{n-s} P_{n-s} - 11 F_{n-1-s} P_{n-1-s} \\
&\qquad - 2F_{n-2-s} P_{n-2-s} + F_{n-3-s} P_{n-3-s} \big) J_s j_s \\
&= J_n j_n - 3 J_{n-1} j_{n-1} + 4 J_{n-2} j_{n-2}, \\
&2F_n J_n + \sum_{s=1}^{n-3} \big(5F_{n-s} J_{n-s} - 8F_{n-1-s} J_{n-1-s} \\
&\qquad - 2F_{n-2-s} J_{n-2-s} + 4F_{n-3-s} J_{n-3-s} \big) P_s Q_s \\
&= P_n Q_n - 5 P_{n-1} Q_{n-1} + P_{n-2} Q_{n-2}.
\end{aligned}$$

Taking $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(2,p_2,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in Theorem 2.15 yields the following result.

Corollary 2.18 (Sequence pairs $\big(\boldsymbol{U_n^{(1)} V_n^{(1)}}\big)$ and $\big(\boldsymbol{V_n^{(2)} V_n^{(3)}}\big)$)**.** *For $n \geq 4$,*

$$\sum_{s=1}^{n-4} \Big((p_1^2 - p_2 p_3 + 2q_1) V_{n-1-s}^{(2)} V_{n-1-s}^{(3)} - (p_2^2 q_3 + p_3^2 q_2 + q_1^2 + 2q_2 q_3) V_{n-2-s}^{(2)} V_{n-2-s}^{(3)}$$

$$- p_2p_3q_2q_3V^{(2)}_{n-3-s}V^{(3)}_{n-3-s} + q_2^2q_3^2V^{(2)}_{n-4-s}V^{(3)}_{n-4-s}\Big)U^{(1)}_sV^{(1)}_s$$
$$= -p_1V^{(2)}_{n-1}V^{(3)}_{n-1} + 4U^{(1)}_nV^{(1)}_n - (4p_1^2 - p_2p_3 + 8q_1)U^{(1)}_{n-1}V^{(1)}_{n-1}$$
$$- (p_1^2p_2p_3 + 2p_2p_3q_1 - p_2^2p_3^2 - 2p_2^2q_3 - 4q_1^2 - 4q_2q_3 - 2p_3^2q_2)U^{(1)}_{n-2}V^{(1)}_{n-2}$$
$$- \Big(p_1^2p_2^2p_3^2 + 2p_1^2p_2^2q_3 + 2p_1^2p_3^2q_2 + 4p_1^2q_2q_3 - p_2p_3q_1^2 + 2p_2^2p_3^2q_1 + 4p_2^2q_1q_3$$
$$- p_2^3p_3^3 - 3p_2^3p_3q_3 - 3p_2p_3^3q_2 - 9p_2p_3q_2q_3 + 4p_3^2q_1q_2 + 8q_1q_2q_3\Big)U^{(1)}_{n-3}V^{(1)}_{n-3}.$$

Example 2.19.

$$\sum_{s=1}^{n-4}\big(Q_{n-1-s}j_{n-1-s} - 14Q_{n-2-s}j_{n-2-s} - 4Q_{n-3-s}j_{n-3-s} + 4Q_{n-4-s}j_{n-4-s}\big)F_sL_s$$
$$= 4F_nL_n - 10F_{n-1}L_{n-1} + 28F_{n-2}L_{n-2} + 10F_{n-3}L_{n-3} - Q_{n-1}j_{n-1},$$
$$\sum_{s=1}^{n-4}\big(3L_{n-1-s}Q_{n-1-s} - 11L_{n-2-s}Q_{n-2-s}$$
$$- 2L_{n-3-s}Q_{n-3-s} + L_{n-4-s}Q_{n-4-s}\big)J_sj_s$$
$$= 4J_nj_n - 18J_{n-1}j_{n-1} + 24J_{n-2}j_{n-2} - 26J_{n-3}j_{n-3} - L_{n-1}Q_{n-1},$$
$$\sum_{s=1}^{n-4}\big(5L_{n-1-s}j_{n-1-s} - 8L_{n-2-s}j_{n-2-s} - 2L_{n-3-s}j_{n-3-s} + 4L_{n-4-s}j_{n-4-s}\big)P_sQ_s$$
$$= 4P_nQ_n - 23P_{n-1}Q_{n-1} + 13P_{n-2}Q_{n-2} - 61P_{n-3}Q_{n-3} - 2L_{n-1}j_{n-1}.$$

Taking $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in Theorem 2.15, and replacing n by $n+1$, yields the following.

Corollary 2.20 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n V^{(1)}_n}\big)$ and $\big(\boldsymbol{U^{(2)}_n V^{(3)}_n}\big)$). *For $n \geq 3$,*

$$\sum_{s=1}^{n-3}\Big((-p_1^2 + p_2p_3 - 2q_1)U^{(2)}_{n-s}V^{(3)}_{n-s} + (p_2^2q_3 + p_3^2q_2 + q_1^2 + 2q_2q_3)U^{(2)}_{n-1-s}V^{(3)}_{n-1-s}$$
$$+ p_2p_3q_2q_3U^{(2)}_{n-2-s}V^{(3)}_{n-2-s} - q_2^2q_3^2U^{(2)}_{n-3-s}V^{(3)}_{n-3-s}\Big)U^{(1)}_sV^{(1)}_s$$
$$= p_1U^{(2)}_nV^{(3)}_n - p_3U^{(1)}_nV^{(1)}_n + (p_1^2p_3 - p_2p_3^2 - 2p_2q_3 + 2p_3q_1)U^{(1)}_{n-1}V^{(1)}_{n-1}$$
$$+ \Big(p_1^2p_2p_3^2 + 2p_1^2p_2q_3 - p_2^2p_3^3 + 2p_2p_3^2q_1 + 4p_2q_1q_3 - 3p_2^2p_3q_3$$
$$- p_3^3q_2 - p_3q_1^2 - 3p_3q_2q_3\Big)U^{(1)}_{n-2}V^{(1)}_{n-2}.$$

Example 2.21.

$$\sum_{s=1}^{n-3}\big(P_{n-s}j_{n-s} - 14P_{n-1-s}j_{n-1-s} - 4P_{n-2-s}j_{n-2-s} + 4P_{n-3-s}j_{n-3-s}\big)F_sL_s$$
$$= F_nL_n + 7F_{n-1}L_{n-1} + 6F_{n-2}L_{n-2} - P_nj_n,$$

$$\sum_{s=1}^{n-3}\big(5F_{n-s}j_{n-s}-8F_{n-1-s}j_{n-1-s}-2F_{n-2-s}j_{n-2-s}+4F_{n-3-s}j_{n-3-s}\big)P_sQ_s$$
$$=P_nQ_n-P_{n-1}Q_{n-1}-15P_{n-2}Q_{n-2}-2F_nj_n,$$
$$\sum_{s=1}^{n-3}\big(3P_{n-s}L_{n-s}-11P_{n-1-s}L_{n-1-s}-2P_{n-2-s}L_{n-2-s}+P_{n-3-s}L_{n-3-s}\big)J_sj_s$$
$$=J_nj_n+J_{n-1}j_{n-1}-6J_{n-2}j_{n-2}-P_nL_n.$$

Remark 2.22. Taking $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(0,1,p_3,q_3)$ in (2.4) instead of (2.7) yields a more complicated variant of Corollary 2.16 which we will not state here. Similar remarks apply to the identities in Corollaries 2.18 and 2.20.

We now prove a general result in the case when the p and q parameters are the same in both function pairs.

Theorem 2.23. *Suppose $p_1=p_2=p$, $q_1=q_2=q$, $p_3=p_4=y$, and $q_3=q_4=z$. Further, assume that p,q and y,z satisfy $2b_1b_2-2a_1a_2q=(a_1b_2+a_2b_1)p$ and $2b_3b_4-2a_3a_4z=(a_3b_4+a_4b_3)y$. Then for $n\geq 2$,*

$$\sum_{s=0}^{n-2}\Big((p^2-y^2+2q-2z)T^{(3)}_{n-1-s}T^{(4)}_{n-1-s}+(z^2-q^2)T^{(3)}_{n-2-s}T^{(4)}_{n-2-s}\Big)T^{(1)}_sT^{(2)}_s$$
$$=-a_1a_2T^{(3)}_nT^{(4)}_n+(b_1-a_1p)(b_2-a_2p)T^{(3)}_{n-1}T^{(4)}_{n-1}+a_3a_4T^{(1)}_nT^{(2)}_n$$
$$+\big(a_3b_4y+a_4b_3y-b_3b_4-a_3a_4(p^2+2q-2z)\big)T^{(1)}_{n-1}T^{(2)}_{n-1}. \tag{2.9}$$

Proof. By the assumptions on the parameters, we have

$$\sum_{n\geq 0}T^{(1)}_nT^{(2)}_nx^n=\frac{a_1a_2-(b_1-a_1p)(b_2-a_2p)x}{1-(p^2+2q)x+q^2x^2}$$

and

$$\sum_{n\geq 0}T^{(3)}_nT^{(4)}_nx^n=\frac{a_3a_4-(b_3-a_3y)(b_4-a_4y)x}{1-(y^2+2z)x+z^2x^2}.$$

Then the quantity

$$a_3a_4T^{(1)}_nT^{(2)}_n-(b_3-a_3y)(b_4-a_4y)T^{(1)}_{n-1}T^{(2)}_{n-1}-a_1a_2T^{(3)}_nT^{(4)}_n$$
$$+(b_1-a_1p)(b_2-a_2p)T^{(3)}_{n-1}T^{(4)}_{n-1}$$

has generating function given by

$$\sum_{n\geq 0}T^{(1)}_nT^{(2)}_nx^n\cdot\sum_{n\geq 0}T^{(3)}_nT^{(4)}_nx^n\cdot\big((p^2-y^2+2q-2z)x+(z^2-q^2)x^2\big).$$

Extracting the coefficient of x^n gives

$$(p^2-y^2+2q-2z)\sum_{s=0}^{n-1}T^{(3)}_{n-1-s}T^{(4)}_{n-1-s}T^{(1)}_sT^{(2)}_s$$

$$+ (z^2 - q^2) \sum_{s=0}^{n-2} T^{(3)}_{n-2-s} T^{(4)}_{n-2-s} T^{(1)}_s T^{(2)}_s,$$

and equating this with the original quantity leads to (2.9). □

Taking $(0, 1, p_1, q_1)$, $(2, p_1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_2, p_2, q_2)$ in Theorem 2.23, and replacing n by $n + 1$, implies the following result.

Corollary 2.24 (Sequence pairs $\big(U^{(1)}_n V^{(1)}_n\big)$ and $\big(U^{(2)}_n V^{(2)}_n\big)$). *For $n \geq 1$,*

$$\sum_{s=1}^{n-1} \Big((p_2^2 - p_1^2 - 2q_1 + 2q_2) U^{(2)}_{n-s} V^{(2)}_{n-s} + (q_1^2 - q_2^2) U^{(2)}_{n-1-s} V^{(2)}_{n-1-s} \Big) U^{(1)}_s V^{(1)}_s$$
$$= p_1 U^{(2)}_n V^{(2)}_n - p_2 U^{(1)}_n V^{(1)}_n.$$

Example 2.25.

$$3 \sum_{s=1}^{n-1} P_{n-s} Q_{n-s} F_s L_s = P_n Q_n - 2F_n L_n,$$
$$\sum_{s=1}^{n-1} \big(P_{n-s} Q_{n-s} + 3P_{n-1-s} Q_{n-1-s} \big) J_s j_s = P_n Q_n - 2J_n j_n, \tag{2.10}$$
$$\sum_{s=1}^{n-1} \big(2J_{n-s} j_{n-s} - 3J_{n-1-s} j_{n-1-s} \big) F_s L_s = J_n j_n - F_n L_n.$$

Remark 2.26. Taking $(0, 1, p_1, q_1)$, $(2, p_1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_2, p_2, q_2)$ in either (2.4) or (2.7) above instead of (2.9) leads to more complicated variants of Corollary 2.24.

3. Further remarks

In this section, we point out some further extensions of the prior results. We first allow for the indices of the sequences whose terms appear in the identities above to come from an arbitrary arithmetic sequence. Let $k \geq 1$ be fixed and $0 \leq i \leq k - 1$. Then we have the recurrence $U_{nk+i} = V_k U_{(n-1)k+i} - (-q)^k U_{(n-2)k+i}$ for $n \geq 2$, which can be shown using the Binet formulas for U_n and V_n. The same recurrence is seen to hold also for the sequence V_{nk+i}. Thus, taking $a = U_i$, $b = U_{i+k}$, $p = V_k$, $q = -(-q)^k$ or $a = V_i$, $b = V_{i+k}$, $p = V_k$, $q = -(-q)^k$ in Theorem 2.2 gives various formulas involving products of terms derived from the U_{nk+i} and/or the V_{nk+i} sequences.

For example, taking (a_j, b_j, p_j, q_j) for $1 \leq j \leq 4$ to be $\big(0, U^{(1)}_k, V^{(1)}_k, -(-q_1)^k\big)$, $\big(0, U^{(2)}_k, V^{(2)}_k, -(-q_2)^k\big)$, $\big(2, V^{(1)}_k, V^{(1)}_k, -(-q_1)^k\big)$, $\big(2, V^{(2)}_k, V^{(2)}_k, -(-q_2)^k\big)$, respectively, in (2.4) gives

$$U^{(1)}_k U^{(2)}_k V^{(1)}_{(n-1)k} V^{(2)}_{(n-1)k} - (q_1 q_2)^k U^{(1)}_k U^{(2)}_k V^{(1)}_{(n-3)k} V^{(2)}_{(n-3)k}$$

$$
\begin{aligned}
&= 4U_{nk}^{(1)}U_{nk}^{(2)} - 3V_k^{(1)}V_k^{(2)}U_{(n-1)k}^{(1)}U_{(n-1)k}^{(2)} \\
&\quad + 2\Big((-q_1)^k\big(V_k^{(2)}\big)^2 + (-q_2)^k\big(V_k^{(1)}\big)^2 - 2(q_1q_2)^k\Big)U_{(n-2)k}^{(1)}U_{(n-2)k}^{(2)} \\
&\quad - (q_1q_2)^kV_k^{(1)}V_k^{(2)}U_{(n-3)k}^{(1)}U_{(n-3)k}^{(2)}. \qquad (3.1)
\end{aligned}
$$

Note that (3.1) reduces to Corollary 2.3 when $k = 1$. Taking, for instance, $U_k^{(1)} = F_k$, $U_k^{(2)} = P_k$, and $q_1 = q_2 = 1$ in (3.1) gives

$$
\begin{aligned}
F_kP_kL_{(n-1)k}Q_{(n-1)k} - L_{(n-3)k}Q_{(n-3)k} &= 4F_{nk}P_{nk} - 3L_kQ_kF_{(n-1)k}P_{(n-1)k} \\
&\quad + 2\big((-1)^k(L_k^2 + Q_k^2) - 2\big)F_{(n-2)k}P_{(n-2)k} - L_kQ_kF_{(n-3)k}P_{(n-3)k},
\end{aligned}
$$

which reduces to (2.6) when $k = 1$. Further, formula (3.1) represents only the $i = 0$ case of a more general identity, though it is a bit more complex, which involves products of terms from the sequences $U_{nk+i}^{(1)}$, $U_{nk+i}^{(2)}$, $V_{nk+i}^{(1)}$, $V_{nk+i}^{(2)}$ for any $0 \le i \le k-1$.

As another example, letting $\big(0, U_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big)$, $\big(2, V_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big)$, $\big(0, U_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big)$, $\big(2, V_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big)$ in Theorem 2.2, and replacing n by $n+1$, gives

$$
\begin{aligned}
&U_k^{(1)}V_k^{(2)}U_{nk}^{(2)}V_{nk}^{(1)} - 2(-q_2)^kU_k^{(1)}V_k^{(1)}U_{(n-1)k}^{(2)}V_{(n-1)k}^{(1)} \\
&\quad + (q_1q_2)^kU_k^{(1)}V_k^{(2)}U_{(n-2)k}^{(2)}V_{(n-2)k}^{(1)} \\
&= U_k^{(2)}V_k^{(1)}U_{nk}^{(1)}V_{nk}^{(2)} - 2(-q_1)^kU_k^{(2)}V_k^{(2)}U_{(n-1)k}^{(1)}V_{(n-1)k}^{(2)} \\
&\quad + (q_1q_2)^kU_k^{(2)}V_k^{(1)}U_{(n-2)k}^{(1)}V_{(n-2)k}^{(2)}, \qquad (3.2)
\end{aligned}
$$

which reduces to Corollary 2.5 when $k = 1$. From (3.2), one can obtain such identities as

$$
\begin{aligned}
&L_kJ_kF_{nk}j_{nk} - 2(-1)^kJ_kj_kF_{(n-1)k}j_{(n-1)k} + 2^kL_kJ_kF_{(n-2)k}j_{(n-2)k} \\
&= F_kj_kL_{nk}J_{nk} + (-2)^{k+1}F_kL_kL_{(n-1)k}J_{(n-1)k} + 2^kF_kj_kL_{(n-2)k}J_{(n-2)k}.
\end{aligned}
$$

Next, observe the identity

$$
2U_{i+k}V_{i+k} + 2(-q)^kU_iV_i = (U_iV_{i+k} + V_iU_{i+k})V_k, \qquad (3.3)
$$

which follows from combining the formulas

$$
U_iV_{i+k} + V_iU_{i+k} = 2U_{2i+k} \quad \text{and} \quad U_{i+k}V_{i+k} + (-q)^kU_iV_i = U_{2i+k}V_k,
$$

which can be shown using the Binet formulas for U_n and V_n. Thus, by (3.3), the condition $2b_1b_2 - 2a_1a_2q = (a_1b_2 + a_2b_1)p$ in Theorem 2.15 remains satisfied when one considers generalized Fibonacci or Lucas sequences whose indices come from an arbitrary arithmetic progression. Hence, one may apply Theorems 2.15 and 2.23 to obtain analogous identities involving products of terms derived from the sequences U_{nk+i} and V_{nk+i} wherein members of at least one sequence pair share p and q parameter values.

For example, letting

$$\big(0, U_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big), \quad \big(2, V_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big),$$
$$\big(0, U_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big), \quad \big(2, V_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big)$$

in (2.9) gives

$$\begin{aligned}\sum_{s=0}^{n-1}\Big(\Big(\big(V_k^{(2)}\big)^2 - \big(V_k^{(1)}\big)^2 + 2(-q_1)^k - 2(-q_2)^k\Big)U_{(n-s)k}^{(2)}V_{(n-s)k}^{(2)} \\ + (q_1^{2k} - q_2^{2k})U_{(n-1-s)k}^{(2)}V_{(n-1-s)k}^{(2)}\Big)U_{sk}^{(1)}V_{sk}^{(1)} \\ = U_k^{(1)}V_k^{(1)}U_{nk}^{(2)}V_{nk}^{(2)} - U_k^{(2)}V_k^{(2)}U_{nk}^{(1)}V_{nk}^{(1)},\end{aligned} \tag{3.4}$$

which reduces to Corollary 2.24 when $k = 1$. Letting, for instance, $U_k^{(1)} = J_k$, $U_k^{(2)} = P_k$, $q_1 = 2$, and $q_2 = 1$ in (3.4) yields

$$\begin{aligned}\sum_{s=0}^{n-1}\Big((Q_k^2 - j_k^2 - (-2)^{k+1} - 2(-1)^k)P_{(n-s)k}Q_{(n-s)k} \\ + (4^k - 1)P_{(n-1-s)k}Q_{(n-1-s)k}\Big)J_{sk}j_{sk} = J_kj_kP_{nk}Q_{nk} - P_kQ_kJ_{nk}j_{nk},\end{aligned}$$

which reduces to (2.10) when $k = 1$. Formula (3.4) may be generalized to U_{nk+i} and V_{nk+i} for any $0 \le i \le k-1$ by taking $\big(U_i^{(1)}, U_{i+k}^{(1)}, V_k^{(1)}, -(-q_1)^k\big)$ for the first 4-tuple and the analogous quantities for the other three. Generalizations comparable to (3.4) may be given for the identities in Corollaries 2.16, 2.18, and 2.20.

We have the following further general result in the case when $T_n^{(1)}$ and $T_n^{(2)}$ share p and q parameter values.

Theorem 3.1. *Suppose $p_1 = p_2 = p$ and $q_1 = q_2 = q$, with $a_1a_2p = a_1b_2 + a_2b_1$ and $a_1a_2q = -b_1b_2$. Then for $n \ge 4$,*

$$\begin{aligned}\sum_{s=0}^{n-4}\Big((q + p_3p_4)T_{n-1-s}^{(3)}T_{n-1-s}^{(4)} + (p_3^2q_4 + p_4^2q_3 + 2q_3q_4)T_{n-2-s}^{(3)}T_{n-2-s}^{(4)} \\ + p_3p_4q_3q_4T_{n-3-s}^{(3)}T_{n-3-s}^{(4)} - q_3^2q_4^2T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}\Big)T_s^{(1)}T_s^{(2)} \\ = a_1a_2T_n^{(3)}T_n^{(4)} - a_3a_4T_n^{(1)}T_n^{(2)} - (b_3b_4 + a_3a_4q)T_{n-1}^{(1)}T_{n-1}^{(2)} \\ - \big(b_3b_4q + a_3a_4q_3q_4 + a_3b_4p_4q_3 + a_4b_3p_3q_4 + b_3b_4p_3p_4\big)T_{n-2}^{(1)}T_{n-2}^{(2)} \\ - \Big(b_3b_4(p_3^2q_4 + p_4^2q_3 + q_3q_4) + q_3q_4(a_3b_4p_3 + a_4b_3p_4) \\ + (q + p_3p_4)(a_3q_3 + b_3p_3)(a_4q_4 + b_4p_4)\Big)T_{n-3}^{(1)}T_{n-3}^{(2)}.\end{aligned}$$

Proof. Using $a_1a_2p = a_1b_2 + a_2b_1$ and $a_1a_2q = -b_1b_2$, one can show

$$a_1a_2p^2 - b_1b_2 = a_1a_2(p^2 + 2q) + (b_1 - a_1p)(b_2 - a_2p)$$

and

$$q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2 + q)\big) = a_1a_2q^2 + (p^2 + 2q)(b_1 - a_1p)(b_2 - a_2p),$$

from which it follows the factorization

$$\begin{aligned} &a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2 + q)\big)x^2 \\ &\quad - q^2(b_1 - a_1p)(b_2 - a_2p)x^3 \\ &= \big(a_1a_2 - (b_1 - a_1p)(b_2 - a_2p)x\big)\big(1 - (p^2 + 2q)x + q^2x^2\big). \end{aligned}$$

Thus if $T_n^{(1)}$ and $T_n^{(2)}$ are such that their parameters satisfy the required conditions, then by (2.3) the generating function $\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n$ is given by

$$\begin{aligned} &\frac{a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2 + q)\big)x^2}{1 - p^2x - 2q(p^2 + q)x^2 - p^2q^2x^3 + q^4x^4} \\ &\quad - \frac{q^2(b_1 - a_1p)(b_2 - a_2p)x^3}{1 - p^2x - 2q(p^2 + q)x^2 - p^2q^2x^3 + q^4x^4} \\ &= \frac{\big(a_1a_2 - (b_1 - a_1p)(b_2 - a_2p)x\big)\big(1 - (p^2 + 2q)x + q^2x^2\big)}{(1 + 2qx + q^2x^2)\big(1 - (p^2 + 2q)x + q^2x^2\big)} \\ &= \frac{a_1a_2 - (b_1 - a_1p)(b_2 - a_2p)x}{1 + 2qx + q^2x^2} = \frac{a_1a_2 - b_1b_2x}{1 + 2qx + q^2x^2} \\ &= \frac{a_1a_2(1 + qx)}{(1 + qx)^2} = \frac{a_1a_2}{1 + qx}. \end{aligned}$$

The proof is completed in a similar manner as before upon considering the generating function of the quantity

$$\begin{aligned} &a_1a_2T_n^{(3)}T_n^{(4)} - a_3a_4T_n^{(1)}T_n^{(2)} - (b_3b_4 - a_3a_4p_3p_4)T_{n-1}^{(1)}T_{n-1}^{(2)} \\ &\quad - \big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4)\big)T_{n-2}^{(1)}T_{n-2}^{(2)} \\ &\quad + q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4)T_{n-3}^{(1)}T_{n-3}^{(2)}. \end{aligned}$$

□

Remark 3.2. If $p = q = 1$ in the prior theorem, then the a_i and b_i satisfy $a_1a_2 = a_1b_2 + a_2b_1 = -b_1b_2$. Replacing b_2 with $-b_2$, we then have $a_1a_2 = a_1b_2 - a_2b_1 = b_1b_2$. If all variables are positive in the last system, then eliminating b_2 leads to the equality $a_1b_1 = a_1^2 - b_1^2$, where a_2 can be chosen arbitrarily and $b_2 = \frac{a_1a_2}{b_1}$. This essentially covers all the cases when a_1a_2 is non-zero, upon considering separately when a_1a_2 is positive or negative and renaming quantities as needed. Note that the case when a_1a_2 is zero is trivial since one (or both) of $T_n^{(1)}$ and $T_n^{(2)}$ is seen to be the sequence of all zeros in that case.

In analogy with Theorem 2.23 above, we have the following further result when $T_n^{(3)}$ and $T_n^{(4)}$ also share p and q parameter values.

Theorem 3.3. *Suppose $p_1 = p_2 = p$, $q_1 = q_2 = q$, $p_3 = p_4 = y$, and $q_3 = q_4 = z$. Further, assume that p, q and y, z satisfy $a_1a_2p = a_1b_2 + a_2b_1$, $a_1a_2q = -b_1b_2$ and $a_3a_4y = a_3b_4 + a_4b_3$, $a_3a_4z = -b_3b_4$. Then for $n \geq 1$,*

$$(z - q)\sum_{s=0}^{n-1} T_{n-1-s}^{(3)}T_{n-1-s}^{(4)}T_s^{(1)}T_s^{(2)} = a_3a_4T_n^{(1)}T_n^{(2)} - a_1a_2T_n^{(3)}T_n^{(4)}.$$

Acknowledgements. The authors are grateful to the referee for useful comments that helped to improve the quality of the paper.

References

[1] K. Adegoke: *An identity for second order sequences obeying the same recurrence relation*, Preprint (2019), URL: https://arxiv.org/abs/1901.08662.

[2] K. Adegoke: *Identities for second order recurrence sequences*, Preprint (2018), URL: https://arxiv.org/abs/1805.06765.

[3] K. Adegoke, R. Frontczak, T. Goy: *Some special sums with squared Horadam numbers and generalized tribonacci numbers*, Palestine Math. J. 11.1 (2022), pp. 66–73.

[4] A. Bramham, M. Griffiths: *Combinatorial interpretations of some convolution identities*, Fibonacci Quart. 54.4 (2016), pp. 335–339.

[5] R. Frontczak: *Identities for generalized balancing numbers*, Notes Number Theory Discrete Math. 25.2 (2019), pp. 169–180, DOI: https://doi.org/10.7546/nntdm.2019.25.2.169-180.

[6] R. Frontczak: *Relations for generalized Fibonacci and Tribonacci sequences*, Notes Number Theory Discrete Math. 25.1 (2019), pp. 178–192, DOI: https://doi.org/10.7546/nntdm.2019.25.1.178-192.

[7] R. Frontczak: *Some Fibonacci-Lucas-Tribonacci-Lucas identities*, Fibonacci Quart. 56.3 (2018), pp. 263–274.

[8] R. Frontczak, T. Goy: *Mersenne-Horadam identities using generating functions*, Carpathian Math. Publ. 12.1 (2020), pp. 34–45, DOI: https://doi.org/10.15330/cmp.12.1.34-45.

[9] G. C. Greubel, R. Frontczak: *Solution to Problem B-756*, Fibonacci Quart. 58.1 (2020), pp. 82–84.

[10] T. Koshy: *Sums of Fibonacci-Pell-Jacobsthal products*, Int. J. Math. Educ. Sci. Technol. 44.4 (2012), pp. 559–568, DOI: https://doi.org/10.1080/0020739X.2012.729679.

[11] T. Koshy, M. Griffiths: *Some gibonacci convolutions with dividends*, Fibonacci Quart. 56.3 (2018), pp. 237–245.

[12] S. Rabinowitz: *Problem B-756*, Fibonacci Quart. 32.1 (1994), p. 85; Reprinted in Fibonacci Quart. 57.1 (2019), p. 81.

[13] H.-J. Seiffert: *Problem B-759*, Fibonacci Quart. 32.1 (1994), p. 86.

[14] N. J. A. Sloane: *The On-Line Encyclopedia of Integer Sequences*, Published electronically at https://oeis.org.

[15] T. Szakács: *Convolution of second order linear recursive sequences I*, Ann. Math. Inform. 46 (2016), pp. 205–216.

[16] T. Szakács: *Convolution of second order linear recursive sequences II*, Commun. Math. 25 (2017), pp. 137–148, DOI: https://doi.org/10.1515/cm-2017-0011.

Annales Mathematicae et Informaticae
55 (2022) pp. 49–75
DOI: https://doi.org/10.33039/ami.2022.02.001
URL: https://ami.uni-eszterhazy.hu

Exploring self-intersected N-periodics in the elliptic billiard

Ronaldo Garcia[a*], **Dan Reznik**[b]

[a]Fed. Univ. of Goiás, Goiânia, Brazil
ragarcia@ufg.br

[b]Data Science Consulting, Rio de Janeiro, Brazil
dreznik@gmail.com

Abstract. This is a continuation of our simulation-based investigation of N-periodic trajectories in the elliptic billiard. With a special focus on self-intersected trajectories we (i) describe new properties of $N = 4$ family, (ii) derive expressions for quantities recently shown to be conserved, and to support further experimentation, we (iii) derive explicit expressions for vertices and caustic semi-axes for several families. Finally, (iv) we include links to several animations of the phenomena.

Keywords: Invariant, elliptic, billiard, turning number, self-intersected

AMS Subject Classification: 51M04, 51N20, 51N35, 68T20

1. Introduction

This is a continuation of our simulation-based investigation of periodic trajectories in the elliptic billiard, i.e., Poncelet families of polygons interscribed between two confocal conics (see Appendix A for a review).

Here we focus on trajectories which are self-intersected, i.e., which wrap around the inner conic, or *caustic*, more than once (i.e., their turning number is greater than one [24]). Figure 1 (resp. 2) illustrate cases where the caustic is an ellipse (resp. hyperbola).

Specifically, we (i) describe some curious Euclidean properties, loci, and invariants of the $N = 4$ self-intersected family (Section 3); (ii) derive expressions for some conserved quantities presented in [20, 21], for both simple and self-intersected cases

*Supported by PRONEX/CNPq/FAPEG 2017-10-26-7000-508.

Submitted: April 5, 2021
Accepted: February 11, 2022
Published online: February 22, 2022

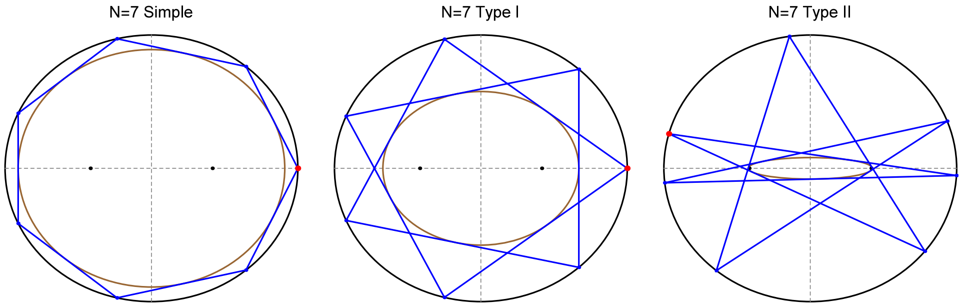


Figure 1. A simple (left), and two self-intersecting (middle, right) 7-periodics in the elliptic billiard. In the article the latter two are labeled "type I", and "type II" (turning number 2 and, 3 respectively). Video 1 Video 2

(Section 4). Interestingly, we identify a few situations where quantities conserved elsewhere become variable (reasons are unclear).

One of our goals is to encourage and support further simulation work. Toward that end, we include links to animated phenomena in the caption of most figures, all of which appear on Table 2. In Appendix B we provide expressions for both vertices and caustics for several families. Appendix C lists most symbols used herein.

1.1. Related work

Birkhoff provides a method to compute the number of possible Poncelet N-periodics, simple or not [7]. For example, for $N = 5, 6, 7, 8$ there are $1, 2, 2, 3$ distinct self-intersected closed trajectories, respectively. In [18] expressions are derived for caustic parameters which produce various types of N-periodics in the elliptic billiard. Points of self-intersections of Poncelet N-periodics are located on confocal conics of the associated Poncelet grid [16, 22, 25]. A kinematic analysis of the geometry of N=periodics using Jacobian elliptic functions is proposed in [24]. Works [13, 19] derive explicit expressions for some invariants in the $N = 3$ case (billiard triangles). Additional constructions derived from N-periodics (e.g., pedals, antipedals, etc.) are considered in [20], augmenting the list of elliptic billiard invariants to 80. In recent publications [13, 19, 21] we have described several Euclidean quantities which remain invariant over a given family, some of which have been subsequently proved [2, 5, 8].

2. Preliminaries

Throughout this article we assume the elliptic billiard is the ellipse:

$$f(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \quad a > b > 0.$$

Below we refer to trajectories with turning number 1, 2, 3, and 4, by simple, type I, type II, and type III, respectively.

2.1. A word about our proof method

We omit most proofs as they have been produced by a consistent process, namely: (i) using the expressions in Appendix B, find the vertices an axis-symmetric N-periodic, i.e., whose first vertex $P_1 = (a, 0)$; (ii) obtain a symbolic expression for the invariant of interest; (iii) derive an expression for the given quantity for a generic trajectory parametrized by t; (iv) using CAS simplification, show that t can be eliminated, i.e., that (iii) reduces to (ii).

3. Properties of self-intersected 4-periodics

The family of simple 4-periodics in the elliptic billiard are parallelograms [9]. In this section consider self-intersected 4-periodics whose caustic is a confocal hyperbola; see Figure 2. We start deriving simple facts about them and then proceed to certain elegant properties.

Proposition 3.1. *The perimeter L of the self-intersected 4-periodic is given by:*

$$L = \frac{4a^2}{c}, \quad \text{with} \quad c^2 = a^2 - b^2. \tag{3.1}$$

Proof. Since perimeter is constant, use as the $N = 4$ candidate the centrally-symmetric one, Figure 2 (right). Its upper-right vertex $P_1 = (x_1, y_1)$ is such that it reflects a vertical ray toward $-P_1$, and this yields:

$$P_1 = (x_1, y_1) = \left[\frac{a\sqrt{a^2 - 2b^2}}{bc}, \frac{b}{c}\right].$$

Since $P_2 = -P_1$ its perimeter is $L = 2(|2P1| + 2y_1)$ and this can be simplified to (3.1), invariant over the family. □

with $a/b \geq \sqrt{2}$. At $a/b = \sqrt{2}$ the family is a straight line from top to bottom vertex of the elliptic billiard, Figure 2 (left).

Observation 3.2. *At $a/b = \sqrt{1 + \sqrt{2}} \simeq 1.55377$ the two self-intersecting segments of the bowtie do so at right-angles.*

Observation 3.3. *At $a/b \simeq 1.55529$ the perimeter of the bowtie equal that of the elliptic billiard.*

Referring to Figure 3:

Proposition 3.4. *The $N = 4$ self-intersected family has zero signed orbit area and zero sum of signed cosines, i.e., both are invariant. The same two facts are true for its outer polygon. Furthermore the latter has zero sum of double-angle signed cosines.*

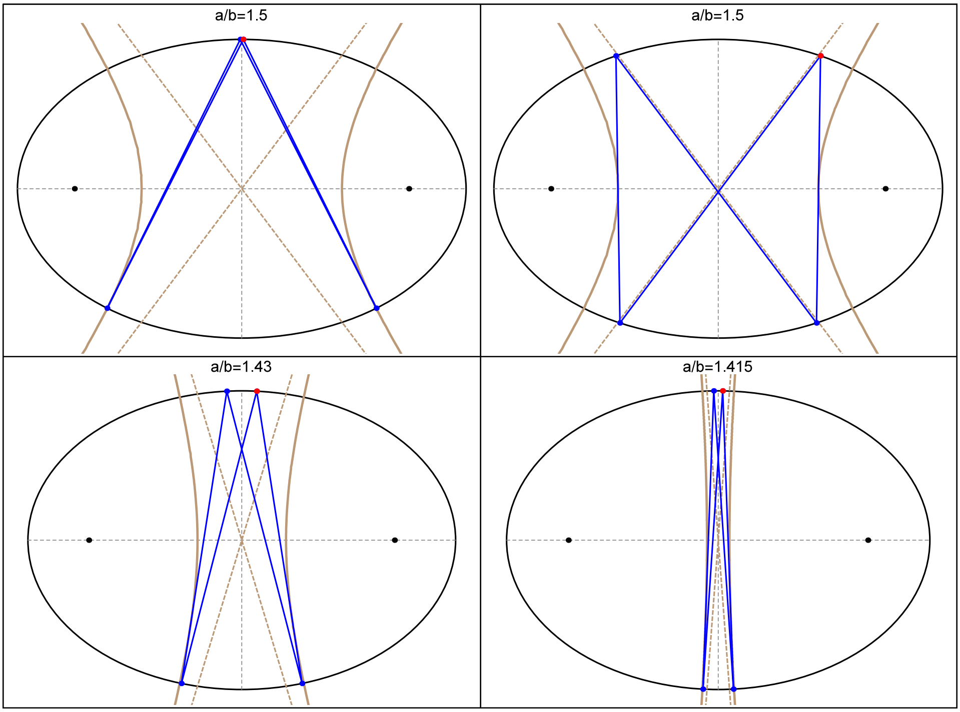


Figure 2. In the top left (resp. top right) an $N = 4$ self-intersected trajectory is shown near its "doubled up" (resp. almost symmetric) position. In both cases trajectory segments are tangent to a hyperbolic caustic (brown). As the aspect ratio of the elliptic billiard decreases (bottom left and right), the family is squeezed into an ever narrower space between the approaching branches of the caustic. Video

Proof. This stems from the fact all self-intersected 4-periodics are symmetric with respect to the elliptic billiard's minor axis. □

Referring to Figure 3, as in Appendix B.2, let vertex P_1 of the self-intersected 4-periodic be parametrized as $P_1(u) = \left[au, b\sqrt{1-u^2}\right]$, with $|u| \leq \frac{a}{c^2}\sqrt{a^2 - 2b^2}$. Then:

Theorem 3.5. *The four vertices of the self-intersected 4-periodic (resp. outer polygon) are concyclic with the two foci of the elliptic billiard, on a circle $\mathcal{C}$ of variable radius R (resp. R') whose center C (resp. C') lies on the y axis. These are given by:*

$$C = \left[0, \frac{c^2u^2 - a^2 + 2b^2}{2b\sqrt{1-u^2}}\right], \qquad C' = \left[0, -\frac{2bc^2\sqrt{1-u^2}}{a^2 + (u^2-2)c^2}\right],$$

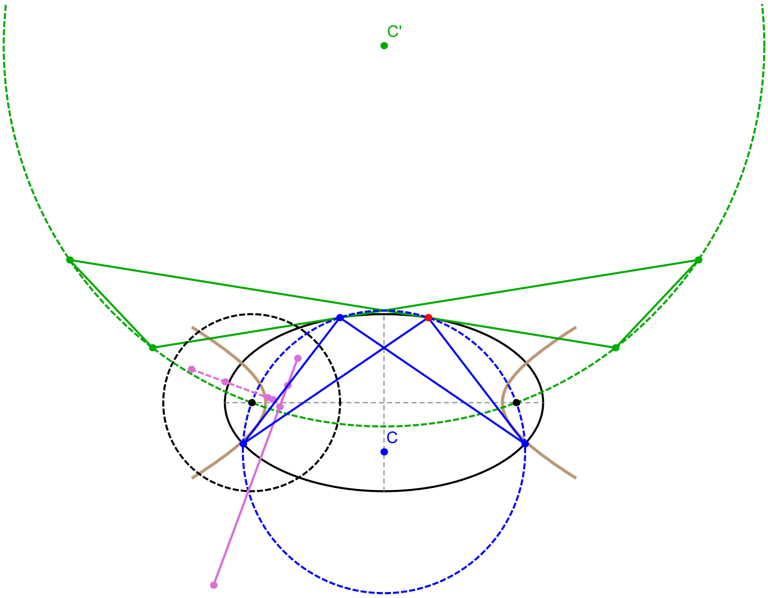


Figure 3. The vertices of the self-intersected 4-periodic (blue) are concyclic with the foci of the elliptic billiard on a circle (dashed blue) centered on C. The inversive polygon (pink segment) with respect to a unit circle $C^\dagger$ (dashed black) centered on the left focus degenerates to a segment along the radical axis of the two circles. The vertices of the outer polygon (green) are also concyclic with the foci on a distinct circle (dashed green) centered on C'. Therefore the outer's inversive polygon (dotted pink) is also a segment along the radical axis of this circle with $C^\dagger$. Note the two radical axes are dynamically perpendicular. Video 1 Video 2

$$R = \frac{a^2 - c^2u^2}{2b\sqrt{1-u^2}}, \qquad R' = \frac{c(c^2u^2 - a^2)}{a^2 + (u^2-2)c^2}.$$

Corollary 3.6. *The half harmonic mean of R^2 and R'^2 is invariant and equal to $c^2 = a^2 - b^2$, i.e., $1/R^2 + 1/R'^2 = 1/c^2$.*

Note: the above Pythagorean relation implies that the polygon whose vertices are a focus, and the inversion of C, C', O (center of the elliptic billiard) with respect to a unit circle centered on said focus, is a rectangle of sides $1/R$ and $1/R'$ and diagonal $1/c$.

Remarkably:

Corollary 3.7. *Over the self-intersected $N = 4$ family, the power of the origin with respect to both $\mathcal{C}$ and $\mathcal{C}'$ is invariant and equal to $b^2 - a^2$.*

Referring to Figure 4, $N = 4$ self-intersected trajectories are *anti-parallelograms* [27]: these are images of vertices of a parallelogram reflected on opposite diagonals.

A well-know property is that the midpoints of its four segments are collinear on a horizontal line parallel to a diagonal.

Observation 3.8. *The locus of midpoints of $N = 4$ self-intersected segments is an ∞-shaped quartic curve given by:*

$$c^2(b^2x^2 + a^2y^2)^2 - b^4a^2\big((a^2 - 2b^2)x^2 - a^2y^2\big) = 0.$$

Furthermore, the above quartic is tangent to the confocal hyperbolic caustic at its vertices $[\pm a\sqrt{a^2 - 2b^2}/c, 0]$.

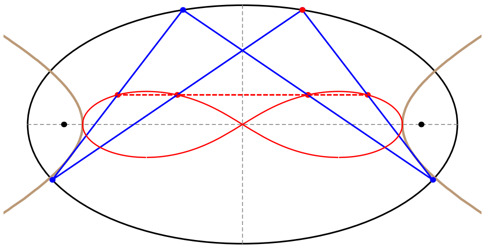


Figure 4. The midpoints of each of the four segments of self-intersected 4-periodics are collinear on a horizontal line. Their locus is an ∞-shaped quartic which touches the caustic at its vertices. Video

Let $\mathcal{P}^\dagger$ (resp. $\mathcal{Q}^\dagger$) denote the inversive polygon of 4-periodics (resp. its outer polygon) wrt a unit circle $\mathcal{C}^\dagger$ centered on one focus. From properties of inversion:

Corollary 3.9. *$\mathcal{P}^\dagger$ (resp. $\mathcal{Q}^\dagger$) has four collinear vertices, i.e., it degenerates to a segment along the radical axis of $\mathcal{C}^\dagger$ and $\mathcal{C}$ (resp. $\mathcal{C}'$).*

Proposition 3.10. *The two said radical axes are perpendicular.*

Proof. It is enough to check that the vectors $C - [-c, 0]$ and $C' - [-c, 0]$ are orthogonal. Observe that when $u^2 = (a^2 - 2b^2)/c = (2c^2 - a^2)/c$ the outer polygon is contained in the horizontal axis. □

Observation 3.11. *The pairs of opposite sides of the outer polygon to self-intersected 4-periodics intersect at the top and bottom of the circle (C, R) on which the 4-periodic vertices are concyclic.*

4. Deriving both simple and self-intersected invariants

In this section, we derive expressions for selected invariants introduced in [20], specifically for "low-N" cases, e.g., $N = 3, 4, 5, 6, 8$. In that publication, each invariant is identified by a 3-digit code, e.g., k_{101}, k_{102}, etc. Table 1 lists the invariants considered herein. The quantities involved are defined next.

Table 1. List of selected invariants taken from [20] as well as the low-N cases (column "derived") for expressions are derived herein, where N refers to simple N-periodics, and N_i (resp. N_{ii}) refers to type I (resp. type II) N-periodics. Refer to Table 3 for the meaning of symbols in column "invariant". † L_1 was co-discovered with P. Roitman. A closed-form expression for k_{119} was derived by H. Stachel; see (A.1).

code	invariant	valid N	derived	proofs
k_{101}	$\sum \cos\theta_i$	all	$JL - N$	[2, 5]
k_{102}	$\prod \cos\theta_i'$	all	3,4,5,5_i,6,6_i,6_{ii}	[2, 5]
k_{103}	A'/A	odd	3,5,5_i	[2, 8]
k_{104}	$\sum \cos(2\theta_i')$	all	3,4,4_i,5,5_i,6,6_i,8	[1]
k_{105}	$\prod \sin(\theta_i/2)$	odd	3,5,5_i	[1]
k_{106}	$A'A$	even	4,4_i,6,6_i,6_{ii}	[8]
k_{110}	$A\,A''$	even	4,4_i,6,6_i,6_{ii}	?
$^\dagger k_{119}$	$\sum \kappa_i^{2/3}$	all	3,4,6	[2, 23]
$k_{802,a}$	$\sum 1/d_{1,i}$	all	3,4,6	[2]
k_{803}	$L_1^\dagger$	all	3,4,6	?
$^\dagger k_{804}$	$\sum \cos\theta_{1,i}^\dagger$	$\neq$4	3	?
$k_{805,a}$	$A\,A_1^\dagger$	$\equiv 0 \pmod 4$	4,4_i,8	?
k_{806}	$A/A_1^\dagger$	$\equiv 2 \pmod 4$	6	?
k_{807}	$A_1^\dagger.A_2^\dagger$	odd	3	?

Let θ_i denote the ith N-periodic angle. Let A the signed area of an N-periodic. Referring to Figure 5, singly-primed quantities (e.g., θ_i', A', etc.), etc., always refer to the *outer polygon*: its sides are tangent to the elliptic billiard at the P_i. Likewise, doubly-primed quantities (θ_i'', A'', etc.) refer to the *inner polygon*: its vertices lie at the touchpoints of N-periodic sides with the caustic. More details on said quantities appear in Appendix A.

Recall $k_{101} = JL - N$, as introduced in [5, 19].

Referring to Figure 6, the f_1-*inversive polygon* has vertices at inversions of the P_i with respect to a unit circle centered on f_1. Quantities such as $L_1^\dagger$, $A_1^\dagger$, etc., refer to perimeter, area, etc. of said polygon.

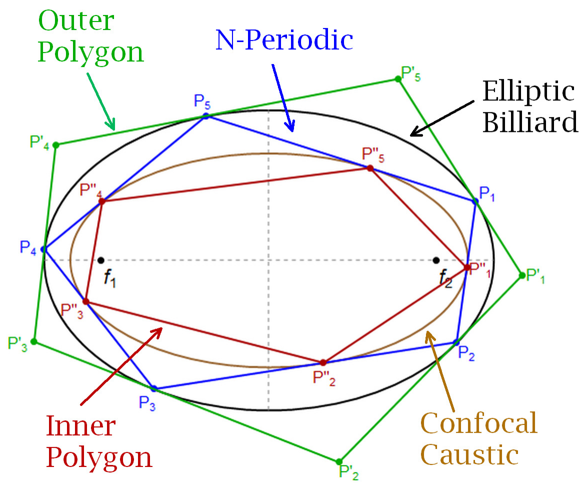


Figure 5. The N-periodic (blue), is associated with an outer (green) and an inner (red) polygons. The former's sides are tangent to the billiard (black) at each N-periodic vertex; the latter's vertices are the tangency points of N-periodics sides to the confocal caustic (brown). Video

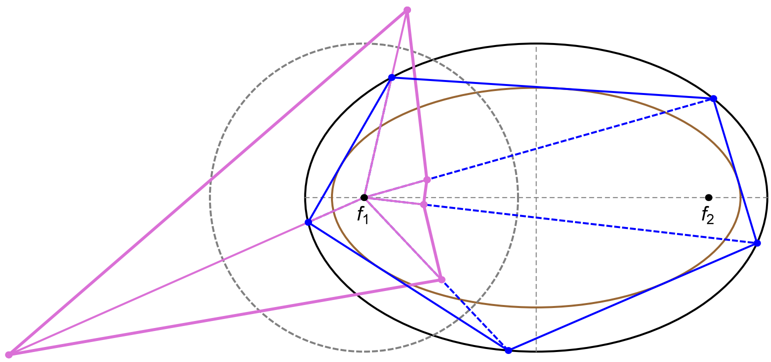


Figure 6. Focus-inversive 5-periodic (pink) whose vertices are inversions of the P_i (blue) with respect to a unit circle (dashed black) centered on f_1. It turns out its perimeter is also invariant over the family as is the sum of its spoke lengths (pink lines). Video

4.1. Invariants for $N = 3$

As before, let $\delta = \sqrt{a^4 - a^2b^2 + b^4}$. For $N = 3$ explicit expressions for J and L have been derived [13]:

$$J = \frac{\sqrt{2\delta - a^2 - b^2}}{c^2}, \quad L = 2(\delta + a^2 + b^2)J. \tag{4.1}$$

When $a = b$, $J = \sqrt{3}/2$ and when $a/b \to \infty$, $J \to 0$.

Proposition 4.1. *For $N = 3$, $k_{102} = (JL)/4 - 1$.*

Proof. We've shown $\sum_{i=1}^{3} \cos\theta_i = JL - 3$ is invariant for the $N = 3$ family [13]. For any triangle $\sum_{i=1}^{3} \cos\theta_i = 1 + r/R$ [28], so it follows that $r/R = JL - 4$ is also invariant. Let r_h, R_h be the Orthic Triangle's Inradius and Circumradius. The relation $r_h/R_h = 4\prod_{i=1}^{3} |\cos\theta_i|$ is well-known [28, Orthic Triangle]. Since a triangle is the Orthic of its Excentral Triangle, we can write $r/R = 4\prod_{i=1}^{3} \cos\theta_i'$, where θ_i' are the Excentral angles which are always acute [28] (absolute value can be dropped), yielding the claim. □

Proposition 4.2. *For $N = 3$, $k_{103} = k_{109} = 2/(k_{101} - 1) = 2/(JL - 4)$.*

Proof. Given a triangle A' (resp. A'') refers to the area of the Excentral (resp. Extouch) triangles. The ratios A'/A and A/A'' are equal. Actually, $A'/A = A/A'' = (s_1 s_2 s_3)/(r^2 L)$, where s_i are the sides, L the perimeter, and r the Inradius [28, Excentral, Extouch]. Also known is that $A'/A = 2R/r$ [14]. Since $r/R = \sum_{i=1}^{3} \cos\theta_i - 1 = k_{101} - 1$ [28, Inradius], the result follows. □

Proposition 4.3. *For $N = 3$, $k_{104} = -k_{101}$ and is given by:*

$$k_{104} = \frac{(a^2 + b^2)(a^2 + b^2 - 2\delta)}{c^4} = 3 - JL.$$

Proposition 4.4. *For $N = 3$, $k_{105} = (JL)/4 - 1 = k_{102}$.*

Proof. Let r, R be a triangle's Inradius and Circumradius. The identity $r/R = 4\prod_{i=1}^{3} \sin(\theta_i/2)$ holds for any triangle [28, Inradius], which with Proposition 4.1 This completes the proof. □

Proposition 4.5. *For $N = 3$, k_{119} is given by:*

$$(k_{119})^3 = \frac{2J^3 L}{(JL - 4)^2}, \quad k_{119} = \frac{a^2 + b^2 + \delta}{(ab)^{\frac{4}{3}}}.$$

Proof. Use the expressions for L, J in (4.1). □

Proposition 4.6. *For $N = 3$:*

$$\begin{aligned}
k_{802,a} &= \frac{a^2 + b^2 + \delta}{ab^2} = \frac{J\sqrt{2}\sqrt{JL + \sqrt{9 - 2JL} - 3}}{JL - 4},\\
k_{803} &= \rho\frac{\sqrt{(8a^4 + 4a^2b^2 + 2b^4)\delta + 8a^6 + 3a^2b^4 + 2b^6}}{a^2b^2},\\
k_{804} &= \frac{\delta(a^2 + c^2 - \delta)}{a^2c^2},\\
k_{807} &= \frac{\rho^8}{8a^8b^2}\left[(a^4 + 2a^2b^2 + 4b^4)\delta + a^6 + (3/2)a^4b^2 + 4b^6\right].
\end{aligned}$$

Note: ρ is the radius of the inversion circle, included above for unit consistency. By default $\rho = 1$.

4.2. Invariants for $N = 4$

Proposition 4.7. *For simple $N = 4$, $k_{102} = 0$.*

Proof. Simple 4-periodics are parallelograms [9] whose outer polygon is a rectangle inscribed in Monge's Orthoptic Circle [19]. This finishes the proof. □

Proposition 4.8. *For simple $N = 4$, $k_{104} = -4$.*

Proof. As in Proposition 4.7, outer polygon is a rectangle. □

Let $\kappa_a = (ab)^{-2/3}$ denote the affine curvature of the ellipse and $r_m = \sqrt{a^2 + b^2}$ the radius of Monge's orthoptic circle [28].

Proposition 4.9. *For simple $N = 4$:*

$$k_{106} = 8a^2b^2, \qquad k_{110} = \frac{2a^4b^4}{(a^2+b^2)^2},$$

$$k_{119} = \frac{2(a^2+b^2)}{(ab)^{\frac{4}{3}}} = 2(\kappa_a r_m)^2, \qquad k_{802,a} = \frac{2(a^2+b^2)}{ab^2},$$

$$k_{803} = \frac{4\rho^2\sqrt{a^2+b^2}}{b^2}, \qquad k_{805,a} = 4.$$

Note: when $b = 1$, k_{803} is equal to the perimeter of the 4-periodic; see (B.1).

Counter-example 4.10. *Experimentally, k_{804} is invariant for all simple N-periodics, except when $N = 4$.*

Restating results from Proposition 3.4:

Observation 4.11. *Over self-intersected $N = 4$, $k_{101} = k_{104} = 0$. Since A is null, so are k_{106}, k_{110}, and $k_{805,a}$.*

We leave as exercises the derivation of expressions for k_{102}, k_{119}, $k_{802,a}$, and k_{803} over self-intersected $N = 4$.

4.3. Invariants for $N = 5$

As seen in Appendix B, the vertices of 5-periodics can only be obtained via an implicitly-defined caustic. Namely, we first numerically obtain the caustic semi-axes and then compute a axis-symmetric polygon tangent to it. Note that both simple and self-intersected 5-periodics possess an elliptic confocal caustic; see Figure 7.

Proposition 4.12. *For simple (resp. self-intersected) $N = 5$, k_{102} is given by the largest negative (resp. positive) real root of the following 6th-degree polynomial:*

$$\begin{aligned} k_{102} : \; & 1024c^{20}x^6 + 2048(a^4 + a^3b - ab^3 + b^4)(a^4 - a^3b + ab^3 + b^4)c^{12}x^5 \\ & + 256(4a^{12} - a^{10}b^2 + 32a^8b^4 - 22a^6b^6 + 32a^4b^8 - a^2b^{10} + 4b^{12})c^8x^4 \end{aligned}$$

$$- 64a^2b^2(4a^{12} - 27a^{10}b^2 + 38a^8b^4 - 126a^6b^6 + 38a^4b^8 - 27a^2b^{10} + 4b^{12})c^4x^3$$
$$- 16a^6b^6(7a^8 - 96a^6b^2 + 114a^4b^4 - 96a^2b^6 + 7b^8)x^2$$
$$- 8a^8b^8(7a^4 + 30a^2b^2 + 7b^4)x - a^{10}b^{10} = 0.$$

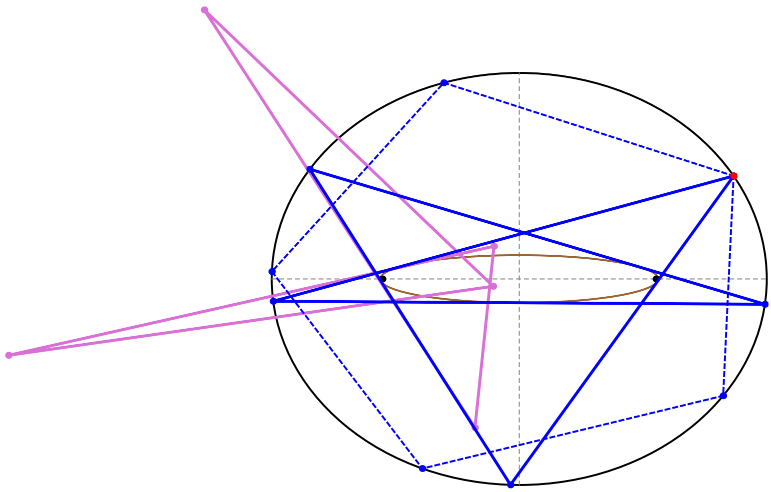


Figure 7. A simple (blue) and self-intersected (dashed blue) 5-periodic, as well as the former's focus-inversive polygon (pink). Video

Proposition 4.13. *For simple (resp. self-intersected) $N = 5$, k_{103} is given by the smallest (resp. largest) real root greater than 1 of the following 6th-degree polynomial:*

$$k_{103} : a^6b^6x^6 - 2b^2a^2\left(4a^8 - a^6b^2 - a^2b^6 + 4b^8\right)x^5$$
$$- b^2a^2\left(4a^8 + 19a^6b^2 - 62a^4b^4 + 19a^2b^6 + 4b^8\right)x^4$$
$$+ 12b^2a^2\left(a^4 + b^4\right)c^4x^3 + \left(4a^8 + 19a^6b^2 + 66a^4b^4 + 19a^2b^6 + 4b^8\right)c^4x^2$$
$$+ \left(2a^8 + 12a^6b^2 + 36a^4b^4 + 12a^2b^6 + 2b^8\right)c^4x - c^{12}.$$

Proposition 4.14. *For simple (resp. self-intersected) $N = 5$, k_{104} is given by the only negative (resp. smallest largest) real root of the following 6th-degree polynomial:*

$$c^{12}x^6 - 2(a^4 + 10a^2b^2 + b^4)c^8x^5 - (37a^4 - 6a^2b^2 + 37b^4)c^8x^4$$
$$+ 4(5a^8 + 92a^6b^2 + 62a^4b^4 + 92a^2b^6 + 5b^8)c^4x^3$$
$$+ (423a^{12} - 354a^{10}b^2 + 2713a^8b^4 - 4796a^6b^6 + 2713a^4b^8 - 354a^2b^{10} + 423b^{12})x^2$$
$$+ (270a^{12} + 740a^{10}b^2 - 3630a^8b^4 + 7160a^6b^6 - 3630a^4b^8 + 740a^2b^{10} + 270b^{12})x$$
$$- 675a^{12} - 850a^{10}b^2 + 1075a^8b^4 - 3900a^6b^6 + 1075a^4b^8 - 850a^2b^{10} - 675b^{12}.$$

Proposition 4.15. *For simple (resp. self-intersected) $N = 5$, k_{105} is given by the largest positive real root (resp. the symmetric value of the largest negative root) of the following 6th-degree polynomial:*

$$\begin{aligned}
&2^{10}c^{20}x^6 + 2^{10}(2a^{12} + a^{10}b^2 + 26a^8b^4 + 70a^6b^6 + 26a^4b^8 + a^2b^{10} + 2b^{12})c^8x^5\\
&+ 2^8(4a^{12} + 30a^{10}b^2 + 71a^8b^4 + 350a^6b^6 + 71a^4b^8 + 30a^2b^{10} + 4b^{12})c^8x^4\\
&+ 2^6a^2b^2(4a^{12} + 9a^{10}b^2 - 318a^8b^4 - 126a^6b^6 - 318a^4b^8 + 9a^2b^{10} + 4b^{12})c^4x^3\\
&- 2^6a^2b^2(8a^{16} - 53a^{14}b^2 + 253a^{12}b^4 - 1041a^{10}b^6 + 1650a^8b^8\\
&\qquad - 1041a^6b^{10} + 253a^4b^{12} - 53a^2b^{14} + 8b^{16})x^2\\
&- 2^4a^2b^2(16a^{16} - 12a^{14}b^2 + 5a^{12}b^4 + a^{10}b^6 + 2a^8b^8 + a^6b^{10}\\
&\qquad + 5a^4b^{12} - 12a^2b^{14} + 16b^{16})x\\
&- a^{10}b^{10}.
\end{aligned}$$

4.4. Invariants for $N = 6$

Referring to Figure 5 (right):

Proposition 4.16. *For simple $N = 6$:*

$$\begin{aligned}
k_{102} &= a^2b^2/(4(a+b)^4) = (JL-4)^2/64,\\
k_{104} &= k_{101} = JL - 6,\\
k_{106} &= \frac{4b^2(2a+b)a^2(a+2b)}{(a+b)^2} = -\frac{(JL-12)(JL-4)^2}{16J^4},\\
k_{110} &= \frac{4a^3b^3(2a+b)^2(a+2b)^2}{(a+b)^6} = -\frac{(JL-12)^2(JL-4)^3}{256J^4},\\
k_{119}^3 &= \frac{2^5J^5L^3}{(JL-4)^4},\\
k_{802,a} &= \frac{2(a^2+ab+b^2)}{ab^2} = \frac{4J^2L\left(1+\sqrt{JL-3}\right)}{(JL-4)^2},\\
k_{803} &= 2\rho^2(2a^2+2ab-b^2)/(ab^2),\\
k_{806} &= 4\rho^{-4}a^3b^4/((2a-b)(a+b)^2).
\end{aligned}$$

4.5. $N = 6$ self-intersecting

Only two $N = 6$ topologies can produce closed trajectories, both with two self-intersections. These will be referred to as type I and type II, and are depicted in Figures 8, and 9, respectively.

Proposition 4.17. *For $N = 6$ type I:*

$$k_{102} = a^2b^2/(4(a-b)^4) = (J\,L-4)^2/64,$$

$$k_{104} = -\frac{2(a^2 - 4ab + b^2)}{(a-b)^2} = JL - 6 = k_{101},$$
$$k_{106} = 4a^2b^2(a-2b)(2a-b)/(a-b)^2 = -(JL-12)(JL-4)^2/(16J^4),$$
$$k_{110} = \frac{-4a^3b^3(a-2b)^2(2a-b)^2}{(a-b)^6} = \frac{(JL-12)^2(JL-4)^3}{2^8J^4}.$$

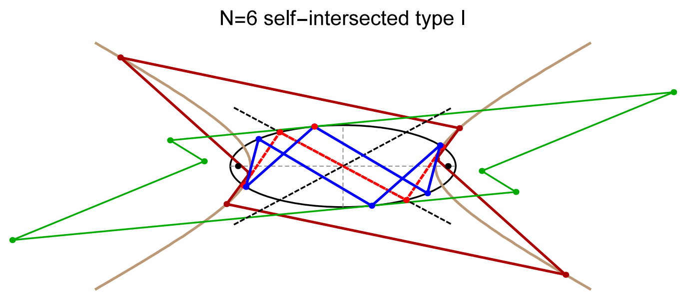


Figure 8. Type I self-intersected 6-periodic (blue) and its doubled-up configuration (dashed red), both tangent to a hyperbolic confocal caustic (brown). Its asymptotes (dashed black) pass through the center of the elliptic billiard. Also shown are the outer (green) and inner (dark red) polygons. Video

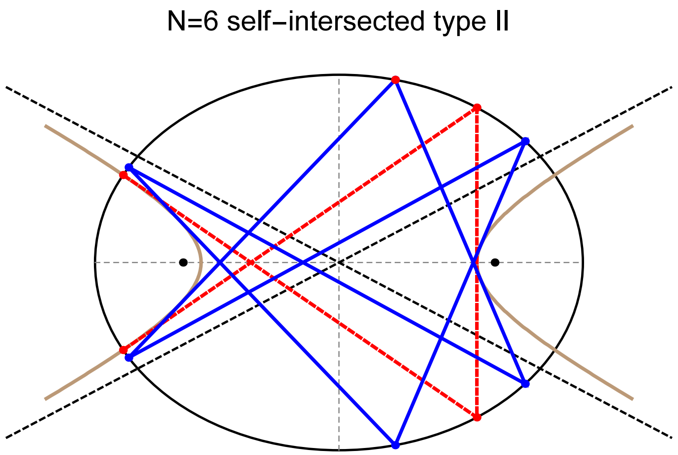


Figure 9. Self-intersected 6-periodic (type II) shown both at one of its doubled-up configurations (dashed red) and in general position (blue). Segments are tangnet to a hyperbolic confocal caustic (brown) whose asymptotes (dashed black) pass through the center of the elliptic billiard. Also shown is the outer polygon (green) which in this case is always simple. Video

Proposition 4.18. *For $N = 6$ type II:*

$$k_{102} = a^2(a-c)^2/(4c^4) = (JL-8)^2(JL-4)^2/1024,$$
$$k_{104} = \frac{2(a^2-ac+c^2)(a^2-ac-c^2)}{c^4} = \frac{(J^2L^2-12JL+16)(J^2L^2-12JL+48)}{128},$$
$$k_{106} = k_{110} = 0.$$

Note (i) in contrast with the above, $k_{104} = k_{101} = JL - 6$ for both $N = 6$ simple and type I, and (ii) k_{106} and k_{110} are nil since both A and A' vanish.

Counter-example 4.19. *Experimentally, k_{804} is invariant for $N = 6$ simple, and type I. However, it is variable for $N = 6$ type II.*

4.6. Invariants for $N = 8$

Referring to Figure 10:

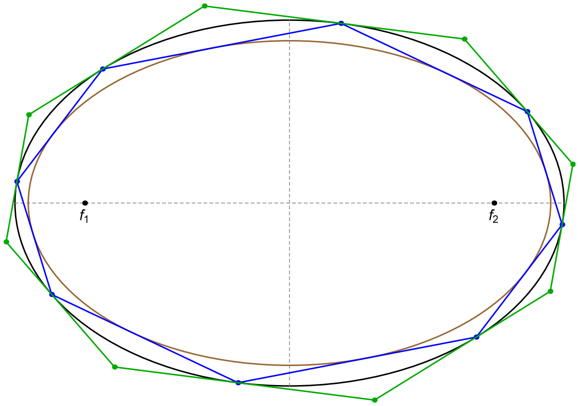


Figure 10. The outer polygon (green) to a simple 8-periodic has null sum of double cosines. Video

Proposition 4.20. *For simple $N = 8$, k_{102} is given by* $(1/2^{12})(JL-4)^2(JL-12)^2$.

Proposition 4.21. *For $N = 8$, $k_{104} = 0$.*

Proof. Using the CAS, we checked that k_{104} vanishes for an 8-periodic in the "horizontal" position, i.e., $P_1 = (a, 0)$. Since k_{104} is invariant [2], this completes the proof. □

4.7. $N = 8$ self-intersected

There are 3 types of self-intersected 8-periodics [6], here called type I, II, and III. These correspond to trajectories with turning numbers of 0, 2, and 3, respectively. These are depicted in Figures 11, 12, and 13.

Observation 4.22. *The signed area of $N = 8$ type I is zero.*

Referring to Figure 13, the following is related to the Poncelet Grid [16] and the Hexagramma Mysticum [3]:

Observation 4.23. *The outer polygon to $N = 8$ type III is inscribed in an ellipse.*

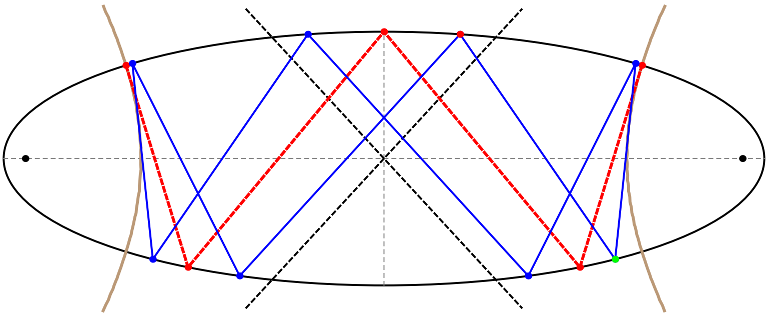


Figure 11. Self-intersecting 8-periodic of type I (blue) and its doubled-up configuration (dashed red) in $a/b = 3$ ellipse. Trajectory segments are tangent to a confocal hyperbolic caustic (brown). Video

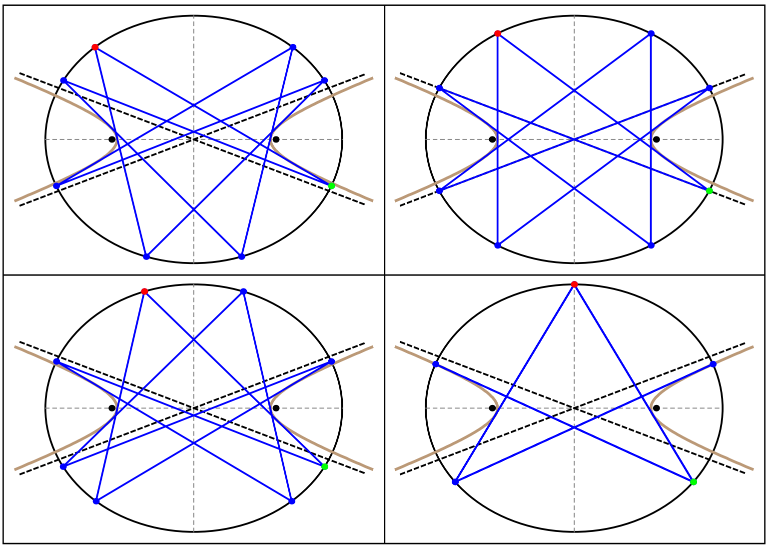


Figure 12. Four positions of a type-II self-intersecting 8-periodic trajectory (blue) in an $a/b = 1.2$ elliptic billiard, at four different locations of a starting vertex (red). In general position, these have turning number 2. Also shown is confocal hyperbolic caustic (brown). Video

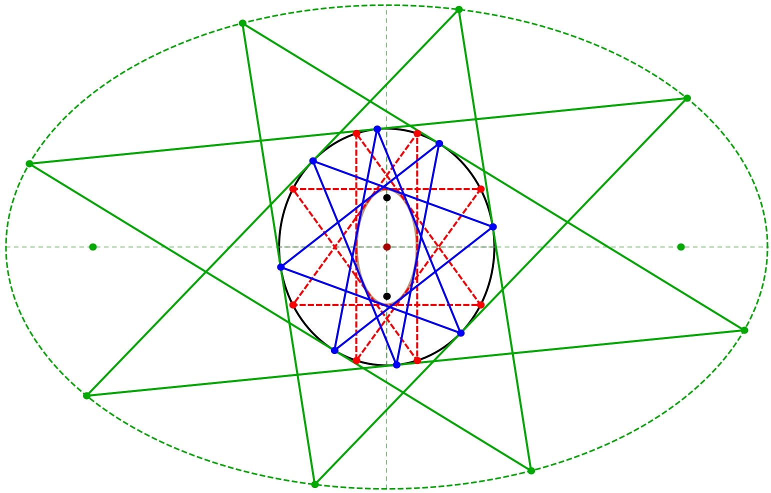


Figure 13. A type-III self-intersected 8-periodic trajectory (blue) and its doubled-up configuration (dashed red) in an $a/b = 1.1$ elliptic billiard (shown rotated by 90° to save space). The turning number is 3. The confocal caustic is an ellipse (brown). Also shown is the outer polygon (green) whose vertices are inscribed in an axis-aligned, concentric ellipse (dashed green), a result related to [3]. Video

5. When invariants are variable

The sum of focus-inversive cosines (k_{804}) is invariant in all N-periodics so far studied, excepting $N = 4$ simple and $N = 6$ type II; see Counter-examples 4.10 and 4.19. Notice that for the former the area of simple 4-periodics is non-zero while the sum of cosines vanishes; in the latter case the situation reverses: the area of type II 6-periodics vanishes and the sum of cosines is non-zero. Could either vanishing quantity be the reason k_{804} becomes variable, e.g., is this introducing a pole in the meromorphic functions on the elliptic curve assumed in the proof of invariants in [2]?

6. Videos

Animations illustrating some of the above phenomena are listed on Table 2.

Acknowledgments

The authors are grateful for the constructive comments by the referee. We would also like to thank Arseniy Akopyan, Jair Koiller, Pedro Roitman, Richard Schwartz,

Table 2. Videos illustrating some phenomena. The last column is clickable and/or provides the YouTube code.

id	Title	youtu.be/<...>
01	$N = 3$–6 orbits and caustics	Y3q35DObfZU
02	$N = 5$ with inner and outer polygons	PRkhrUNTXd8
03	$N = 6$ zero-area antipedal @ $a/b = 2$	HMhZW_kWLGw
04	$N = 8$ outer poly's null sum of double cosines	GEmV_U4eRIE
05	$N = 4$ self-int. and its outer polygon	C8W2e6ftf0w
06	$N = 4$ self-int. vertices concyclic w/ foci	207Ta31P19I
07	$N = 4$ self-int. vertices and outer concyclic w/ foci	4g-JBshX10U
08	$N = 4$ self-int. coll. segment midpts. & 8-shaped locus	GZCrek7RTpQ
09	$N = 5$ self-int. (pentagram)	ECe4DptduJY
10	$N = 6$ self-int. type I	f0D85MNrmdQ
11	$N = 6$ self-int. type II	gQ-FbSq7wWY
12	$N = 7$ self-int. type I and II	yzBG8rgPUP4
13	$N = 8$ self-int. type I	5Lt9atsZhRs
14	$N = 8$ self-int. type II	3xpGnDxyi0
15	$N = 8$ self-int. type III	JwD_w5ecPYs
16	$N = 3$ inversives rigidly-moving circumbilliard	L0JK5izTctI
17	$N = 3$ inversive: invariant area product	0L2uMk2xyKk
18	$N = 5$ inversives: invariant area product	bTkbdEPNU0Y
19	$N = 5$ self-int. inversive: invariant perimenter	LuLtbwkfSbc
20	$N = 5$ and outer inversives: invariant area ratio	eG4UCgMkK18
21	$N = 7$ self-int. type I inversives: invariant area product	BRQ3909ogNE

Hellmuth Stachel, and Sergei Tabachnikov, for useful insights.

A. Review: Elliptic billiard

In *mathematical billiards* one studies the constant-velocity motion of a point mass as it undergoes elastic collisions with a chosen boundary (smooth or polygonal). A special case is when the boundary is an ellipse, called the *elliptic billiard*. Since consecutive trajectory segments are bisected by the ellipse normal, Graves' theorem implies these will be tangent to a confocal caustic [26]. Equivalently, a certain quantity, known as Joachimsthal's constant J, is conserved [4, 10]. Uniquely amongst all planar billiards, the elliptic billiard is an *integrable* dynamical system, i.e., its phase-space if foliated by tori or equivalently, the billiard-map is volume preserving [15].

Referring to Figure 14, for an arbitrary starting condition of a point mass, its trajectory is in general aperiodic (i.e., space filling). But if certain conditions are met, known as *Cayley conditions* [11], a trajectory can be made to close after N reflections or "bounces"; see Figure 15.

The elliptic billiard can be regarded as a special case of Poncelet's porism: if an N-gon can be found inscribed in a first conic $\mathcal{C}$ and circumscribing a second one $\mathcal{C}'$, a 1d family of such N-gons exists with a vertex at an arbitrary point on $\mathcal{C}$.

Therefore, if in the elliptic billiard a certain trajectory is found to close within N segments, a family of such trajectories exists. Remarkably, over such a family, perimeter L is conserved [26].

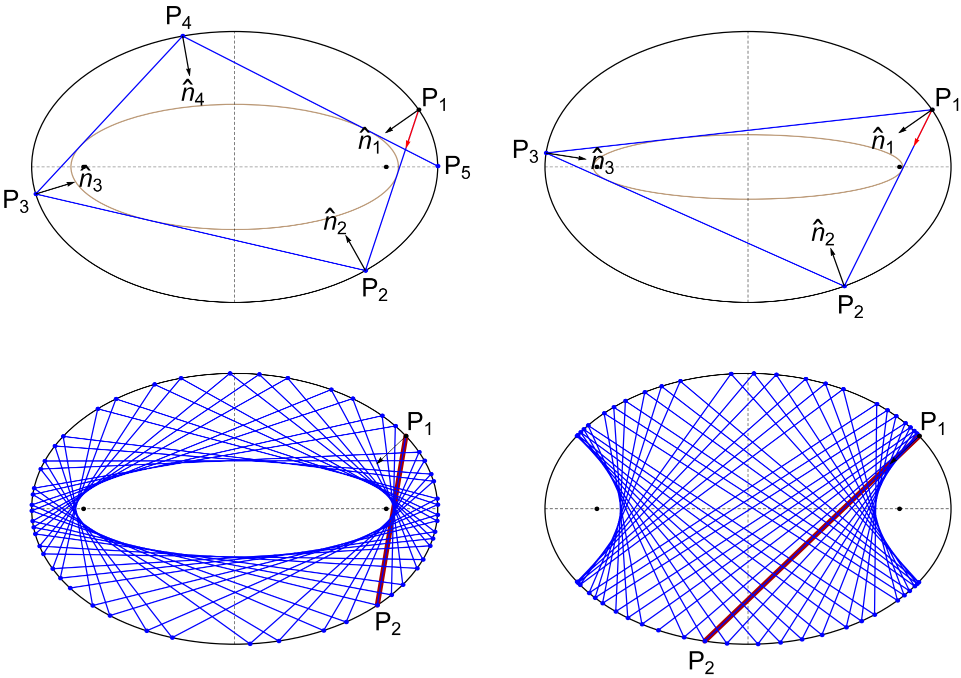


Figure 14. Top left: first four segments of a trajectory in an elliptic billiard. Each bounce is elastic (consecutive segments P_iP_{i+1} are bisected by ellipse normals $\hat{n}_i$). All segments are tangent to a confocal caustic (brown). **Top right**: A trajectory which closes in 3 iterations (it is 3-periodic). **Bottom left**: An aperiodic trajectory such that a first segment P_1P_2 does not pass between the foci; all subsequent segments won't either, and the caustic is an ellipse. **Bottom right**: P_1P_2 now passes between the foci; all subsequent segments will as well, and all will be tangent to hyperbolic caustic.

Joachimsthal's constant is equivalent to stating that every trajectory segment is tangent to a confocal caustic [26]. Equivalently, a positive quantity J remains invariant at every

$$J = \frac{1}{2}\nabla f_i.\hat{v} = \frac{1}{2}|\nabla f_i| \cos\alpha,$$

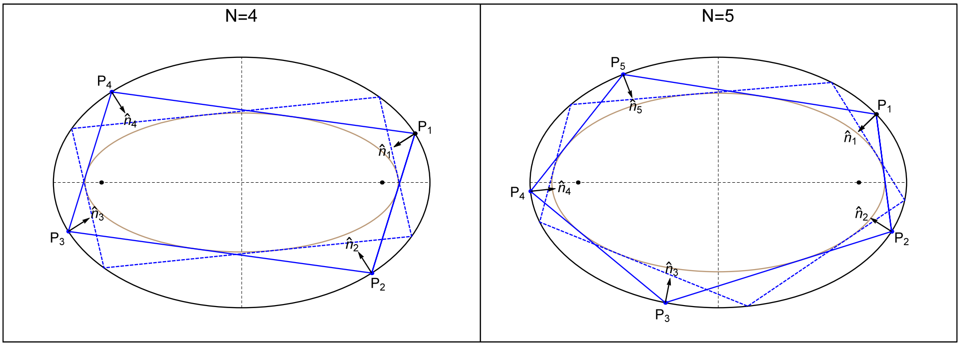


Figure 15. Elliptic Billiard (black) 4- and 5-periodics (blue). Every trajectory vertex P_i (resp. segment P_iP_{i+1}) is bisected by the local normal $\hat{n}_i$ (resp. tangent to the confocal caustic, brown). A second, equi-perimeter member of each family is also shown (dashed blue). Video

where $\hat{v}$ is the unit incoming (or outgoing) velocity vector, and:

$$\nabla f_i = 2\Big(\frac{x_i}{a^2}, \frac{y_i}{b^2}\Big).$$

Joachimsthal's constant J can also be expressed in terms of the billiard semiaxes a, b and the major semiaxis a'' of the caustic [23]:

$$J = \frac{\sqrt{a^2 - a''^2}}{ab}.$$

Let κ_i denote the curvature of the elliptic billiard at P_i given by [28, Ellipse]:

$$\kappa = \frac{1}{a^2 b^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{-3/2}.$$

The signed area of a polygon is given by the following sum of cross-products [17]:

$$A = \frac{1}{2}\sum_{i=1}^{N}(P_{i+1} - P_i) \times (P_i - P_{i+1}).$$

Let $d_{j,i}$ be the distance $|P_i - f_j|$. The inversion $P^{\dagger}_{j,i}$ of vertex P_i with respect to a circle of radius ρ centered on f_j is given by:

$$P^{\dagger}_{j,i} = f_j + \left(\frac{\rho}{d_{j,i}}\right)^2 (P_i - f_j).$$

The following closed-form expression for k_{119} for all N was contributed by H. Stachel [23]:

$$\sum_{i=1}^{N} \kappa_i^{2/3} = L/[2J(ab)^{4/3}]. \tag{A.1}$$

B. Vertices & caustics $N = 3, 4, 5, 6, 8$

The four intersections of an ellipse with semi-axes a, b with a confocal hyperbola with semi-axes a'', b'' are given by:

B.1. $N = 3$ vertices & caustic

Let $P_i = (x_i, y_i)/q_i$, $i = 1, 2, 3$, denote the 3-periodic vertices, given by [12]:

$$\begin{aligned}
q_1 &= 1,\\
x_2 &= -b^4\big((a^2+b^2)k_1 - a^2\big)x_1^3 - 2a^4b^2k_2x_1^2y_1\\
&\quad + a^4\big((a^2-3b^2)k_1 + b^2\big)x_1y_1^2 - 2a^6k_2y_1^3,\\
y_2 &= 2b^6k_2x_1^3 + b^4\big((b^2-3a^2)k_1 + a^2\big)x_1^2y_1\\
&\quad + 2a^2b^4k_2x_1y_1^2 - a^4\big((a^2+b^2)k_1 - b^2\big)y_1^3,\\
q_2 &= b^4\big(a^2 - c^2k_1\big)x_1^2 + a^4\big(b^2 + c^2k_1\big)y_1^2 - 2a^2b^2c^2k_2x_1y_1,\\
x_3 &= b^4\big(a^2 - \big(b^2+a^2\big)\big)k_1x_1^3 + 2a^4b^2k_2x_1^2y_1\\
&\quad + a^4\big(k_1\big(a^2-3b^2\big) + b^2\big)x_1y_1^2 + 2a^6k_2y_1^3,\\
y_3 &= -2b^6k_2x_1^3 + b^4\big(a^2 + \big(b^2-3a^2\big)k_1\big){x_1}^2y_1\\
&\quad - 2a^2b^4k_2x_1y_1^2 + a^4\big(b^2 - \big(b^2+a^2\big)k_1\big)y_1^3,\\
q_3 &= b^4\big(a^2 - c^2k_1\big)x_1^2 + a^4\big(b^2 + c^2k_1\big)y_1^2 + 2a^2b^2c^2k_2x_1y_1,
\end{aligned}$$

where:

$$\begin{aligned}
k_1 &= \frac{d_1^2\delta_1^2}{d_2} = \cos^2\alpha,\\
k_2 &= \frac{\delta_1 d_1^2}{d_2}\sqrt{d_2 - d_1^4\delta_1^2} = \sin\alpha\cos\alpha,\\
c^2 &= a^2 - b^2, \quad d_1 = (ab/c)^2, \quad d_2 = b^4x_1^2 + a^4y_1^2,\\
\delta &= \sqrt{a^4 + b^4 - a^2b^2}, \quad \delta_1 = \sqrt{2\delta - a^2 - b^2},
\end{aligned}$$

where α, though not used here, is the angle of segment P_1P_2 (and P_1P_3) with respect to the normal at P_1. The caustic is the ellipse:

$$\frac{x^2}{a''^2} + \frac{y^2}{b''^2} - 1 = 0, \quad a'' = \frac{a(\delta - b^2)}{a^2 - b^2}, \quad b'' = \frac{b(a^2 - \delta)}{a^2 - b^2}.$$

B.2. $N = 4$ vertices & caustic

B.3. Simple

The vertices of the 4-periodic orbit are given by:

$$P_1 = (x_1, y_1), \qquad P_2 = \left(-\frac{a^4y_1}{\sqrt{b^6x_1^2 + a^6y_1^2}}, \frac{b^4x_1}{\sqrt{b^6x_1^2 + a^6y_1^2}}\right),$$

$$P_3 = -P_1, \qquad P_4 = -P_2.$$

The caustic is the ellipse:

$$\frac{x^2}{a''^2} + \frac{y^2}{b''^2} - 1 = 0, \quad a'' = \frac{a^2}{\sqrt{a^2+b^2}}, \quad b'' = \frac{b^2}{\sqrt{a^2+b^2}}.$$

The area and its bounds are given by:

$$A = \frac{2(b^4x_1^2 + a^4y_1^2)}{\sqrt{b^6x_1^2 + a^6y_1^2}}, \quad \frac{4a^2b^2}{a^2+b^2} \le A \le 2ab.$$

The minimum (resp. maximum) area is achieved when the orbit is a rectangle with $P_1 = (x_1, b^2x_1/a^2)$ (resp. rhombus with $P_1 = (a, 0)$). The perimeter is given by:

$$L = 4\sqrt{a^2+b^2}. \tag{B.1}$$

Note: when $b = 1$, the perimeter is equal to the $N = 4$ k_{803}.

The exit angle α required to close the trajectory from a departing position (x_1, y_1) on the elliptic billiard boundary is given by:

$$\cos\alpha = \frac{a^2b}{\sqrt{a^2+b^2}\sqrt{a^4 - c^2x_1^2}}.$$

B.3.1. Self-intersected

When $a/b > \sqrt{2}$, the vertices of the 4-periodic self-intersecting orbit are given by:

$$P_1 = \left[au, b\sqrt{1-u^2}\right], \quad P_3 = \left[-au, b\sqrt{1-u^2}\right],$$
$$P_2 = \left[-\frac{a\sqrt{a^2(a^2-2b^2) - c^4u^2}}{c^2\sqrt{1-u^2}}, -\frac{b^3}{c^2\sqrt{1-u^2}}\right],$$
$$P_4 = \left[\frac{a\sqrt{a^2(a^2-2b^2) - c^4u^2}}{c^2\sqrt{1-u^2}}, -\frac{b^3}{c^2\sqrt{1-u^2}}\right],$$

where $|u| \le \frac{a}{c^2}\sqrt{a^2-2b^2}$. The confocal hyperbolic caustic is given by:

$$\frac{x^2}{a''^2} - \frac{y^2}{b''^2} = 1, \quad a'' = \frac{a\sqrt{a^2-2b^2}}{c}, \quad b'' = \frac{b^2}{c}.$$

The four intersections of an ellipse with semi-axes a, b with confocal hyperbola with axes a'', b'' are given by:

$$\left[\pm\frac{aa''}{c}, \pm\frac{bb''}{c}\right]. \tag{B.2}$$

The exit angle α required to close the trajectory from a departing position (x_1, y_1) on the elliptic billiard boundary is given by:

$$\cos\alpha = \frac{a^2 b}{c\sqrt{a^4 - c^2 x_1^2}}.$$

The perimeter of the orbit is $L = 4a^2/c$.

B.4. $N = 5$ vertices & caustic

Let $a > b$ be the semi-axes of the elliptic billiard.

Proposition B.1. *The major semiaxis length a'' of the caustic for $N = 5$ simple (resp. self-intersecting, i.e., pentagram) is given by the root of the largest (resp. smallest) real root $x \in (0, a)$ of the following bi-sextic polynomial:*

$$\begin{aligned} P_5(x) = c^{12}x^{12} &- 2c^4a^2\big(3a^8 - 9a^6b^2 + 31a^4b^4 + a^2b^6 + 6b^8\big)x^{10} \\ &+ c^4a^4\big(15a^8 - 30a^6b^2 + 191a^4b^4 + 16a^2b^6 + 16b^8\big)x^8 \\ &- 4c^4a^{10}\big(5a^4 - 5a^2b^2 + 66b^4\big)x^6 \\ &+ a^{12}\big(15a^8 - 30a^6b^2 + 191a^4b^4 - 368a^2b^6 + 208b^8\big)x^4 \\ &- 2a^{14}\big(3a^8 - 3a^6b^2 + 22a^4b^4 - 48a^2b^6 + 32b^8\big)x^2 + a^{24}. \end{aligned}$$

Proof. Consider a 5-periodic with vertices $P_i,\ i = 1, \ldots, 5$ where P_1 is at $(a, 0)$, i.e., the orbit is "horizontal". The polynomial P_5 is exactly the Cayley condition for the existence of 5-periodic orbits, see [11]. For $c = 0$ the roots are $a_2' = (\sqrt{5}-1)a/4$ and $a_2'' = (\sqrt{5}+1)a/4$ and corresponds to the regular case. For $b = 0$ the roots are coincident in given by $x = a$. By analytic continuation, for $c \in (0, a)$, the two roots are in the interval $(0, a)$. □

For $N = 5$ non-intersecting, the abcissae of vertices $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$ are given by the smallest positive solution (resp. unique negative) of the following equations:

$$\begin{aligned} x_2 : {}& c^6x_2^6 - 2a\big(2a^2 - b^2\big)c^4x_2^5 + a^2\big(5a^2 + 4b^2\big)c^4x_2^4 - 8a^5b^2c^2x_2^3 \\ &- a^8\big(5a^2 - 9b^2\big)x_2^2 + 2a^9\big(2a^2 - b^2\big)x_2 - a^{12} = 0, \\ x_3 : {}& c^6x_3^6 - 2ab^2c^2\big(3a^2 + b^2\big)x_3^5 - a^2c^2\big(3a^4 - 3a^2b^2 + 4b^4\big)x_3^4 + 12a^5b^2c^2x_3^3 \\ &+ a^6\big(3a^4 - 3a^2b^2 + 4b^4\big)x_3^2 - 2a^7b^2\big(3a^2 - 4b^2\big)x_3 - a^{12} = 0. \end{aligned}$$

Joachimsthal's constant J for the simple orbit is the smallest positive root of:

$$\begin{aligned} &4096c^{12}J^{12} + 2048(3a^2 + b^2)(a^2 + 3b^2)(a^2 + b^2)c^4J^{10} \\ &- 256(29a^4 + 54a^2b^2 + 29b^4)c^4J^8 + 2304(a^2 + b^2)c^4J^6 \\ &- 16(3a^2 - 4ab - 3b^2)(3a^2 + 4ab - 3b^2)J^4 - 40(a^2 + b^2)J^2 + 5 = 0. \end{aligned}$$

The perimeter of the simple orbit is given by $L = p/q$ where:

$$
\begin{aligned}
p = &\left(1024\left(a^2+b^2\right)c^4b^2J^7 - 256c^4b^2J^5 - 64\left(a^2+b^2\right)b^2J^3 + 16Jb^2\right)\sqrt{1-4a^2J^2}\\
&- 1024c^2\left(5a^4+2a^2b^2+b^4\right)b^2J^7 + 256c^2\left(3a^2+b^2\right)b^2J^5 + 64c^2b^2J^3 + 16Jb^2\\
q = &\,256c^8J^8 - 256c^2\left(a^2+b^2\right)^2J^6 + 32c^2\left(3a^2+5b^2\right)J^4 - 16c^2J^2 + 1.
\end{aligned}
$$

B.5. $N = 6$ vertices & caustic

B.5.1. Simple

Vertices $P_i, i = 2, ..., 6$ with $P_1 = (a, 0)$ are given by:

$$
\begin{aligned}
&P_4 = [-a, 0], \quad P_2 = [k_x, k_y], \quad P_5 = -P_2, \;\; P_3 = [-k_x, k_y], \quad P_6 = -P_3,\\
&k_x = \frac{a^2}{a+b}, \;\; k_y = \frac{b\sqrt{b(2a+b)}}{a+b}.
\end{aligned}
$$

The confocal, elliptic caustic is given by:

$$
\frac{x^2}{a''^2} + \frac{y^2}{b''^2} = 1, \quad a'' = \frac{a\sqrt{a(a+2b)}}{a+b}, \quad b'' = \frac{b\sqrt{b(2a+b)}}{a+b}.
$$

The perimeter is given by:

$$
L = \frac{4(a^2+ab+b^2)}{a+b}.
$$

B.5.2. Self-Intersected (type I)

This orbit only exists for $a > 2b$. Vertices $P_i,\ i = 2, \ldots, 6$ with $P_1 = (0, b)$ are given by:

$$
\begin{aligned}
&P_4 = [0, -b], \quad P_2 = [k_x, k_y], \quad P_5 = -P_2, \quad P_3 = [k_x, -k_y], \quad P_6 = -P_3,\\
&k_x = \frac{a\sqrt{a(a-2b)}}{b-a}, \quad k_y = \frac{b^2}{b-a}.
\end{aligned}
$$

The confocal, hyperbolic caustic is given by:

$$
\frac{x^2}{a''^2} - \frac{y^2}{b''^2} = 1, \quad a'' = \frac{a^{3/2}\sqrt{a-2b}}{a-b}, \quad b'' = \frac{b^{3/2}\sqrt{2a-b}}{a-b}.
$$

The 4 intersections of the above caustic with the elliptic billiard are given by (B.2). The perimeter is given by:

$$
L = \frac{4(a^2-ab+b^2)}{a-b}.
$$

B.5.3. Self-intersected (type II)

This orbit only exists for $a > \frac{2b\sqrt{3}}{3}$. Vertices $P_i,\ i = 2,\ldots,6$ with $P_1 = [0, b]$ are given by:

$$P_4 = [0, -b], \quad P_2 = [k_x, k_y], \quad P_3 = -P_2, \quad P_5 = [k_x, -k_y], \quad P_6 = -P_5$$
$$k_x = -\frac{a^{\frac{3}{2}}\sqrt{2c-a}}{c}, \quad k_y = \frac{(c-a)b}{c}.$$

The confocal hyperbolic caustic is given by:

$$\frac{x^2}{a''^2} - \frac{y^2}{b''^2} = 1, \quad a''^2 = \frac{a^3\left(3ac - 2b^2\right)}{c(3a^2+b^2)}, \quad b''^2 = \frac{b^2\left(2a^2(a-c) - b^2c\right)}{c(3a^2+b^2)}.$$

The 4 intersections of the above caustic with the elliptic billiard are given by (B.2). The perimeter is given by:

$$L = 4(a+c)\sqrt{2a/c - 1}.$$

B.6. $N = 7$ caustic

Referring to Figure 1, there are three types of 7-periodics: (i) non-intersecting, (ii) self-intersecting type I, i.e., with turning number 2, (iii) self-intersecting type II.

Proposition B.2. *The caustic semiaxis for non-intersecting 7-periodics (resp. self-intersecting type I, and type II self-intersecting) are given by the smallest (resp. second and third smallest) root of the following degree-12 polynomial:*

$$\begin{aligned}
&c^{12}x_1^{12} - 4\left(a^2+b^2\right)c^6a\left(3a^2+b^2\right)b^2x_1^{11} - 2c^6a^2\left(3a^6 - 6a^4b^2 + 13a^2b^4 - 2b^6\right)x_1^{10}\\
&+ c^6a^3\left(60a^4 + 60b^2a^2 + 8b^4\right)x_1^9\\
&+ a^6c^2\left(15a^8 - 45a^6b^2 + 125a^4b^4 - 143a^2b^6 + 112b^8\right)x_1^8\\
&- 8a^7b^2c^2\left(15a^6 - 20a^4b^2 - 7a^2b^4 + 8b^6\right)x_1^7\\
&- 4a^8c^2\left(5a^8 - 10a^6b^2 + 35a^4b^4 - 30a^2b^6 + 36b^8\right)x_1^6\\
&+ 8a^9b^2c^2\left(15a^6 - 25a^4b^2 - 2a^2b^4 + 4b^6\right)x_1^5\\
&+ a^{10}c^2\left(15a^8 - 15a^6b^2 + 80a^4b^4 - 32a^2b^6 + 64b^8\right)x_1^4\\
&- 4a^{15}b^2\left(15a^4 - 45b^2a^2 + 32b^4\right)x_1^3\\
&- 2a^{16}\left(3a^6 - 3a^4b^2 + 10a^2b^4 - 8b^6\right)x_1^2 + 4a^{17}b^2\left(3a^2 - 4b^2\right)\left(a^2 - 2b^2\right)x_1 + a^{24}\\
&= 0.
\end{aligned}$$

It can shown the first two smallest (resp. third smallest) roots of the above polynomial are negative (resp. positive), and all have absolute values within $(0, a)$.

For $a = b$ the polynomial equation above is given by $a^3 + 4a^2x_1 - 4ax_12 - 8x_1^3 = 0$ with roots $-0.9009688680a,\ -0.2225209340a,\ 0.6234898025a$.

B.7. $N = 8$ vertices & caustic

B.7.1. Simple

Let P_i, $i = 1, \ldots, 8$ be the vertices of a simple 8-periodic, Figure 10. Set $P_1 = [a, 0]$. Then:

$$P_5 = -[a, 0], \quad P_3 = [0, b], \quad P_7 = [0, -b], \quad P_{2,4,6,8} = [\pm az, \pm b\sqrt{1 - z^2}],$$

where z, which plays the role of a cosine, is the only positive root of $c^4z^4 - 2a^2c^2z^3 + 2a^2b^2z^2 + 2a^2c^2z - a^4 = 0$.

Remark B.3. Given an ellipse with semi-axes (a, b), let $P_1 = [a, 0]$ and $P_2 = [a\cos t, b\sin t]$. Let $z = \cos t$. The unique confocal ellipse which is tangent to the chord P_1P_2 has major axis a'' given by:

$$a'' = a\sqrt{\frac{a^2 - z(a^2 - 2b^2)}{a^2 + b^2 - zc^2}}.$$

Recall confocality implies $b'' = \sqrt{a''^2 - c^2}$. Therefore, one can use the above to compute from z the semi-axes of the caustic for simple 8-periodics.

B.7.2. Self-intersected (type I and type II)

These have hyperbolic confocal caustics; see Figures 11 and 12. Let $P_1 = (x_1, y_1)$ be at the intersection of the hyperbolic caustic with the elliptic billiard for each case. For type-I (resp. type-II) x_1 is given by the smallest (resp. largest) positive root $x_1 \in (0, a)$ of the following degree-8 polynomial:

$$\begin{aligned} x_1: \ & c^{16}x_1^8 - 4a^4c^8(a^6 - 4a^4b^2 + a^2b^4 - 2b^6)x_1^6 + 2a^8c^6(3a^6 - 15a^4b^2 - 4b^6)x_1^4 \\ & - 4a^{16}c^4(a^2 - 6b^2)x_1^2 + a^{20}(a^4 - 8a^2b^2 + 8b^4) = 0. \end{aligned}$$

From (B.2) obtain the major semi-axis for confocal caustic: $a'' = (x_1c)/a$, and $b'' = \sqrt{c^2 - a''^2}$.

B.7.3. Self-intersected (type III)

The confocal caustic is an ellipse with semi-axes (a'', b''). Let $P_1 = (a'', y_1)$ and $P_2 = (a'', -y_1)$ be two consecutive vertices in the doubled-up type III 8-periodic (dashed red in Figure 13) connected by a vertical line (the figure is rotated, therefore this line will appear horizontal). Let $\alpha = a/b$. The square of a'' is given by the smallest positive root of the following quartic polynomial:

$$\begin{aligned} & \alpha^{16} + (\alpha^2 - 1)^2(\alpha^4 + 6\alpha^2 + 1)x^4 - 4(\alpha^2 - 1)^2(\alpha^2 + 5)\alpha^4x^3 + \\ & (6(\alpha^4 + 2\alpha^2 - 7)\alpha^2 + 32)\alpha^6x^2 - 4(\alpha^6 + \alpha^4 - 4\alpha^2 + 4)\alpha^8x = 0. \end{aligned}$$

Note: $b'' = \sqrt{a''^2 - c^2}$.

C. Table of symbols

Table 3. Symbols used in the invariants.
Note $i \in [1, N]$ and $j = 1, 2$.

symbol	meaning
O, N	center of elliptic billiard and vertex count
L, J	inv. perimeter and Joachimsthal's constant
a, b	elliptic billiard major, minor semi-axes
a'', b''	caustic major, minor semi-axes
f_1, f_2	foci
P_i, P_i', P_i''	N-periodic, outer, inner polygon vertices
$d_{j,i}$	distance $\lvert P_i - f_j \rvert$
κ_i	ellipse curvature at P_i
θ_i, θ_i'	N-periodic, outer polygon angles
A, A', A''	N-periodic, outer, inner areas
ρ	radius of the inversion circle
$P^{\dagger}_{j,i}$	vertices of the inversive polygon wrt f_j
$L^{\dagger}_j, A^{\dagger}_j$	perimeter, area of inversive polygon wrt f_j
$\theta^{\dagger}_{j,i}$	ith angle of inversive polygon wrt f_j

References

[1] A. Akopyan: *Corollary of Theorem 6 in Akopyan et al., "Billiards in Ellipses Revisited" (2020).* Private Communication, Jan. 2020.

[2] A. Akopyan, R. Schwartz, S. Tabachnikov: *Billiards in Ellipses Revisited*, Eur. J. Math. (Sept. 2020), doi: https://doi.org/10.1007/s40879-020-00426-9.

[3] D. Baralić, I. Spasojević: *Illumination of Pascal's Hexagrammum and Octagrammum Mysticum*, Discrete & Computational Geometry 53.2 (2015).

[4] M. V. Berry: *Regularity and chaos in classical mechanics, illustrated by three deformations of a circular 'billiard'*, Eur. J. Phys. 2.91 (1981).

[5] M. Bialy, S. Tabachnikov: *Dan Reznik's Identities and More*, Eur. J. Math. (Sept. 2020), doi: https://doi.org/10.1007/s40879-020-00428-7.

[6] G. Birkhoff: *On the periodic motions of dynamical systems*, Acta Mathematica 50.1 (1927), pp. 359–379, doi: https://doi.org/10.1007/BF02421325.

[7] G. D. Birkhoff: *Dynamical Systems*, 1966th ed., vol. 9, Colloqium Publications, Providence, RI: American Mathematical Society, 1927.

[8] A. Chavez-Caliz: *More About Areas and Centers of Poncelet Polygons*, Arnold Math J. (Aug. 2020), doi: https://doi.org/10.1007/s40598-020-00154-8.

[9] A. Connes, D. Zagier: *A Property of Parallelograms Inscribed in Ellipses*, The American Mathematical Monthly 114.10 (2007), pp. 909–914.

[10] D. R. da Costa, C. P. Dettmann, E. D. Leonel: *Circular, elliptic and oval billiards in a gravitational field*, Comm. in Nonlin. Sci. and Num. Sim. 22.1 (2015), pp. 731–746.

[11] V. Dragović, M. Radnović: *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*, Frontiers in Mathematics, Basel: Springer, 2011.

[12] R. Garcia: *Elliptic Billiards and Ellipses Associated to the 3-Periodic Orbits*, American Mathematical Monthly 126.06 (2019), pp. 491–504, doi: https://doi.org/10.1080/00029890.2019.1593087.

[13] R. Garcia, D. Reznik, J. Koiller: *New Properties of Triangular Orbits in Elliptic Billiards*, Amer. Math. Monthly (Oct. 2021), doi: https://doi.org/10.1080/00029890.2021.1982360.

[14] R. A. Johnson: *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Boston, MA: Houghton Mifflin, 1929.

[15] V. Kaloshin, A. Sorrentino: *On the Integrability of Birkhoff Billiards*, Phil. Trans. R. Soc. A.376 (2018), doi: https://doi.org/10.1098/rsta.2017.0419.

[16] M. Levi, S. Tabachnikov: *The Poncelet Grid and Billiards in Ellipses*, The American Mathematical Monthly 114.10 (2007), pp. 895–908, doi: https://doi.org/10.1080/00029890.2007.11920482.

[17] F. Preparata, M. Shamos: *Computational Geometry - An Introduction*, 2nd, Basel: Springer-Verlag, 1988.

[18] R. Ramírez-Ros: *On Cayley conditions for billiards inside ellipsoids*, Nonlinearity 27.5 (2014), pp. 1003–1028, doi: https://doi.org/10.1088/0951-7715/27/5/1003.

[19] D. Reznik, R. Garcia, J. Koiller: *Can the Elliptic Billiard still surprise us?*, Math Intelligencer 42 (2020), pp. 6–17, doi: https://doi.org/10.1007/s00283-019-09951-2.

[20] D. Reznik, R. Garcia, J. Koiller: *Eighty New Invariants of N-Periodics in the Elliptic Billiard*, arXiv:2004.12497, Nov. 2020.

[21] D. Reznik, R. Garcia, J. Koiller: *Fifty New Invariants of N-Periodics in the Elliptic Billiard*, Arnold Math. J. 7 (2021), pp. 341–355, doi: https://doi.org/10.1007/s40598-021-00174-y.

[22] R. Schwartz: *The Poncelet grid*, Advances in Geometry 7.2 (2007), pp. 157–175, doi: https://doi.org/10.1515/ADVGEOM.2007.010.

[23] H. Stachel, Private Communication, 2020.

[24] H. Stachel: *On the motion of billiards in ellipses*, Eur. J. of Math. (2021), doi: https://doi.org/10.1007/s40879-021-00524-2.

[25] H. Stachel: *The geometry of billiards in ellipses and their Poncelet grids*, Journal of Geometry 112.40 (2021).

[26] S. Tabachnikov: *Geometry and Billiards*, vol. 30, Student Mathematical Library, Providence, RI: American Mathematical Society, 2005.

[27] Various contributors: *Antiparallelogram*, Wikipedia, 2021, url: https://en.wikipedia.org/wiki/Antiparallelogram.

[28] E. W. Weisstein: *CRC concise encyclopedia of mathematics (2nd ed.)* Boca Raton, FL: Chapman and Hall/CRC, 2002.

Annales Mathematicae et Informaticae
55 (2022) pp. 76–87
DOI: https://doi.org/10.33039/ami.2021.12.001
URL: https://ami.uni-eszterhazy.hu

Some identities of Gaussian binomial coefficients

Tian-Xiao He[a], **Anthony G. Shannon**[b], **Peter J.-S. Shiue**[c]

[a]Department of Mathematics, Illinois Wesleyan University,
Bloomington, Illinois 61702, USA
the@iwu.edu

[b]Warrane College, University of New South Wales,
Kensington, NSW 2033, Australia
t.shannon@warrane.unsw.edu.au

[c]Department of Mathematical Sciences, University of Nevada,
Las Vegas, Nevada, 89154-4020, USA
shiue@unlv.nevada.edu

Abstract. In this paper, we present some identities of Gaussian binomial coefficients with respect to recursive sequences, Fibonomial coefficients, and complete functions by use of their relationships.

Keywords: Gaussian binomial coefficient, Fibonomial coefficient, complete homogenous symmetric polynomial, complete function, recursive sequence, Fibonacci number sequence, Newton interpolation

AMS Subject Classification: 05A15, 11B83, 05A05, 05A19

1. Introduction

q-series are defined by

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n) \tag{1.1}$$

for integer $n > 0$ and $(q)_0 = 1$. Arising out of these are Gaussian binomial coefficients (or Gaussian coefficients as an abbreviation) for integers $n, k \geq 0$,

$$\binom{n}{k}_q = \begin{cases} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(q)_k}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

Submitted: August 18, 2021
Accepted: December 18, 2021
Published online: January 3, 2022

$$= \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k \le n, \tag{1.2}$$

where the q-factorial $[m]_q!$ is defined by $[m]_q! = \Pi_{k=1}^m [k]_q = [1]_q[2]_q \cdots [m]_q$, and

$$[k]_q = \sum_{i=0}^{k-1} q^i = 1 + q + q^2 + \cdots + q^{k-1} = \begin{cases} \frac{1-q^k}{1-q} & \text{for } q \ne 1, \\ k & \text{for } q = 1. \end{cases}$$

From (1.2) we have $\binom{n}{0}_q = \binom{n}{n}_q = 1$, $\binom{n}{k}_q = \binom{n}{n-k}_q$,

$$(1-q^k)\binom{n}{k}_q = (1-q^n)\binom{n-1}{k-1}_q, \tag{1.3}$$

and for $0 < k < n$

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q, \tag{1.4}$$

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q. \tag{1.5}$$

Identities (1.4) and (1.5) are analogs of Pascal's identities. Alternatively using (1.4) and (1.5), we obtain the identity

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q - (1-q^{n-1})\binom{n-2}{k-1}_q \tag{1.6}$$

More precisely, by substituting (1.4) with the transformation $n \to n-1$ and $k \to k-1$ into (1.5), we have

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-1}\binom{n-2}{k-1}_q + q^{n-k}\binom{n-2}{k-2}_q.$$

Substituting (1.5) with the transformation $n \to n-1$ and $k \to k-1$ into the last term of the above identity, we have

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-1}\binom{n-2}{k-1}_q + \binom{n-1}{k-1}_q - \binom{n-2}{k-1}_q,$$

which implies (1.6).

In 1915 Georges Fontené (1848–1928) published a one page note [8] suggesting a generalization of binomial coefficients, replacing the natural numbers by an arbitrary sequence (A_n) of real or complex numbers, namely,

$$\binom{n}{k}_A = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1} \tag{1.7}$$

with $\binom{n}{0}_A = \binom{n}{n}_A = 1$, where A stands for (A_n). He gave the fundamental recurrence relation for these generalized coefficients and include the ordinary binomial coefficients as a special case for $A_n = n$, while for $A_n = q^n - 1$ we obtain the Gaussian binomial coefficients (or q-binomial coefficients) (1.6) studied by Gauss (as well as Euler, Cauchy, F . H, Jackson, and many others later). The history of Gaussian binomial coefficients can be seen in a recent paper by Shannon [18] and its references.

These generalized coefficients of Fontené were rediscovered by Morgan Ward (1901–1963) in a remarkable paper [23] in 1936 which developed a symbolic calculus of sequences without mentioning Fontené. In that paper, Ward posed the problem whether a suitable definition for generalized Bernoulli numbers could be framed so that a generalized Staudt-Clausen theorem [7] existed for them within the framework of the Jackson calculus [14]; the Staudt-Clausen theorem deals with the fractional part of Bernoulli numbers [20]. Rado [17] and Carlitz [4, 5] outlined partial generalizations of the theorem with the Jackson operators for q-Bernoulli numbers, and Horadam and Shannon completed this proof [13]. We shall follow Gould [10] and call the generalized coefficients (1.7) the Fontené-Ward generalized binomial coefficients.

Since 1964, there has been an accelerated interest in Fibonomial coefficients, which correspond to the choice $A_n = F_n$, where F_n are the Fibonacci numbers defined by $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$, and $F_1 = 1$. For instance, see Trojovský [21] and its references. As far as we know, the first person to name them (not utilize them) was Stephen Jerbic, a research Master student of Verner Hoggatt, who completed his thesis in 1968 [15]. One of the authors of this paper read his MA thesis in 1975 when the author visited Verner Hoggatt in San Jose.

If the recursive number sequence $(U_n(a, b; p_1, p_2))$ that satisfies $U_{n+2} = p_1U_{n+1} - p_2U_n$ $(n \geq 0)$ and has initials $U_0 = a$ and $U_1 = b$ is used to replace (A_n) in the Fontené-Ward generalized binomial coefficients, then the corresponding Gaussian binomial coefficients are called the generalized Fibonacci binomial coefficients, which are shown in the recent paper [18] by Shannon. $U_n(0, 1; p_1, p_2)$ can be represented by its Binet from $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ (cf. the authors [11]), where α and β are two distinct roots of the $(U_n)'s$ characteristic equation $x^2 - p_1x + p_2 = 0$. Throughout this paper, we always assume the characteristic equation $x^2 - p_1x + p_2 = 0$ has non-zero constant term p_2 and two distinct roots α and β. Since $\alpha\beta = p_2$, we have $\alpha, \beta \neq 0$. Shannon's paper starts from a nice relationship between the Gaussian binomial coefficients defined by (1.7) with $q = \beta/\alpha$ $(\alpha \neq 0, i.e., p_2 \neq 0)$ for $U_n(0, 1; p_1, p_2)$ and the generalized Fibonacci binomial coefficients

$$\binom{n}{k}_U = \frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k}, \tag{1.8}$$

where U stands for $(U_n(0, 1; p_1, p_2))$, represented by

$$\binom{n}{k}_q = \alpha^{-k(n-k)}\binom{n}{k}_U, \tag{1.9}$$

where $q = \beta/\alpha$, and $\alpha \neq 0$ and β are two distinct roots of the $(U_n)'s$ characteristic equation $x^2 - p_1 x + p_2 = 0$ assumed before. In fact, we have

$$\begin{aligned}\binom{n}{k}_q &= \frac{(1-(\beta/\alpha)^n)(1-(\beta/\alpha)^{n-1})\cdots(1-(\beta/\alpha)^{n-k+1})}{(1-\beta/\alpha)(1-(\beta/\alpha)^2)\cdots(1-(\beta/\alpha)^k)} \\ &= \frac{(\alpha^n-\beta^n)(\alpha^{n-1}-\beta^{n-1})\cdots(\alpha^{n-k+1}-\beta^{n-k+1})}{(\alpha-\beta)(\alpha^2-\beta^2)\cdots(\alpha^k-\beta^k)} \\ &\quad \frac{(1/\alpha^n)(1/\alpha^{n-1})\cdots(1/\alpha^{n-k+1})}{(1/\alpha^k)(1/\alpha^{k-1})\cdots(1/\alpha)} \\ &= \frac{U_n U_{n-1}\cdots U_{n-k+1}}{U_1 U_2 \cdots U_k}\left(\frac{1}{\alpha^{n-k}}\right)^k,\end{aligned}$$

which implies (1.9).

Based on the relationship (1.9), several interesting identities are established. For instance, [18] used (1.9) to establish the following identity.

$$\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \frac{2-q^k-q^{n-k}}{1-q^n}\binom{n}{k}_q. \tag{1.10}$$

Obviously, identity (1.10) can also be proved by using (1.3) and

$$\begin{aligned}(1-q^{n-k})\binom{n}{k}_q &= (1-q^{n-k})\binom{n}{n-k}_q \\ &= (1-q^n)\binom{n-1}{n-k-1}_q = (1-q^n)\binom{n-1}{k}_q.\end{aligned}$$

Consequently, combining $(1-q^k)\binom{n}{k}_q = (1-q^n)\binom{n-1}{k-1}_q$ on the leftmost side and the rightmost side of the last equation yields

$$(1-q^n)\left(\binom{n-1}{k}_q + \binom{n-1}{k-1}_q\right) = (2-q^k-q^{n-k})\binom{n}{k}_q.$$

In this paper, we will continue Shannon's work to construct a few more identities.

The second part of this paper concerns complete homogenous symmetric functions, which have a natural connection with Gaussian coefficients. A good source of information for the early history of symmetric functions, such as the fundamental theorem of symmetric functions and the symmetry of the matrix, is [22] by Vahlen. In particular, the first published work on symmetric functions is due to Girard [9] in 1629, who gave an explicit formula expressing symmetric polynomials. The complete homogeneous symmetric polynomials are a specific kind of symmetric polynomials. Every symmetric polynomial can be expressed as a polynomial expression in complete homogeneous symmetric polynomials. The fundamental relation between the elementary symmetric polynomials and the complete homogeneous ones can be found in [16] by Macdonald. More historical context on the symmetric functions and the complete homogeneous symmetric polynomials can be found in [16] and Stanley [19]. The complete functions are q analogies of the complete homogenous symmetric polynomials.

The complete homogeneous symmetric polynomial of degree k in n variables $x_1, x_2, \ldots, x_n$, written h_k for $k = 0, 1, 2, \ldots$, is the sum of all monomials of total degree k in the variables. More precisely, for integers i_1, $i_2, \ldots$, i_k,

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{1.11}$$

or equivalently, for integers l_1, $l_2, \ldots$, l_k

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{l_1 + l_2 + \cdots + l_n = k,\, l_i \ge 0} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}. \tag{1.12}$$

Here, l_p is the multiplicity of p in the sequence i_k. The first few of these polynomials are

$$\begin{aligned} h_0(x_1, x_2, \ldots, x_n) &= 1, \\ h_1(x_1, x_2, \ldots, x_n) &= \sum_{1 \le j \le n} x_j, \\ h_2(x_1, x_2, \ldots, x_n) &= \sum_{1 \le j \le k \le n} x_j x_k, \\ h_3(x_1, x_2, \ldots, x_n) &= \sum_{1 \le j \le k \le \ell \le n} x_j x_k x_\ell. \end{aligned}$$

Thus, for each nonnegative integer k, there exists exactly one complete homogeneous symmetric polynomial of degree k in n variables. Further results about complete homogeneous symmetric polynomials can be expressed in terms of their generating function (see, for example, Bhatnagar [1])

$$H(t) = \sum_{n \ge 0} h_n t^n = \Pi_{r=1}^n (1 - x_r t)^{-1}.$$

If $x_i = q^{i-1}$, from Cameron [3] (cf. P. 224), (1.11) defines the following relationship between $h_r(1, q, q^2, \ldots, q^{n-1})$ and Gaussian coefficients, where $h_r(1, q, q^2, \ldots, q^{n-1})$ is called the complete function of order (n, k).

$$h_r(1, q, q^2, \ldots, q^{n-1}) = \binom{n + r - 1}{r}_q. \tag{1.13}$$

From (1.9), we also have a relationship between $h_r(1, q, q^2, \ldots, q^{n-1})$ and generalized Fibonomial coefficients as follows:

$$\binom{n}{k}_U = \alpha^{k(n-k)} h_k(1, q, q^2, \ldots, q^{n-k}), \tag{1.14}$$

where $q = \beta/\alpha$ (recall that $\alpha \neq 0$ and β are two distinct roots of the equation $x^2 - p_1x + p_2 = 0$), and U is referred to as recursive sequence $(U_n(a_0, a_1; p_1, p_2))_{n\geq 0}$.

In the next section, we give identities of Gaussian coefficients and generalized Fibonomial coefficients. In Section 3, by using formula (1.13) we will transfer the results between Gaussian coefficients and the complete functions.

2. Identities of Gaussian coefficients and Fibonomial coefficients

Theorem 2.1. *Let $\binom{n}{k}_q$ be the Gaussian binomial coefficients defined by (1.2). Then*

$$(1-q^k)(1-q^{n-k})\binom{n}{k}_q = (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q \tag{2.1}$$

for $1 \leq k \leq n-1$.

Proof. By applying (1.3) we have

$$\begin{aligned}(1-q^k)(1-q^{n-k})\binom{n}{k}_q &= (1-q^{n-k})(1-q^n)\binom{n-1}{k-1}_q \\ = (1-q^n)(1-q^{n-k})\binom{n-1}{n-k}_q &= (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q.\end{aligned}$$

An alternative proof may provides an example of the use of (1.6). Starting from (1.6) and noting (1.10), we have

$$\begin{aligned}&(1-q^n)\binom{n}{k}_q \\ &= (1-q^n)\left(\binom{n-1}{k}_q + \binom{n-1}{k-1}_q\right) - (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q \\ &= (2-q^k-q^{n-k})\binom{n}{k}_q - (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q,\end{aligned}$$

or equivalently,

$$(1-q^n-(2-q^k-q^{n-k}))\binom{n}{k}_q = -(1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q,$$

which implies (2.1). □

By applying mathematical induction to the recursive relation (2.1), we may prove the following formula.

Corollary 2.2. *Let* $\binom{n}{k}_q$ *be the Gaussian binomial coefficients defined by* (1.2). *Then for* $0 \le j \le n$ *and* $j \le k \le n-j$

$$\left(\Pi_{\ell=0}^{j-1}(1-q^{k-\ell})(1-q^{n-k-\ell})\right)\binom{n}{k}_q = \left(\Pi_{\ell=0}^{2j-1}(1-q^{n-\ell})\right)\binom{n-2j}{k-j}_q. \tag{2.2}$$

Relationship (1.9) can be used to change an identity for Gaussian binomial coefficients to an identity for generalized Fibonomial coefficients and vice versa.

Corollary 2.3. *Let* $\binom{n}{k}_q$ *be the Gaussian binomial coefficients defined by* (1.2) *with* $q=\beta/\alpha$, *and let* $\binom{n}{k}_U$ *be the generalized Fibonomial coefficients defined by* (1.8). *Then*

$$\alpha^k U_{n-k} + \alpha^{n-k} U_k = \frac{2-q^k-q^{n-k}}{1-q^n} U_n. \tag{2.3}$$

Proof. Substituting

$$\begin{aligned}
\binom{n-1}{k}_q &= \alpha^{-k(n-k-1)}\binom{n-1}{k}_U = \alpha^{-k(n-k-1)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k}}{U_1U_2\cdots U_k}\\
\binom{n-1}{k-1}_q &= \alpha^{-(k-1)(n-k)}\binom{n-1}{k-1}_U = \alpha^{-(k-1)(n-k)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k+1}}{U_1U_2\cdots U_{k-1}}\\
\binom{n}{k}_q &= \alpha^{-k(n-k)}\binom{n}{k}_U = \alpha^{-k(n-k)}\frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k}
\end{aligned}$$

into (1.10), we have

$$\begin{aligned}
&\alpha^{-k(n-k-1)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k}}{U_1U_2\cdots U_k} + \alpha^{-(k-1)(n-k)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k+1}}{U_1U_2\cdots U_{k-1}}\\
&= \frac{2-q^k-q^{n-k}}{1-q^n}\alpha^{-k(n-k)}\frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k},
\end{aligned}$$

which implies (2.3). □

From [10], we have analogues of identities (1.4) and (1.5) for the generalized coefficients defined by (1.7).

Proposition 2.4. *Let* $\binom{n}{k}_q$ *be the Gaussian binomial coefficients defined by* (1.2), *and let* $\binom{n}{k}_A$ *be the generalized coefficients defined by* (1.7). *Then we have*

$$\binom{n}{k}_A - \binom{n-1}{k-1}_A = \binom{n-1}{k}_A \frac{A_n - A_k}{A_{n-k}} \quad \text{and} \tag{2.4}$$

$$\binom{n}{k}_A - \binom{n-1}{k}_A = \binom{n-1}{k-1}_A \frac{A_n - A_{n-k}}{A_k}, \tag{2.5}$$

which generate the identities (1.4) *and* (1.5), *respectively, as the special cases for* $A_n = q^n - 1$.

Proof. From definition (1.7), we may write the left-hand side of (2.4) as

$$\frac{A_nA_{n-1}\cdots A_{n-k+1}}{A_1A_2\cdots A_k}-\frac{A_{n-1}A_{n-2}\cdots A_{n-k+1}}{A_1A_2\cdots A_{k-1}}$$
$$=\frac{A_{n-1}A_{n-2}\cdots A_{n-k+1}A_{n-k}}{A_1A_2\cdots A_k}\frac{A_n-A_k}{A_{n-k}}=\binom{n-1}{k}_A\frac{A_n-A_k}{A_{n-k}},$$

which proves (2.4). Identity (2.5) can be proved similarly. To show (1.4) is a special case of (2.4) for$A_n=q^n-1$, we only need to notice that $\binom{n}{k}_A=\binom{n}{k}_q$ and

$$\frac{A_n-A_k}{A_{n-k}}=\frac{q^n-1-(q^k-1)}{q^{n-k}-1}=q^k,$$

which will convert identity (2.4) to (1.4). Similarly, the transformation $A_n=q^n-1$ will convert identity (2.5) to (1.5). □

Identities of Fibonomial coefficients can be changed to the identities of Fibonacci number sequence and vice versa. For instance, Hoggatt [12] (cf. formula (D)) gives the following identity for Fibonomial coefficients $\binom{n}{k}_F$, where $F=(F_n(0,1,1,-1))$ is the Fibonacci number sequence.

$$\binom{n}{k}_F=F_{k+1}\binom{n-1}{k}_F+F_{n-k-1}\binom{n-1}{k-1}_F. \tag{2.6}$$

By substituting $\binom{n}{k}_F=(F_nF_{n-1}\cdots F_{n-k+1})/(F_1F_2\cdots F_k)$ into the above identity and cancelling the same terms on the both sides of the equation, we obtain the following well-known identity for the Fibonacci number sequence:

$$F_n=F_{n-k}F_{k+1}+F_{n-k-1}F_k, \tag{2.7}$$

which presents a Fibonacci number in terms of smaller Fibonacci numbers. Conversely, from an identity of recursive number sequences, one may obtain identities of Fibonacci coefficients. For instance from Cassini's identity

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n \tag{2.8}$$

we may obtain the following identity for Fibonacci coefficients:

$$F_nF_{n-k}\binom{n}{k}_F=(F_{n+1}F_{n-1}-(-1)^n)\binom{n-1}{k}_F, \tag{2.9}$$

which returns to Cassini's identity when $k=0$. Hence, we have the following results that can also be extended to other transformation between the identities of recursive sequences and the identities of Gaussian coefficients.

Proposition 2.5. *From Cassini's identity (2.8) and the identity (2.7) presenting Fibonacci numbers in terms of smaller Fibonacci numbers, we may derive the corresponding Gaussian Coefficient identities (2.9) and (2.6), respectively, and vice versa.*

3. Identities of the complete functions

Using the relationship (1.13), $h_r(1, q, q^2, \ldots, q^{n-1}) = \binom{n+r-1}{r}_q$, we may re-write the identities of Gaussian coefficients in terms of the complete functions. For instance, from the property of Gaussian coefficients $\binom{n+r-1}{r}_q = \binom{n+r-1}{n-1}_q$ and identities (1.3)-(1.6), we immediately have the following results.

Proposition 3.1. *Let $h_r(1, q, q^2, \ldots, q^{n-1})$ and $\binom{n}{k}_q$ be defined as before. Then*

$$h_r(1, q, q^2, \ldots, q^{n-1}) = h_{n-1}(1, q, q^2, \ldots, q^r), \tag{3.1}$$

$$(1 - q^k)h_k(1, q, q^2, \ldots, q^{n-k}) = (1 - q^n)h_{k-1}(1, q, q^2, \ldots, q^{n-k}), \tag{3.2}$$

$$h_k(1, q, q^2, \ldots, q^{n-k}) = q^k h_k(1, q, q^2, \ldots, q^{n-k-1}) + h_{k-1}(1, q, q^2, \ldots, q^{n-k}), \tag{3.3}$$

$$h_k(1, q, q^2, \ldots, q^{n-k}) = h_k(1, q, q^2, \ldots, q^{n-k-1}) + q^{n-k}h_{k-1}(1, q, q^2, \ldots, q^{n-k}), \tag{3.4}$$

$$h_k(1, q, q^2, \ldots, q^{n-k}) = h_k(1, q, q^2, \ldots, q^{n-k-1}) + h_{k-1}(1, q, q^2, \ldots, q^{n-k}) + (q^{n-1} - 1)h_{k-1}(1, q, q^2, \ldots, q^{n-k-1}). \tag{3.5}$$

From (3.1), we have

$$h_1(1, q, q^2, \ldots, q^{n-1}) = h_{n-1}(1, q).$$

Then, by using (1.9) the recursive sequence $U_n = U_n(a_0, a_1; p_1, p_2) = (\alpha^n - \beta^n)/(\alpha - \beta)$, where α and β are two distinct roots of the equation $x^2 - p_1 x + p_2 = 0$, can be written as

$$\begin{aligned} U_n &= \alpha^{n-1}\frac{1 - q^n}{1 - q} = \alpha^{n-1}\binom{n}{1}_q \\ &= \alpha^{n-1}h_1(1, q, q^2, \ldots, q^{n-1}) = \alpha^{n-1}h_{n-1}(1, q), \end{aligned} \tag{3.6}$$

where $q = \beta/\alpha$. From the definition of $h_r(1, q, \ldots, q^{n-1})$ given by (1.12), we obtain

$$\begin{aligned} U_n &= \alpha^{n-1}h_{n-1}(1, q) = \alpha^{n-1}\sum_{l_1+l_2=n-1,\, l_1,l_2\geq 0} 1^{l_1}q^{l_2} \\ &= \alpha^{n-1}\sum_{l_1+l_2=n-1,\, l_1,l_2\geq 0} 1^{l_1}\left(\frac{\beta}{\alpha}\right)^{l_2} \\ &= \sum_{l_1+l_2=n-1,\, l_1,l_2\geq 0} \alpha^{l_1}\beta^{l_2} = h_{n-1}(\alpha, \beta). \end{aligned}$$

For Fibonacci numbers

$$F_{k+1} = h_k(\alpha, \beta),$$

where $\alpha = (1+\sqrt{5})/2$ and $q = (1-\sqrt{5})/(1+\sqrt{5})$, from (1.9) and (2.6) we obtain

$$\begin{aligned}&h_k(1,q,\dots,q^{n-k})\\&=\alpha^{-k}h_k(\alpha,\beta)h_k(1,q,\dots,q^{n-k-1})\\&\quad+\alpha^{-n+k}h_{n-k+2}(\alpha,\beta)h_{k-1}(1,q\dots,q^{n-k}).\end{aligned}$$

From (3.6) we may establish the following theorem.

Theorem 3.2. *Let $(U_n = U_n(a,b;p_1,p_2))$ be the recursive sequence defined by $U_{n+2} = p_1U_{n+1} - p_2U_n$ $(n \geq 0)$ with the initials $U_0 = a$ and $U_1 = b$, and let α and β be two distinct roots of the characteristic equation $x^2 - p_1x + p_2 = 0$. Then*

$$\alpha^2\binom{n+2}{1}_q = \alpha p_1\binom{n+1}{1}_q - p_2\binom{n}{1}_q, \tag{3.7}$$

or equivalently,

$$\alpha^2 h_{n+1}(1,q) = \alpha p_1 h_n(1,q) - p_2 h_{n-1}(1,q). \tag{3.8}$$

Proof. Noting $p_1 = \alpha+\beta$, $p_2 = \alpha\beta$, and $q = \beta/\alpha$, where $\alpha \neq 0$ (i.e., $p_2 \neq 0$), the right-hand side of (3.7) can be re-written as

$$\begin{aligned}&\alpha p_1\binom{n+1}{1}_q - p_2\binom{n}{1}_q\\&=\alpha p_1\frac{1-q^{n+1}}{1-q} - p_2\frac{1-q^n}{1-q}\\&=\frac{1}{1-q}\big(\alpha p_1(1-q^{n+1}) - p_2(1-q^n)\big)\\&=\frac{1}{1-q}((\alpha p_1 - p_2) - q^n(\alpha p_1 q - p_2))\\&=\frac{1}{1-q}(\alpha^2-\beta^2q^n) = \alpha^2\frac{1-q^{n+2}}{1-q},\end{aligned}$$

which implies (3.7). Consequently, we obtain (3.8) by substituting

$$\binom{m}{1}_q = h_1(1,q,\dots,q^{m-1}) = h_{m-1}(1,q) \tag{3.9}$$

into (3.7) for $m = n, n+1$, and $n+2$, respectively. □

Chen and Louck [6] and Bhatnagar [1] present different approaches to Sylvester's identity related to the complete homogeneous symmetric functions.

Theorem 3.3 (Sylvester's identity)**.** *For each integer $m \geq 0$, we have*

$$\sum_{i=1}^{n}\frac{x_i^m}{\Pi_{j\neq i}(x_i-x_j)} = h_{m-n+1}(x_1,x_2,\dots,x_n), \tag{3.10}$$

where h_k is the kth homogeneous symmetric function, which is defined to be zero for $k < 0$.

Divided differences is a recursive division process. The method can be used to calculate the coefficients of the interpolation polynomial in the Newton form. The divided difference of a function f at knots $x_1, x_2, \ldots, x_n$ has the formula (see, for example, Burden and Faires [2])

$$[x_1, x_2, \ldots, x_n]f = \sum_{i=1}^{n} \frac{f(x_i)}{\Pi_{j\neq i}(x_i - x_j)} = \sum_{i=1}^{n} \frac{f(x_i)}{g'(x_i)}. \tag{3.11}$$

where $g(t) = (t - x_1)(t - x_2)\cdots(t - x_n)$. Thus, from formulas (3.10) and (3.11) we obtain a corollary of Theorem 3.3.

Corollary 3.4. *The value of the complete homogeneous symmetric polynomial, $h_{m-n+1}(x_1, x_2, \ldots, x_n)$, of degree $m-n+1$ at n distinct points $x_1, x_2, \ldots, x_n$ is the coefficient of the highest power term of the Newton interpolation of function $f(x) = x^m$ at points $x_1, x_2, \ldots, x_n$. Particularly, if $m = n$, then the coefficient of power n in the Newton interpolation of $f(x) = x^n$ is $h_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$.*

Corollary 3.5. *If evaluating points of an interpolation are arranged geometrically as $x_i = q^{i-1}$, $i = 1, 2, \ldots, n$, then the coefficient of power n in the Newton interpolation of $f(x) = x^n$ is the Gaussian coefficient $h_1(1, q, \ldots, q^{n-1}) = \binom{n}{1}_q = (1 - q^n)/(1 - q)$.*

Corollary 3.6. *If evaluating points of an interpolation are arranged geometrically as $x_i = q^{i-1}$, $i = 1, 2, \ldots, n$, where $q = \beta/\alpha$ and $\alpha \neq 0$ and β are two distinct roots of the equation $x^2 - p_1 x + p_2 = 0$, then the coefficient of power n in the Newton interpolation of $f(x) = x^n$ is the $\alpha^{-(n-1)}$ multiple of the Fibonacci binomial coefficient $\binom{n}{1}_U$, i.e., $h_1(1, q, \ldots, q^{n-1}) = \alpha^{-(n-1)}\binom{n}{1}_U$. Here, U is referred to as recursive sequence $(U_n(a_0, a_1; p_1, p_2))_{n\geq 0}$.*

Acknowledgments. The authors wish to express their sincere thanks to the Editor and the referee for helpful comments and remarks that lead to an improved and revised version of the original manuscript. The authors also thank Professor William Y. C. Chen for kindly reading our draft and providing the references [1, 14].

References

[1] G. Bhatnagar: *A short proof of an identity of Sylvester*, Internat. J. Math. & Math. Sci. 22.2 (1999), pp. 431–435, doi: https://doi.org/10.1155/S0161171299224313.

[2] R. L. Burden, J. D. Faires: *Numerical analysis, 8th Edition*, Belmont, CA: Thomson Brooks/Cole, 2005.

[3] P. J. Cameron: *Combinatorics: topics, techniques, algorithms*, Cambridge: Cambridge University Press, 1994.

[4] L. Carlitz: *q-Bernoulli numbers and polynomials*, Duke Math. J. 15.4 (1948), pp. 987–1000, doi: https://doi.org/10.1215/S0012-7094-48-01588-9.

[5] L. Carlitz: *The coefficients of the reciprocals of a series*, Duke Math. J. 8.4 (1941), pp. 689–700, doi: https://doi.org/10.1215/S0012-7094-41-00859-1.

[6] W. Y. C. Chen, J. D. Louck: *Interpolation for symmetric functions*, Adv. Math. 117.1 (1996), pp. 147–156, doi: https://doi.org/10.1006/AIMA.1996.0004.

[7] T. Clausen: *Theorem*, Astron. Nachr. 17.22 (1840), pp. 351–352, doi: https://doi.org/10.1002/asna.18400172204.

[8] G. Fontené: *Generalisation d'une formule connue*, Nouv. Ann. Math. 15.4 (1915), p. 112.

[9] A. Girard: *Invention Nouvelle en l'Algébre*, Amsterdam: Imprimé chez Muré frères, 1629.

[10] H. Gould: *The bracket function and Fontené–Ward generalized binomial coefficients with application to Fibonomial coefficients*, Fibonacci Quart. 7.1 (1969), pp. 23–40.

[11] T.-X. He, P. J.-S. Shiue: *On sequences of numbers and polynomials defined by linear recurrence relations of order* 2, Internat. J. of Math. & Math. Sci. (2009), Art. ID 709386, 21 pp. doi: https://doi.org/10.1155/2009/709386.

[12] V. E. Hoggatt: *Fibonacci numbers and generalized binomial coefficients*, Fibonacci Quart. 5.4 (1967), pp. 383–400.

[13] A. F. Horadam, A. G. Shannon: *Ward's Staudt-Clausen problem*, Math. Scand. 29.4 (1976), pp. 239–250.

[14] F. H. Jackson: *q-difference equations*, Amer. J. Math. 32.4 (1910), pp. 305–314.

[15] S. K. Jerbic: *Fibonomial Coefficients*, California: Master of Arts Thesis, San Jose State College, 1968.

[16] I. G. Macdonald: *Symmetric Functions and Hall Polynomials, second ed.* Oxford: Oxford: Clarendon Press, 1995.

[17] R. Rado: *A new proof of a theorem of V. Staudt*, J. Lond. Math. Soc. 9.1 (1934), pp. 85–88, doi: https://doi.org/10.1112/jlms/s1-9.2.85.

[18] A. G. Shannon: *Gaussian binomial coefficients*, Notes Number Theory Discrete Math. 26.1 (2020), pp. 225–229, doi: https://doi.org/10.7546/nntdm.2020.26.1.225-229.

[19] R. P. Stanley: *Enumerative Combinatorics, Vol. 2.* Cambridge: Cambridge University Press, 1999.

[20] C. von Staudt: *Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend*, J. Reine Angew. Math. 21 (1840), pp. 372–374.

[21] P. Trojovský: *On some identities for the Fibonomial coefficients via generating function*, Discrete Appl. Math. 155.15 (2007), pp. 2017–2024, doi: https://doi.org/10.1016/j.dam.2007.05.003.

[22] K. T. Vahlen: *IB3b. Rationale Funktionen der Wurzeln; symmetrische und Affektfunktionen*, in: Encyklopädie der Mathematischen Wissenschaften, Erster Band, Leipzig: Teubner, 1898, pp. 449–479, doi: https://doi.org/10.1007/978-3-663-16017-5_12.

[23] M. Ward: *A Calculus of Sequences*, Amer. J. Math. 58.2 (1936), pp. 255–266, doi: https://doi.org/10.2307/2371035.

Annales Mathematicae et Informaticae
55 (2022) pp. 88–92
DOI: https://doi.org/10.33039/ami.2022.05.002
URL: https://ami.uni-eszterhazy.hu

Generalized Mersenne numbers of the form cx^2

Azizul Hoque

Department of Mathematics,
Rangapara College, Rangapara-784505, Assam, India
ahoque.ms@gmail.com

Abstract. Generalized Mersenne numbers are defined as $M_{p,n} = p^n - p + 1$, where p is a prime and n is a positive integer. Here, we prove that for each pair (c, p) with $c \geq 1$ an integer, there is at most one $M_{p,n}$ of the form cx^2 with a few exceptions.

Keywords: Generalized Mersene number, Diophantine equation, integer solution

AMS Subject Classification: 11D61, 11N32

1. Introduction

Mersenne numbers are positive integers of the form $2^n - 1$ with $n \geq 1$. These numbers have attracted a great deal of interest since the seventeenth century. Furthermore, the primes of this form; so called Mersenne primes are traceable back to Euclid, who in his *"Elements"* connected primes of the form $2^n - 1$ to even perfect numbers. In particular, Euclid-Euler theorem states that an even number is perfect if and only if it has the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime number. A perfect number is a positive integer that is equal to the sum of its proper divisors. Several earliest results spawned from attempts to understand these numbers. Although some modern researchers continue to attribute the same mystical significance to these numbers that the ancient people once did, these numbers remain a substantial inspiration for research in number theory (see [5, 8, 10, 11]). One of the challenging unsolved problems in number theory is Lenstra–Pomerance–Wagstaff conjecture, which states that there are infinitely many Mersenne primes.

In [7], the author and Saikia studied a generalization of Mersenne numbers; so called generalized Mersenne numbers. These numbers are defined as $M_{p,n} =$

Submitted: February 26, 2022
Accepted: May 11, 2022
Published online: May 13, 2022

$p^n - p + 1$, where p is a prime and n is a positive integer. In this case too, we expect an extended version of Lenstra-Pomerance-Wagstaff conjecture. The precise problem is *whether there are infinitely many primes of the form* $M_{p,n}$ *for each prime* p. A weaker version of this problem was posted in [7]. Here, we investigate the problem: *How many generalized Mersenne numbers are there of the form* cx^2 *for each pair of integers* (c, p) *with* $c \geq 1$ *an integer and* p *a prime?* Precisely, we prove:

Theorem 1.1. *For any odd integer* $c \geq 1$ *and a prime* p*, the generalized Mersenne numbers of the form* cx^2 *are* $1, 25$ *and* 121*, with at most one more possibility for each pair* (c, p)*. Further for even integer* $c \geq 2$*, there is no generalized Mersenne number of the form* cx^2*.*

Assume that $2^n - 1 = cx^2$. Then for $n \geq 3$, we have $c \equiv 7 \pmod 8$ and thus by [3, p. 1], $cx^2 + 1 = 2^n$ has at least one solution (x, n). Therefore we have the following straightforward corollary.

Corollary 1.2. *Let* c *be a positive integer. If* $c \not\equiv 7 \pmod 8$*, then* 1 *is the only Mersenne number of the form* cx^2*. Further for* $c \equiv 7 \pmod 8$*, there is exactly one Mersenne number of the form* cx^2*.*

The proof of Theorem 1.1 largely relies on a remarkable result of Bugeaud and Shorey [2, Theorem 1] on the positive integer solutions of certain Diophantine equations.

2. Preliminary descent

We begin this section with a classical result of Bugeaud and Shorey [2] on the number of positive integer solutions of certain Diophantine equations. Before stating this result, we need to introduce some definitions and notations.

Let F_k (resp. L_k) denote the k-th term in the Fibonacci (resp. Lucas) sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_k + F_{k+1}$ (resp. $L_0 = 2$, $L_1 = 1$, and $L_{k+2} = L_k + L_{k+1}$), where $k \geq 0$ is an integer. Given $\lambda \in \{1, \sqrt{2}, 2\}$, we define the sets $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ as follows:

$$\mathcal{F} := \{(F_{k-2\varepsilon}, L_{k+\varepsilon}, F_k) \mid k \geq 2, \varepsilon \in \{\pm 1\}\},$$

$$\mathcal{G} := \{(1, 4p^r - 1, p) \mid p \text{ is an odd prime}, r \geq 1\},$$

$$\mathcal{H} := \left\{ (D_1, D_2, p) \left| \begin{array}{l} D_1,\ D_2 \text{ and } p \text{ are mutually coprime positive integers with } p \\ \text{an odd prime and there exist positive integers } r,\ s \text{ such that} \\ D_1 s^2 + D_2 = \lambda^2 p^r \text{ and } 3D_1 s^2 - D_2 = \pm\lambda^2 \end{array} \right. \right\}.$$

Note that for $\lambda = 2$, the condition "odd" on the prime p should be removed from the above notations.

Theorem A ([2], Theorem 1)**.** *Assume that D_1 and D_2 are coprime positive integers, and p is a prime satisfying* $\gcd(D_1D_2, p) = 1$*. Then for $\lambda \in \{1, \sqrt{2}, 2\}$, the number of positive integer solutions (x, y) of the Diophantine equation*

$$D_1x^2 + D_2 = \lambda^2 p^y \tag{2.1}$$

is at most one, except for

$$(\lambda, D_1, D_2, p) \in \Omega := \begin{Bmatrix} (2,13,3,2), (\sqrt{2},7,11,3), (1,2,1,3), (2,7,1,2), \\ (\sqrt{2},1,1,5), (\sqrt{2},1,1,13), (2,1,3,7) \end{Bmatrix}$$

and $(D_1, D_2, p) \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$.

Note that the authors in [2] were unable to determine $(\lambda, D_1, D_2, p) = (2, 7, 25, 2)$ in the set Ω due to a mild error in calculation. It gives two solutions to (2.1), namely, $(x, y) = (1, 3), (17, 9)$. This comes from simple computation, and it can also be confirmed by [9] that these are the only solutions in positive integers corresponding to the above quadruple.

We also need the following result of the author.

Lemma 2.1 ([6], Lemma 2.1)**.** *For an integer $k \geq 0$, let F_k (resp. L_k) denote the k-th Fibonacci (resp. Lucas) number. Then for $\varepsilon = \pm 1$, we have $4F_k - F_{k-2\varepsilon} = L_{k+\varepsilon}$.*

In [4], Cohn completely solved the Diophantine equation $x^2 + 2^k = y^n$ in positive integers x, y and n, when $k \geq 1$ is an odd integer. We deduce the following lemma from his result (see [4, Theorem]).

Lemma 2.2. *The solutions of the equation*

$$x^2 + 2 = 3^n, \quad x, y, n \in \mathbb{N} \tag{2.2}$$

are $(x, n) = (1, 1), (5, 3)$.

On the other hand, for even positive integer k, Arif and Abu Muriefah gave the complete solution of the Diophantine equation $x^2 + 2^k = y^n$ in positive integers x, y and n. The next lemma can easily be deduced from their result [1, Theorem 1].

Lemma 2.3. *The solutions of the equation*

$$x^2 + 4 = 5^n, \quad x, y, n \in \mathbb{N} \tag{2.3}$$

are $(x, n) = (1, 1), (11, 3)$.

3. Proof of Theorem 1.1

Assume that N is a generalized Mersenne number such that $N = cx^2$. Then for some prime p and positive integer n, we have $M_{p,n} = cx^2$. This can be written as

$$cx^2 + p - 1 = p^n. \tag{3.1}$$

Clearly $p \nmid cx$ and $\gcd(c, p-1) = 1$. It is easy to see from (3.1) that there is no generalized Mersenne number of the form cx^2 when $2 \mid c$.

Let the sets $\Omega, \mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be as in Theorem A. If $(c, p-1, p) \in \Omega$, then $(c, p-1, p) = (2, 1, 3)$ which is not possible.

Now let $(c, p-1, p) \in \mathcal{F}$. Then $(F_{k-2\varepsilon}, L_{k+\varepsilon}, F_k) = (c, p-1, p)$, and thus by Lemma 2.1 $4p - c = p - 1$. This implies that $c + p \equiv 1 \pmod 4$.

If $p > 2$, then (3.1) modulo 4 gives

$$c \equiv \begin{cases} 1 \pmod 4 & \text{if } 2 \nmid n, \\ 2 - p \pmod 4 & \text{if } 2 \mid n. \end{cases}$$

Thus $c + p \equiv 1 \pmod 4$ implies that either $p \equiv 0 \pmod 4$ or $2 \equiv 1 \pmod 4$, and none of these is possible.

For $p = 2$, we have $F_k = 2$ and $L_{k+\varepsilon} = 1$, which are again not possible. Therefore, $(c, p-1, p) \notin \mathcal{F}$.

Assume that $(c, p-1, p) \in \mathcal{G}$. Then $p - 1 = 4p^r - 1$ for some positive integer r. This implies that $p(4p^{r-1} - 1) = 0$ and thus $4p^{r-1} = 1$ which is not possible. Thus, $(c, p-1, p) \notin \mathcal{G}$.

Now if $(c, p-1, p) \in \mathcal{H}$, then for odd prime p, we have

$$cs^2 + p - 1 = p^r \tag{3.2}$$

and

$$3cs^2 - p + 1 = \pm 1, \tag{3.3}$$

where r and s are positive integers.

From (3.3), we have $3cs^2 = p$ and thus $(c, p, s) = (1, 3, 1)$. Therefore (3.1) becomes $x^2 + 2 = 3^n$, and hence by Lemma 2.2 we have $(x, n) = (1, 1), (5, 3)$. This shows that 1 and 25 are only generalized Mersenne numbers $M_{3,n}$ which can be written in the form cx^2.

Again from (3.3), we have $3cs^2 = p - 2$ and thus (3.2) gives $p(4 - 3p^{r-1}) = 5$, which implies that $(p, r) = (5, 1)$. This gives $(c, s) = (1, 1)$ and hence (3.1) becomes $x^2 + 4 = 5^n$. By Lemma 2.3, we get $(x, n) = (1, 1), (11, 3)$, which shows that 1 and 121 are only generalized Mersenne numbers $M_{5,n}$ that can be written in the form cx^2. Thus, we complete the proof by Theorem A.

Acknowledgements. The author would like to thank the anonymous referee for carefully reading this paper. This work was supported by SERB-MATRICS Project (MTR/2021/000762), Govt. of India.

References

[1] S. A. Arif, F. S. A. Muriefah: *On the Diophantine equation $x^2 + 2^k = y^n$ II*, Arab J. Math. Sc. 7.1 (2001), pp. 67–71.

[2] Y. Bugeaud, T. N. Shorey: *On the number of solutions of the generalized Ramanujan-Nagell equation*, J. Reine Angew. Math. 539 (2001), pp. 55–74, doi: https://doi.org/10.1515/crll.2001.079.

[3] J. H. E. Cohn: *On the class number of certain imaginary quadratic fields*, Proc. Amer. Math. Soc. 130.5 (2002), pp. 1275–1277, doi: https://doi.org/10.1090/S0002-9939-01-06255-4.

[4] J. H. E. Cohn: *The Diophantine equation* $x^2 + 2^k = y^n$, Arch. Math. 59 (1992), pp. 341–344, doi: https://doi.org/10.1007/BF01197049.

[5] S. J. Dittmer: *Spoof of odd perfect numbers*, Math. Comp. 83 (2014), pp. 2575–2582, doi: https://doi.org/10.1090/S0025-5718-2013-02793-7.

[6] A. Hoque: *On a class of Lebesgue-Ramanujan-Nagell equations*, (Submitted for publication).

[7] A. Hoque, H. Saikia: *On generalized Mersenne prime*, SeMA J. 66.1 (2014), pp. 1–7, doi: https://doi.org/10.1007/s40324-014-0019-4.

[8] F. L. K. Ford, I. E. Shparlinski: *On the largest prime factor of the Mersenne numbers*, Bull. Austr. Math. Soc. 79.3 (2009), pp. 455–463, doi: https://doi.org/10.1017/S0004972709000033.

[9] M. Le: *On the Diophantine equation* $D_1x^2 + D_2 = 2^{n+2}$, Acta Arith. 64.1 (1993), pp. 29–41, doi: https://doi.org/10.4064/aa-64-1-29-41.

[10] C. Pomerance: *On primitive divisors of Mersenne numbers*, Acta Arith. 46.4 (1986), pp. 355–367, doi: https://doi.org/10.4064/aa-46-4-355-367.

[11] J. S. S. Wagstaff: *Divisors of Mersenne numbers*, Math. Comp. 40 (1983), pp. 385–397, doi: https://doi.org/10.1090/S0025-5718-1983-0679454-X.

Annales Mathematicae et Informaticae
55 (2022) pp. 93–113
DOI: https://doi.org/10.33039/ami.2022.09.001
URL: https://ami.uni-eszterhazy.hu

The monogenity of power-compositional Eisenstein polynomials

Lenny Jones

Shippensburg University
doctorlennyjones@gmail.com

Abstract. We construct infinite collections of monic Eisenstein polynomials $f(x) \in \mathbb{Z}[x]$ such that the power-compositional polynomials $f(x^{d^n})$ are monogenic for all integers $n \geq 0$ and any integer $d > 1$, where d has the property that $f(x)$ is Eisenstein with respect to every prime divisor of d. We also investigate extending these ideas to power-compositional Eisenstein polynomials $f(x^{s^n})$, where s has a prime divisor p such that $f(x)$ is not Eisenstein with respect to p.

Keywords: Eisenstein, irreducible, monogenic, power-compositional

AMS Subject Classification: Primary 11R04, Secondary 11R09, 12F05

1. Introduction

Let $f(x) \in \mathbb{Z}[x]$ be monic. We define $f(x)$ to be *monogenic* if $f(x)$ is irreducible over $\mathbb{Q}$ and $\{1, \theta, \theta^2, \ldots, \theta^{\deg(f)-1}\}$ is a basis for the ring of integers of $K = \mathbb{Q}(\theta)$, where $f(\theta) = 0$. We say that $f(x)$ is *p-Eisenstein*, or simply *Eisenstein*, if there exists a prime p such that $f(x) \equiv x^{\deg(f)} \pmod{p}$, but $f(0) \not\equiv 0 \pmod{p^2}$. It is well known that Eisenstein polynomials are irreducible over $\mathbb{Q}$. Throughout this article, we use the following notation:

- $\mathcal{P}(z)$ is the set of all prime divisors of the integer $z > 1$,
- $\mathcal{E}_f$ is the set of all primes p for which $f(x)$ is p-Eisenstein,
- Π_f is the product of all primes in $\mathcal{E}_f$,
- Γ_f is the set of all integers $d > 1$ such that $\mathcal{P}(d) \subseteq \mathcal{E}_f$,
- Λ_f is the set of all integers $\lambda > 1$ such that the power-compositional polynomials $f(x^{\lambda^n})$ are monogenic for all integers $n \geq 0$.

Submitted: July 25, 2022
Accepted: September 1, 2022
Published online: September 13, 2022

The main purpose of this article is the construction of infinite collections of monic Eisenstein polynomials $f(x) \in \mathbb{Z}[x]$ such that the power-compositional polynomials $f(x^{d^n})$ are monogenic for all integers $n \geq 0$ and all integers $d \in \Gamma_f$. We divide the main investigation section (Section 3) into subsections according to trinomials, quadrinomials, quintinomials and sextinomials. Binomials, which are fully understood, are discussed briefly in Section 4. The approach we use for trinomials utilizes a result of Jakhar, Khanduja and Sangwan [11] that is tailored specifically for the determination of the monogenity of trinomials. For quadrinomials and beyond, we use a different approach that is based partly on ideas found in [12]. To facilitate our methods in these cases, we also prove a new result that establishes the fact that $\Gamma_f \subseteq \Lambda_f$ for any monogenic Eisenstein polynomial with $|f(0)| = \Pi_f$ (see Lemma 3.1). The following theorem, which is an excerpt taken from Theorem 3.9 in Section 3.2, represents a typical result from Section 3.

Theorem 1.1. *Let $N, \mathcal{K}, t, C \in \mathbb{Z}$ with $N \geq 3$, $\gcd(\mathcal{K}, N) = 1$ and $\mathcal{K}$ squarefree. Let*

$$f(x) = x^N + \mathcal{K}t\big((2CN - 2C + 1)x^2 + (2CN^2 - 4CN + N - 1)x + 1\big).$$

Then there exist infinitely many prime values of t such that $f(x^{d^n})$ is monogenic for all $d \in \Gamma_f$ and all integers $n \geq 0$.

Remark 1.2. We point out that infinite families of monogenic power-compositional trinomials were given in [8]. However, Eisenstein polynomials were not specifically addressed there.

2. Preliminaries

We first require some standard tools and notation. Let $\Delta(f(x))$, or simply $\Delta(f)$, and $\Delta(K)$ denote the discriminants over $\mathbb{Q}$, respectively, of $f(x) \in \mathbb{Z}[x]$ and a number field K. If $f(x)$ is irreducible over $\mathbb{Q}$ with $f(\theta) = 0$, then [1]

$$\Delta(f) = [\mathbb{Z}_K : \mathbb{Z}[\theta]]^2 \Delta(K). \tag{2.1}$$

Observe then, from (2.1), that $f(x)$ is monogenic if and only if $\Delta(f) = \Delta(K)$. We also see from (2.1) that if $\Delta(f)$ is squarefree, then $f(x)$ is monogenic. However, the converse is false in general, and when $\Delta(f)$ is not squarefree, it can be quite difficult to determine whether $f(x)$ is monogenic.

Definition 2.1. [1] Let $\mathcal{R}$ be an integral domain with quotient field K, and let $\overline{K}$ be an algebraic closure of K. Let $f(x), g(x) \in \mathcal{R}[x]$, and suppose that $f(x) = a\prod_{i=1}^{m}(x - \alpha_i) \in \overline{K}[x]$ and $g(x) = b\prod_{i=1}^{n}(x - \beta_i) \in \overline{K}[x]$. Then the *resultant* $R(f, g)$ of f and g is:

$$R(f, g) = a^n \prod_{i=1}^{m} g(\alpha_i) = (-1)^{mn} b^m \prod_{i=1}^{n} f(\beta_i).$$

The following theorem is a well-known result in algebraic number theory [2].

Theorem 2.2. *Let p be a prime and let $f(x) \in \mathbb{Z}[x]$ be a monic p-Eisenstien polynomial with $\deg(f) = N$. Let $K = \mathbb{Q}(\theta)$, where $f(\theta) = 0$. Then*

1. $p^{N-1} || \Delta(K)$ *if* $N \not\equiv 0 \pmod p$,
2. $p^N | \Delta(K)$ *if* $N \equiv 0 \pmod p$.

Theorem 2.3. *Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{Q}[x]$, with respective leading coefficients a and b, and respective degrees m and n. Then*

$$\Delta(f \circ g) = (-1)^{m^2 n(n-1)/2} \cdot a^{n-1} b^{m(mn-n-1)} \Delta(f)^n R(f \circ g, g').$$

Remark 2.4. As far as we can determine, Theorem 2.3 is originally due to John Cullinan [3]. A proof of Theorem 2.3 can be found in [7].

The following theorem, known as *Dedekind's Index Criterion*, or simply *Dedekind's Criterion* if the context is clear, is a standard tool used in determining the monogenity of a polynomial.

Theorem 2.5 (Dedekind [1]). *Let $K = \mathbb{Q}(\theta)$ be a number field, $T(x) \in \mathbb{Z}[x]$ the monic minimal polynomial of θ, and $\mathbb{Z}_K$ the ring of integers of K. Let q be a prime number and let $\overline{*}$ denote reduction of $*$ modulo q (in $\mathbb{Z}$, $\mathbb{Z}[x]$ or $\mathbb{Z}[\theta]$). Let*

$$\overline{T}(x) = \prod_{i=1}^{k} \overline{\tau_i}(x)^{e_i}$$

be the factorization of $T(x)$ modulo q in $\mathbb{F}_q[x]$, and set

$$g(x) = \prod_{i=1}^{k} \tau_i(x),$$

where the $\tau_i(x) \in \mathbb{Z}[x]$ are arbitrary monic lifts of the $\overline{\tau_i}(x)$. Let $h(x) \in \mathbb{Z}[x]$ be a monic lift of $\overline{T}(x)/\overline{g}(x)$ and set

$$F(x) = \frac{g(x)h(x) - T(x)}{q} \in \mathbb{Z}[x].$$

Then

$$[\mathbb{Z}_K : \mathbb{Z}[\theta]] \not\equiv 0 \pmod q \iff \gcd\left(\overline{F}, \overline{g}, \overline{h}\right) = 1 \text{ in } \mathbb{F}_q[x].$$

The following theorem appears as Theorem 1 in [12].

Theorem 2.6. *Let N and k be integers with $N > k \geq 1$. Let*

$$f(x) = x^N + \mathcal{T}u(x), \text{ where } \mathcal{T} \in \mathbb{Z} \text{ and}$$

$$u(x) = a_k x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] \text{ with } a_0, a_k \neq 0.$$

Suppose that $f(x)$ is irreducible over $\mathbb{Q}$, and let $K=\mathbb{Q}(\theta)$, where $f(\theta)=0$. Then

$$\Delta(f)=\frac{(-1)^{\frac{N(N+2k-1)}{2}}\mathcal{T}^{N-1}\mathcal{N}(\widehat{u}(\theta))}{a_0},$$

where

$$\widehat{u}(x)=a_k(N-k)x^k+a_{k-1}(N-(k-1))x^{k-1}+\cdots+a_1(N-1)x+a_0N,$$

and $\mathcal{N}:=\mathcal{N}_{K/Q}$ is the algebraic norm. Moreover, if

$$\widehat{u}(x)=\prod_{i=1}^{k}(A_ix+B_i),$$

where the $A_ix+B_i\in\mathbb{Z}[x]$ are not necessarily distinct, then

$$\mathcal{N}(\widehat{u}(\theta))=\prod_{i=1}^{k}\left(\mathcal{T}\sum_{j=0}^{k}a_jA_i^{N-j}(-B_i)^j+(-B_i)^N\right).$$

The following corollary of Theorem 2.6 will be useful in this article.

Corollary 2.7. *Let $f(x)$, $u(x)$ and $\widehat{u}(x)$ be as defined in Theorem 2.6. Suppose that $f(x)$ is irreducible over $\mathbb{Q}$, $K=\mathbb{Q}(\theta)$ where $f(\theta)=0$, and the content of $u(x)$ is 1. If $\mathcal{T}\mathcal{N}(\widehat{u}(\theta))/a_0$ is squarefree, then $\gcd(\mathcal{T},N)=1$ and $f(x)$ is monogenic.*

Proof. If $\mathcal{T}=1$, the corollary is obviously true. If $|\mathcal{T}|\geq 2$, then $f(x)$ is Eisenstein with $\Pi_f=|\mathcal{T}|$, since the content of $u(x)$ is 1. Let p be a prime divisor of Π_f. If $p\mid N$, then $p^N\mid\Delta(K)$ by Theorem 2.2, which contradicts the fact that $\Pi_f\mathcal{N}(\widehat{u}(\theta))/a_0$ is squarefree. Hence, $p\nmid N$ and $p^{N-1}\mid\mid\Delta(K)$ by Theorem 2.2, which completes the proof. □

The next theorem follows from Corollary (2.10) in [14].

Theorem 2.8. *Let K and L be number fields with $K\subset L$. Then*

$$\Delta(K)^{[L:K]}\mid\Delta(L).$$

Theorem 2.9. *Let $G(t)\in\mathbb{Z}[t]$, and suppose that $G(t)$ factors into a product of distinct irreducibles, such that the degree of each irreducible is at most 3. Define*

$$N_G(X)=|\{p\leq X: p \textit{ is prime and } G(p) \textit{ is squarefree}\}|.$$

Then,

$$N_G(X)\sim C_G\frac{X}{\log(X)},$$

where

$$C_G=\prod_{\ell\ \text{prime}}\left(1-\frac{\rho_G(\ell^2)}{\ell(\ell-1)}\right)$$

and $\rho_G(\ell^2)$ is the number of $z\in(\mathbb{Z}/\ell^2\mathbb{Z})^*$ such that $G(z)\equiv 0 \pmod{\ell^2}$.

Remark 2.10. Theorem 2.9 follows from work of Helfgott, Hooley and Pasten [9, 10, 15]. For more details, see [13].

Definition 2.11. In the context of Theorem 2.9, for $G(t) \in \mathbb{Z}[t]$ and a prime ℓ, if $G(z) \equiv 0 \pmod{\ell^2}$ for all $z \in \left(\mathbb{Z}/\ell^2\mathbb{Z}\right)^*$, we say that $G(t)$ has a *local obstruction* at ℓ.

The following immediate corollary of Theorem 2.9 is a tool used to establish the main results in this article.

Corollary 2.12. *Let $G(t) \in \mathbb{Z}[t]$, and suppose that $G(t)$ factors into a product of distinct irreducibles, such that the degree of each irreducible is at most 3. To avoid the situation when $C_G = 0$, we suppose further that $G(t)$ has no local obstructions. Then there exist infinitely many primes q such that $G(q)$ is squarefree.*

We make the following observation concerning $G(t)$ from Corollary 2.12 in the special case when each of the distinct irreducible factors of $G(t)$ is of the form $\alpha_i t + \beta_i \in \mathbb{Z}[t]$ with $\gcd(\alpha_i, \beta_i) = 1$. In this situation, it follows that the minimum number of distinct factors required in $G(t)$ so that $G(t)$ has a local obstruction at the prime ℓ is $2(\ell - 1)$. More precisely, in this minimum scenario, we have

$$G(t) = \prod_{i=1}^{2(\ell-1)} (\alpha_i t + \beta_i) \equiv C(t-1)^2(t-2)^2 \cdots (t-(\ell-1))^2 \pmod{\ell},$$

where $C \not\equiv 0 \pmod{\ell}$. Then each zero r of $G(t)$ modulo ℓ lifts to the ℓ distinct zeros

$$r, \quad r+\ell, \quad r+2\ell, \quad \ldots\ldots, \quad r+(\ell-1)\ell \in \left(\mathbb{Z}/\ell^2\mathbb{Z}\right)^*$$

of $G(t)$ modulo ℓ^2 [4, Theorem 4.11]. That is, $G(t)$ has exactly $\ell(\ell-1) = \phi(\ell^2)$ distinct zeros $z \in \left(\mathbb{Z}/\ell^2\mathbb{Z}\right)^*$. Therefore, if the number of factors k of $G(t)$ satisfies $k < 2(\ell-1)$, then there must exist $z \in \left(\mathbb{Z}/\ell^2\mathbb{Z}\right)^*$ for which $G(z) \not\equiv 0 \pmod{\ell^2}$, and we do not need to check such primes ℓ for a local obstruction. Consequently, only finitely many primes need to be checked for local obstructions. They are precisely the primes ℓ such that $\ell \le (k+2)/2$.

3. The main results

This section is devoted to the construction of infinite collections of monic Eisenstein polynomials $f(x) \in \mathbb{Z}[x]$ with the property that the power-compositional polynomials $f(x^{d^n})$ are monogenic for all integers $n \geq 0$ and all integers $d \in \Gamma_f$. We begin with a new result that is key to our investigations in Sections 3.2, 3.3 and 3.4.

Lemma 3.1. *Let $f(x) \in \mathbb{Z}[x]$ be Eisenstein with $\deg(f) = N$. If $f(x)$ is monogenic and $|f(0)| = \Pi_f$, then $\Gamma_f \subseteq \Lambda_f$.*

Proof. Note first that we can write

$$f(x) = x^N + \Pi_f w(x), \text{ for some } w(x) \in \mathbb{Z}[x], \text{ with } |w(0)| = 1. \tag{3.1}$$

Let $d \in \Gamma_f$. For $n \geq 0$, define

$$\mathcal{F}_n(x) := f(x^{d^n}), \quad \theta_n := \theta^{1/d^n} \quad \text{and} \quad K_n := \mathbb{Q}(\theta_n),$$

where $f(\theta) = 0$. Then $\theta_0 = \theta$, $\mathcal{F}_0(x) = f(x)$ and, since $f(x)$ is monogenic, we have that $\Delta(f) = \Delta(\mathcal{F}_0) = \Delta(K_0)$. Additionally, for all $n \geq 0$,

$$\mathcal{F}_n(\theta_n) = 0, \quad [K_{n+1} : K_n] = d \quad \text{and} \quad \mathcal{F}_n(x) \text{ is Eisesntein with } |\mathcal{F}_n(0)| = \Pi_f.$$

We have that $\mathcal{F}_0(x)$ is monogenic by hypothesis, and we need to show that $\mathcal{F}_n(x)$ is monogenic for all integers $n \geq 1$. Assume that $\mathcal{F}_n(x)$ is monogenic, so that $\Delta(\mathcal{F}_n) = \Delta(K_n)$, and proceed by induction on n to show that $\mathcal{F}_{n+1}(x)$ is monogenic. Let $\mathbb{Z}_{K_n}$ denote the ring of integers of K_n. Consequently, by Theorem 2.8, it follows that

$$\Delta(\mathcal{F}_n)^d \text{ divides } \Delta(K_{n+1}) = \frac{\Delta(\mathcal{F}_{n+1})}{[\mathbb{Z}_{K_{n+1}} : \mathbb{Z}[\theta_{n+1}]]^2},$$

which implies that

$$[\mathbb{Z}_{K_{n+1}} : \mathbb{Z}[\theta_{n+1}]]^2 \text{ divides } \frac{\Delta(\mathcal{F}_{n+1})}{\Delta(\mathcal{F}_n)^d}.$$

Since $|f(0)| = \Pi_f$, we see from Theorem 2.3 that

$$\begin{aligned} \left|\Delta(\mathcal{F}_n)^d\right| &= \left|\Delta(f)^{d^{n+1}} d^{nd^{n+1}N} (\Pi_f)^{d^{n+1}-d}\right| \quad \text{and} \\ |\Delta(\mathcal{F}_{n+1})| &= \left|\Delta(f)^{d^{n+1}} d^{(n+1)d^{n+1}N} (\Pi_f)^{d^{n+1}-1}\right|. \end{aligned}$$

Hence,

$$\left|\frac{\Delta(\mathcal{F}_{n+1})}{\Delta(\mathcal{F}_n)^d}\right| = d^{d^{n+1}N} (\Pi_f)^{d-1}.$$

Since $\mathcal{P}(d) \subseteq \mathcal{E}_f$, it is enough to show that $\gcd(\Pi_f, [\mathbb{Z}_{K_{n+1}} : \mathbb{Z}[\theta_{n+1}]]) = 1$. To establish this fact, we apply Theorem 2.5 to $T := \mathcal{F}_{n+1}(x)$, with q a prime divisor of Π_f. Then we see from (3.1) that $\overline{T}(x) = x^{d^{n+1}N}$, and so we can let $g(x) = x$ and $h(x) = x^{d^{n+1}N-1}$. Hence

$$F(x) = \frac{g(x)h(x) - T(x)}{q} = -\frac{\Pi_f}{q} w(x^{d^{n+1}}).$$

Since Π_f is squarefree, and $|w(0)| = 1$, we deduce that $\overline{F}(0) \neq 0$ and therefore, $\gcd(\overline{F}, \overline{g}) = 1$. Thus, by Theorem 2.5, we conclude that

$$[\mathbb{Z}_{K_{n+1}} : \mathbb{Z}[\theta_{n+1}]] \not\equiv 0 \pmod{q}$$

and, consequently, $\mathcal{F}_{n+1}(x)$ is monogenic, which completes the proof. □

We see from Lemma 3.1 that we only need to focus on finding infinite collections of monogenic Eisenstein polynomials $f(x)$ with $|f(0)| = \Pi_f$ to produce infinite collections of Eisenstein polynomials with the desired power-compositional properties. Lemma 3.1 will be used for quadrinomials and beyond, but a separate approach is used for trinomials.

3.1. Trinomials

The formula for the discriminant of an arbitrary monic trinomial, due to Swan [16], is given in the following theorem.

Theorem 3.2. *Let $f(x) = x^N + Ax^M + B \in \mathbb{Z}[x]$, where $0 < M < N$. Let $r = \gcd(N, M)$, $N_1 = N/r$ and $M_1 = M/r$. Then*

$$\Delta(f) = (-1)^{N(N-1)/2} B^{M-1} D^r,$$

where

$$D := N^{N_1} B^{N_1 - M_1} - (-1)^{N_1} M^{M_1} (N - M)^{N_1 - M_1} A^{N_1}.$$

Applying Theorem 3.2 to the power-compositional trinomial

$$\mathcal{F}_n(x) := f(x^{d^n}) = x^{d^n N} + Ax^{d^n M} + B \tag{3.2}$$

we get the following immediate corollary.

Corollary 3.3. *Let $f(x)$ and D be as given in Theorem 3.2, and let $\mathcal{F}_n(x)$ be as defined in (3.2). Let $d, n \in \mathbb{Z}$ with $d \geq 1$ and $n \geq 0$. Then*

$$\Delta(\mathcal{F}_n) = (-1)^{d^n N(d^n N - 1)/2} B^{d^n M - 1} d^{n d^n N} D^{d^n r}.$$

The next result is essentially an algorithmic adaptation of Dedekind's index criterion for trinomials.

Theorem 3.4. [11] *Let $N \geq 2$ be an integer. Let $K = \mathbb{Q}(\theta)$ be an algebraic number field with $\theta \in \mathbb{Z}_K$, the ring of integers of K, having minimal polynomial $f(x) = x^N + Ax^M + B$ over $\mathbb{Q}$, with $\gcd(M, N) = r$, $N_1 = N/r$ and $M_1 = M/r$. Let D be as defined in Theorem 3.2. A prime factor q of $\Delta(f)$ does not divide $[\mathbb{Z}_K : \mathbb{Z}[\theta]]$ if and only if q satisfies one of the following conditions:*

1. *when $q \mid A$ and $q \mid B$, then $q^2 \nmid B$;*
2. *when $q \mid A$ and $q \nmid B$, then*

$$\text{either} \quad q \mid A_2 \text{ and } q \nmid B_1 \quad \text{or} \quad q \nmid A_2\Big((-B)^{M_1} A_2^{N_1} - (-B_1)^{N_1}\Big),$$

where $A_2 = A/q$ and $B_1 = \frac{B + (-B)^{q^e}}{q}$ with $q^e \mid\mid N$;

3. *when* $q \nmid A$ *and* $q \mid B$, *then*

   either $q \mid A_1$ *and* $q \nmid B_2$ *or* $q \nmid A_1B_2^{M-1}\left((-A)^{M_1}A_1^{N_1-M_1}-(-B_2)^{N_1-M_1}\right)$,

   where $A_1 = \frac{A+(-A)^{q^j}}{q}$ *with* $q^j \mid\mid (N-M)$, *and* $B_2 = B/q$;

4. *when* $q \nmid AB$ *and* $q \mid M$ *with* $N = s'q^k$, $M = sq^k$, $q \nmid \gcd(s', s)$, *then the polynomials*

   $$x^{s'} + Ax^s + B \quad \textit{and} \quad \frac{Ax^{sq^k} + B + (-Ax^s - B)^{q^k}}{q}$$

   are coprime modulo q;

5. *when* $q \nmid ABM$, *then* $q^2 \nmid D/r^{N_1}$.

The following theorem lays the groundwork for the construction of infinite collections of monogenic power-compositional Eisenstein trinomials.

Theorem 3.5. *Suppose that* $f(x) = x^N + Ax^M + B \in \mathbb{Z}[x]$ *is Eisenstein with* $N > M > 0$ *and* B *squarefree. Suppose further that* $\Pi_f \equiv 0 \pmod{\kappa}$, *where* κ *is the squarefree kernel of* $r := \gcd(N, M)$. *Let*

$$\rho = \prod_{\substack{p \mid \Pi_f \\ p \text{ prime}}} p^{\nu_p(D)} \quad \textit{and} \quad \mathfrak{D} = D/\rho,$$

where ν_p *is the* p*-adic valuation, and* D *is as defined in Theorem 3.2. If* $\mathfrak{D}$ *is squarefree and* $d \in \Gamma_f$, *then* $\mathcal{F}_n(x) := f(x^{d^n})$ *is monogenic for all integers* $n \geq 0$.

Proof. Note that $\mathcal{F}_n(x)$ is Eisenstein, and hence is irreducible over $\mathbb{Q}$. Suppose that $\mathcal{F}_n(\theta) = 0$, and let $\mathbb{Z}_K$ be the ring of integers of $K = \mathbb{Q}(\theta)$. To establish monogenity, we use Theorem 3.4 to show that $[\mathbb{Z}_K : \mathbb{Z}[\theta]] \not\equiv 0 \pmod{q}$ for all primes q dividing $\Delta(\mathcal{F}_n)$ in Corollary 3.3. Since $\mathcal{P}(d) \subseteq \mathcal{P}(B)$, we only have to address primes dividing BD.

Suppose first that $q \mid B$. If $q \mid A$, then condition (1) of Theorem 3.4 is satisfied since B is squarefree. Suppose then that $q \nmid A$, and we examine condition (3). Note that $q \nmid \Pi_f$. If $q \nmid A_1$, then $q^j \mid\mid (d^nN - d^nM)$ for some integer $j \geq 1$. Thus, since $q \nmid d$, we conclude that

$$N - M = q^j c, \tag{3.3}$$

for some integer $c \geq 1$. If $N_1 - M_1 > 1$, then

$$q^2 \mid B^{N_1-M_1} \quad \text{and} \quad q^2 \mid (N-M)^{N_1-M_1}.$$

Thus, $q^2 \mid D$, which implies that $q^2 \mid \mathfrak{D}$ since $q \nmid \Pi_f$, contradicting the fact that $\mathfrak{D}$ is squarefree. Therefore, $N_1 - M_1 = 1$ and we deduce from (3.3) that

$$1 = N_1 - M_1 = (N-M)/r = q^j c/r,$$

which contradicts the fact that $q \nmid r$. Hence, $q \mid A_1$, and since B is squarefree, condition (3) of Theorem 3.4 is satisfied.

Suppose next that $q \mid D$ and $q \nmid B$. If $q \mid A$, then $q \mid N$ so that

$$q^2 \mid N^{N_1} \quad \text{and} \quad q^2 \mid A^{N_1}.$$

Then $q^2 \mid D$, which implies that $q^2 \mid \mathfrak{D}$ since $q \nmid \Pi_f$, contradicting the fact that $\mathfrak{D}$ is squarefree. If $q \nmid AB$ and $q \mid M$, then $q \mid N$. We then conclude that

$$q^2 \mid N^{N_1} \quad \text{and} \quad q^2 \mid M^{M_1}(N-M)^{N_1-M_1}.$$

Then, as before, $q^2 \mid D$, which implies that $q^2 \mid \mathfrak{D}$ since $q \nmid \Pi_f$, contradicting the fact that $\mathfrak{D}$ is squarefree. Finally, suppose that $q \nmid ABM$. Thus, $q^2 \nmid D/r^{N_1}$ since $q \nmid \Pi_f$ and $\mathfrak{D}$ is squarefree, so that condition (5) is satisfied. □

The following corollary illustrates how Theorem 3.5 can be used to construct infinite collections of monic Eisenstein trinomials with the desired power-compositional properties.

Corollary 3.6. *Let $N, C, t \in \mathbb{Z}$ be such that $N \geq 2$ and Ct is squarefree. Then, in each of the following situations, there exist infinitely many prime values of t such that $f(x^{d^n})$ is monogenic for any $d \in \Gamma_f$ and all integers $n \geq 0$:*

1. *$f(x) = x^N + Ctx + Ct$, where $|Ct| \geq 2$ and $\gcd(Ct, N) = 1$,*
2. *$f(x) = x^N + Cx^{N-1} + Ct$, where $|C| \geq 2$ and $\gcd(C, Nt) = 1$,*
3. *$f(x) = x^p + px^{p-1} + pt$, where p is prime.*

Proof. Observe that $f(x)$ is Eisenstein for all situations. For (1), in the setting of Theorem 3.5, we have

$$A = B = Ct, \quad \Pi_f = |Ct|, \quad r = \kappa = 1 \quad \text{and}$$

$$D = (-1)^{N-1}(Ct)^{N-1}\big((N-1)^{N-1}Ct - (-1)^N N^N\big),$$

so that $\Pi_f \equiv 0 \pmod{\kappa}$ and

$$\mathfrak{D} = (1-N)^{N-1}Ct + N^N,$$

since $\gcd(Ct, N) = 1$. Thinking of t as an indeterminate, let $G(t) = \mathfrak{D}$. Since $G(t)$ has no local obstructions, we conclude from Corollary 2.12 that there exist infinitely many primes q such that $G(q)$ is squarefree. Since Ct is also squarefree for any such prime $t = q > C$, part (1) follows from Theorem 3.5.

For (2), in the setting of Theorem 3.5, we have

$$A = C, \quad B = Ct, \quad \Pi_f = |C|, \quad r = \kappa = 1 \quad \text{and}$$

$$D = C\big(N^N t - (-1)^N (N-1)^{N-1} C^{N-1}\big),$$

so that $\Pi_f \equiv 0 \pmod{\kappa}$ and

$$\mathfrak{D} = N^N t - (-1)^N (N-1)^{N-1} C^{N-1}.$$

The remainder of the proof for this part is identical to part (1), and we omit the details.

Finally, for (3), we have that

$$N = p, \quad M = p-1, \quad A = p, \quad B = pt, \quad \Pi_f = p, \quad r = \kappa = 1 \quad \text{and}$$

$$D = p^p(pt - (-1)^p(p-1)^{p-1}) = p^p \mathfrak{D}.$$

Again, the remainder of the proof for this part is identical to part (1), and we omit the details. □

Note that part (3) of Corollary 3.6 is similar to part (2), except that we have lifted the restriction $\gcd(C, Nt) = 1$. Indeed, this restriction is really unnecessary in part (2). However, we have added it there to make the computation of $\mathfrak{D}$ more transparent. Similarly, the restriction $\gcd(Ct, N) = 1$ can be lifted from part (1) as well.

3.2. Quadrinomials

The following lemma contains two special cases of Theorem 2.6.

Lemma 3.7. *Let $N, \mathcal{T}, C \in \mathbb{Z}$ with $N \geq 3$.*

1. *Suppose that*

$$f(x) = x^N + \mathcal{T}\big((2CN - 2C + 1)x^2 + (2CN^2 - 4CN + N - 1)x + 1\big)$$

is irreducible over $\mathbb{Q}$. Then $|\Delta(f)| = \left|\mathcal{T}^{N-1} T_1 T_2\right|$, where

$$T_1 = (2CN^2 + N + 1)\mathcal{T} + (-N)^N \text{ and}$$
$$T_2 = -(N-2)^{N-2}(2CN - 2C + 1)^{N-1}(2CN^2 - 8CN + N + 8C - 3)\mathcal{T} + (-1)^N.$$

2. *Suppose that*

$$f(x) = x^N + \mathcal{T}((CN - C + 1)x^2 + (CN + 2)x + 1)$$

is irreducible over $\mathbb{Q}$. Then $|\Delta(f)| = \left|\mathcal{T}^{N-1} T_1 T_2\right|$, where

$$T_1 = (N-2)^{N-2}(CN^2 + 4)\mathcal{T} + (-N)^N \text{ and}$$
$$T_2 = -C(CN - C + 1)^{N-1}\mathcal{T} + (-1)^N.$$

Proof. We give details only for *(1)* since the details for *(2)* are similar. For *(1)*, we have in the setting of Theorem 2.6 that

$$\begin{aligned} u(x) &= (2CN - 2C + 1)x^2 + (2CN^2 - 4CN + N - 1)x + 1 \text{ and} \\ \widehat{u}(x) &= \big(x + N\big)\big((N-2)(2CN - 2C + 1)x + 1\big), \end{aligned}$$

so that

$$\begin{aligned} &a_2 = 2CN - 2C + 1, \quad a_1 = 2CN^2 - 4CN + N - 1, \quad a_0 = 1, \\ &A_1 = 1, \quad B_1 = N, \quad A_2 = (N-2)(2CN - 2C + 1) \quad \text{and} \quad B_2 = 1. \end{aligned}$$

Then $|\Delta(f)|$ in *(1)* can be calculated easily using Theorem 2.6. □

Remark 3.8. Both cases of Lemma 3.7 provide generalizations of the example $v(x) := x^N + \mathcal{T}(x^2 + (N-1)x + 1)$ given in [12, Corollary 1] for the construction of infinite families of monogenic quadrinomials. For example, $f(x)$ in *(1)* of Lemma 3.7 specializes to $v(x)$ at $C = 0$.

The following theorem uses Lemma 3.1 and Lemma 3.7 to construct monogenic power-compositional Eisenstein quadrinomials.

Theorem 3.9. *Let $N, \mathcal{K}, t, C \in \mathbb{Z}$, where $N \geq 3$, $\gcd(\mathcal{K}, N) = 1$ and $\mathcal{K}$ is squarefree. With $\mathcal{T} = \mathcal{K}t$, let $f(x)$ be as given in either (1) or (2) of Lemma 3.7. Then there exist infinitely many prime values of t such that $f(x^{d^n})$ is monogenic for any $d \in \Gamma_f$ and all integers $n \geq 0$, for any $N \geq 3$ in (1), and any $N \equiv 1 \pmod 2$ in (2).*

Proof. Since the two cases are handled in a similar manner, we give details only for *(1)* of Lemma 3.7. Thinking of t as an indeterminate, let $G(t) := T_1T_2$. We claim that there exist infinitely many primes q such that $G(q)$ is squarefree. To see this, we apply Corollary 2.12 to $G(t)$. Observe that $T_1 \neq T_2$, and that each T_i is of the form $\alpha_i t + \beta_i \in \mathbb{Z}[t]$, with $\gcd(\alpha_i, \beta_i) = 1$. We need to check for local obstructions. According to the discussion following Corollary 2.12, we only need to check the prime $\ell = 2$. An easy computer calculation reveals that either $G(1) \not\equiv 0 \pmod 4$ or $G(3) \not\equiv 0 \pmod 4$ for every one of the 48 possible combinations of $[N \pmod 4, C \pmod 4, \mathcal{K} \pmod 4]$, noting that $\mathcal{K} \not\equiv 0 \pmod 4$ since $\mathcal{K}$ is squarefree. Thus, $G(t)$ has no local obstructions, and we conclude from Corollary 2.12 that there exist infinitely many primes q such that $G(q)$ is squarefree, and the claim is verified. Then, for such a prime q with $t = q > \mathcal{K}N$, it follows that $\mathcal{K}qG(q)$ is squarefree. Thus, since $f(x)$ is Eisenstien with $|f(0)| = \Pi_f = |\mathcal{K}t|$, we deduce that $f(x)$ is monogenic by Corollary 2.7. Hence, by Lemma 3.1, we have that $f(x^{d^n})$ is monogenic for any $d \in \Gamma_f$ and all integers $n \geq 0$. □

3.3. Quintinomials

In this section, we use Theorem 2.6 with $\mathcal{T} = \mathcal{K}t$, where $\mathcal{K}, t \in \mathbb{Z}$ with $\mathcal{K}t \equiv 1 \pmod 2$, $\mathcal{K}t$ squarefree and $|\mathcal{K}t| \geq 3$. Then, a strategy similar to [12] is employed to construct infinite collections of monogenic Eisenstein quintinomials. For an integer $N \geq 4$, suppose that $\gcd(N, \mathcal{K}) = 1$, and let

$$f(x) = x^N + \mathcal{T}u(x) = x^N + \mathcal{K}t(a_3x^3 + a_2x^2 + a_1x + a_0),$$

where $a_i \in \mathbb{Z}$ with $a_0 = 1$. Thus, $f(x)$ is Eisenstein. In the context of Theorem 2.6, we have that

$$\widehat{u}(x) = a_3(N-3)x^3 + a_2(N-2)x^2 + a_1(N-1)x + N. \tag{3.4}$$

Suppose that $\widehat{u}(x)$ factors as

$$\widehat{u}(x) = (a_3x + 1)(x + N)((N-3)x + 1). \tag{3.5}$$

Then, in Theorem 2.6, we have

$$\begin{array}{ll} A_1 = a_3, & B_1 = 1 \\ A_2 = 1, & B_2 = N \\ A_3 = N-3, & B_3 = 1, \end{array}$$

so that $|\Delta(f)| = \left|(\mathcal{K}t)^{N-1}T_1T_2T_3\right|$, where

$$\begin{aligned} T_1 &= a_3^{N-3}(a_3^3 - a_1a_3^2 + a_2a_3 - a_3)\mathcal{K}t + (-1)^N, \\ T_2 &= (1 - a_1N + a_2N^2 - a_3N^3)\mathcal{K}t + (-N)^N, \\ T_3 &= (N-3)^{N-3}\big((N-3)^3 - a_1(N-3)^2 + a_2(N-3) - a_3\big)\mathcal{K}t + (-1)^N. \end{aligned} \tag{3.6}$$

Thinking of t as an indeterminate, we define $G(t) := T_1T_2T_3$. Note that each T_i is of the form $\alpha_it + \beta_i$, where $\gcd(\alpha_i, \beta_i) = 1$. To show that there exist infinitely many primes q such that $G(q)$ is squarefree, we use Corollary 2.12. However, we must first show that $G(t)$ has no obstruction at the prime $\ell = 2$. Expanding $\widehat{u}(x)$ in (3.5) gives

$$\widehat{u}(x) = a_3(N-3)x^3 + (a_3(N^2-3N+1) + N-3)x^2 + (a_3N + N^2 - 3N + 1)x + N. \tag{3.7}$$

Equating coefficients in (3.4) and (3.7) then yields the system of linear Diophantine equations

$$\begin{aligned} (N-1)a_1 - Na_3 &= N^2 - 3N + 1 \\ (N-2)a_2 - (N^2 - 3N + 1)a_3 &= N - 3, \end{aligned} \tag{3.8}$$

which has infinitely many solutions since

$$\gcd(N-1, N) = \gcd(N-2, N^2 - 3N + 1) = 1.$$

Using a parity argument on (3.8), we conclude that:

$$a_1 \equiv a_3 \equiv 1 \pmod 2 \quad \text{when } N \equiv 0 \pmod 2,$$

$$a_2 \equiv a_3 \equiv 1 \pmod 2 \quad \text{when } N \equiv 1 \pmod 2.$$

Upon closer inspection of (3.6), we see that if $a_2 \equiv 0 \pmod 2$ when $N \equiv 0 \pmod 2$, then $T_1 \equiv T_3 \equiv t+1 \pmod 2$ and $T_2 \equiv t \pmod 2$. Thus, $G(1) \equiv G(3) \equiv 0 \pmod 4$ so that $G(t)$ has a local obstruction at $\ell = 2$. Similarly, if $a_1 \equiv 0 \pmod 2$ when $N \equiv 1 \pmod 2$, then $G(t)$ has a local obstruction at $\ell = 2$. However, if $a_1 \equiv a_2 \equiv a_3 \equiv 1 \pmod 2$, then it is easy to verify that $G(t)$ has no local obstruction at $\ell = 2$. To isolate such solutions of (3.8), we let $a_i = 2b_i + 1$ for each $i \in \{1,2,3\}$ and substitute into (3.8) to get

$$\begin{aligned} (N-1)b_1 - Nb_3 &= \frac{N^2-3N+2}{2} \\ (N-2)b_2 - (N^2-3N+1)b_3 &= \frac{N^2-3N}{2}. \end{aligned} \tag{3.9}$$

Unimodular row reduction produces the following parametric solutions of (3.9):

$$\begin{aligned} b_1 &= -\left(\frac{N^4-7N^3+15N^2-9N+2}{2}\right) - (N^2-2N)z, \\ b_2 &= -(N-2)\left(\frac{N^4-7N^3+15N^2-9N+2}{2}\right) - (N^3-4N^2+4N-1)z, \\ b_3 &= \frac{-N^4+8N^3-22N^2+23N-8}{2} - (N^2-3N+1)z, \end{aligned} \tag{3.10}$$

where $z \in \mathbb{Z}$. Thus, for any (a_1, a_2, a_3), where $a_i = 2b_i + 1$ and (b_1, b_2, b_3) is a solution to (3.10), it follows that there exist infinitely many primes q such that $G(q)$ is squarefree. Consequently, Corollary 2.7 implies the following theorem.

Theorem 3.10. *Let $N, \mathcal{K}, t \in \mathbb{Z}$ with $N \geq 4$, $\mathcal{K}t \equiv 1 \pmod 2$, $\mathcal{K}t$ squarefree, $|\mathcal{K}t| \geq 3$ and $\gcd(\mathcal{K}, N) = 1$. Then, for each (a_1, a_2, a_3), where $a_i = 2b_i + 1$ with (b_1, b_2, b_3) a solution to (3.10), there exist infinitely many prime values of t such that*

$$f(x) = x^N + \mathcal{K}t(a_3x^3 + a_2x^2 + a_1x + a_0)$$

is monogenic.

Then, the following corollary, which is immediate from Lemma 3.1, gives us our desired collections of quintinomials.

Corollary 3.11. *As described in Theorem 3.10, let $t = q$ be a prime such that $f(x)$ is monogenic. Then $\Pi_f = |\mathcal{K}q|$ and $f(x^{d^n})$ is monogenic for all $d \in \Gamma_f$ and integers $n \geq 0$.*

3.4. Sextinomials

In this section, we show how techniques similar to previous sections can be used to construct sextinomials with the desired properties. Let m be an integer with $m \notin \{-1, 0\}$, and let

$$N = 9m^2 + 9m + 2 = (3m+1)(3m+2),$$

so that $N \geq 20$. Let

$$f(x) = x^N + tu(x) = x^N + t(a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) \in \mathbb{Z}[x],$$

where $|t| > 1$ is squarefree, $a_i \neq 0$ and $a_0 = 1$. Note that $f(x)$ is Eisenstein. We use Theorem 2.6 and assume that

$$\begin{aligned}\widehat{u}(x) &= (a_4x + 1)(x + 3m + 1)(x + 3m + 2)((N - 4)x + 1) \quad (3.11)\\ &= a_4(N - 4)x^4 + a_3(N - 3)x^3 + a_2(N - 2)x^2 + a_1(N - 1) + N. \quad (3.12)\end{aligned}$$

Equating coefficients in (3.11) and (3.12) yields the linear Diophantine system

$$\begin{aligned}(C + 1)a_1 - (C + 2)a_4 &= C^2 + 6m - 1\\ Ca_2 - (C^2 + 6m - 1)a_4 &= (6m + 3)C - 12m - 5 \quad (3.13)\\ (C - 1)a_3 - ((6m + 3)C - 12m - 5)a_4 &= C - 2,\end{aligned}$$

where $C = 9m^2 + 9m$. Straightforward gcd arguments reveal that

$$\begin{aligned}\gcd(C + 1, C + 2) &= 1\\ \gcd(C, C^2 + 6m - 1) &= \begin{cases} 7 & \text{if } m \equiv 6 \pmod 7\\ 1 & \text{otherwise}\end{cases}\\ \gcd(C - 1, (6m + 3)C - 12m - 5) &= 1.\end{aligned}$$

Since $(6m + 3)C - 12m - 5 \equiv 0 \pmod 7$ when $m \equiv 6 \pmod 7$, it follows that the system (3.13) has infinitely many solutions. We give the following example to illustrate how to complete the process of constructing infinite collections of Eisenstein sextinomials $f(x)$ of degree 20, such that $f(x^{d^n})$ is monogenic for all integers $n \geq 0$ and any $d \in \Gamma_f$.

Example 3.12. Let $m = -2$. Then the solutions to (3.13) are given by

$$\begin{aligned}a_1 &= 1729 + 6120z,\\ a_2 &= 28103 + 100453z,\\ a_3 &= -13685 - 48906z,\\ a_4 &= 1627 + 5814z,\end{aligned}$$

where z is any integer. Suppose that $z = -1$. Then

$$f(x) = x^{20} + t(-4187x^4 + 35221x^3 - 72350x^2 - 4391x + 1),$$

and $\Delta(f) = -t^{19}T_1T_2T_3T_4$, where

$$\begin{aligned}T_1 &= 7109t + 1099511627776,\\ T_2 &= 44954t - 95367431640625,\\ T_3 &= 1915235048127301567486361 6t - 1,\end{aligned}$$

$$T_4 = st - 1,$$

with

$$s = 14490849274367198025181113222425709726313412627796432280806900461323 4102.$$

Let $G(t) := T_1T_2T_3T_4$. Since $G(1) \equiv 1 \pmod 4$, we see that $G(t)$ has no local obstruction at the prime $\ell = 2$. We may apply Corollary 2.12 to $G(t)$ to deduce that there exist infinitely many primes q such that $G(q)$ is squarefree, and using the same arguments as before, we conclude that $f(x)$ is monogenic when $t = q$ for each of these primes q.

4. Extending results beyond Γ_f

Up to this point, all results in this article have dealt with power-compositional Eisesntein polynomials $f(x^{d^n})$, where $d \in \Gamma_f$. What drives this situation is that the exponent d^n in this case does not contribute any new prime factors to the discriminant. Indeed, Lemma 3.1 is predicated upon this very fact. It then seems natural to ask if we can improve Lemma 3.1. That is, do there exist monogenic Eisenstein polynomials $f(x)$ such that Γ_f is a proper subset of Λ_f? In particular, can we find monogenic Eisenstein polynomials $f(x)$ such that the polynomials $f(x^{s^n})$ are monogenic for all integers $n \geq 0$ and all integers $s \in \mathcal{S}$, where $\Gamma_f \subset \mathcal{S} \subseteq \Lambda_f$? In general, this is tricky business since new prime factors p would be introduced in the discriminants $\Delta(f(x^{s^n}))$, where $f(x)$ is not p-Eisenstein. However, we are able to present some results that provide an affirmative answer to the questions posed here.

For an integer $a \geq 2$, we say a prime p is a *base-a Wieferich prime* if $a^{p-1} \equiv 1 \pmod{p^2}$. When $a = 2$, such primes are usually referred to as simply as Wieferich primes. Although it is conjectured that the number of base-a Wieferich primes is infinite, the only Wieferich primes up to 6.7×10^{15} are 1093 and 3511 [5]. It is easy to show that p is a base-a Wieferich prime if and only if $a^{p^k} \equiv a \pmod{p^2}$ for any $k \geq 1$.

Our first theorem gives simple examples of binomials $f(x)$ to show that Γ_f can be a proper subset of Λ_f. Moreover, the set Λ_f is completely determined.

Theorem 4.1. *Let $a, s \in \mathbb{Z}$ with $a \geq 2$ and $s \geq 2$. Suppose that a is squarefree, and let $f(x) = x - a$. Then $f(x^{s^n})$ is monogenic for all integers $n \geq 0$ if and only if s has no prime divisors that are base-a Wieferich primes. That is, $\Lambda_f = \mathcal{S}$, where*

$$\mathcal{S} = \{s \in \mathbb{Z} : s \geq 2 \text{ and no prime divisor of } s \text{ is a base-}a \text{ Wieferich prime}\}.$$

Remark 4.2. We do not provide a proof of Theorem 4.1 for two reasons: the first reason is that it can be deduced from results in [6], and the second reason is that the methods used in the proof are similar to, but less complicated than, the methods used to establish the main result of this section (see Theorem 4.5).

We can then use Lemma 3.1 and Theorem 4.1 to construct an infinite collection of binomials with the desired power-compositional properties in the following immediate corollary, whose proof is omitted.

Corollary 4.3. *Let $f(x) = x - a \in \mathbb{Z}[x]$. Then there exist infinitely many prime values of a such that $f(x^{a^n})$ is monogenic for all integers $n \geq 0$.*

The main result of this section (Theorem 4.5) is an attempt to extend the ideas of Theorem 4.1 to monogenic trinomials of the form $f(x) = x^2 + ax + a \in \mathbb{Z}[x]$, where $a \geq 2$ is squarefree. For the sake of completeness, we begin with a basic proposition which gives a simple condition to determine when such trinomials are monogenic.

Proposition 4.4. *Let $f(x) = x^2 + ax + a \in \mathbb{Z}[x]$, with $a \geq 2$ and squarefree. Then $f(x)$ is monogenic if and only if $a - 4$ is squarefree.*

Proof. Note that $f(x)$ is irreducible since $f(x)$ is Eisenstein. Let $K = \mathbb{Q}(\theta)$, where $f(\theta) = 0$. We use Theorem 2.5 with $T(x) := f(x)$, and p a prime divisor of $\Delta(f) = a(a-4)$.

Suppose first that $p \mid a$. Then $\overline{T}(x) = x^2$, and we may let $g(x) = h(x) = x$, so that

$$F(x) = \frac{g(x)h(x) - T(x)}{a} = -x - 1.$$

Hence, $\gcd(\overline{g}, \overline{F}) = 1$ and therefore, $[\mathbb{Z}_K : \mathbb{Z}[\theta]] \not\equiv 0 \pmod{p}$ by Theorem 2.5.

Now suppose that $a \equiv 4 \pmod{p}$. Then

$$\overline{T}(x) = x^2 + 4x + 4 = (x+2)^2,$$

and we may let $g(x) = h(x) = x + 2$. Thus,

$$F(x) = \frac{g(x)h(x) - T(x)}{p} = \left(\frac{4-a}{p}\right)(x+1).$$

It follows that

$$\overline{F}(-2) = -\overline{\left(\frac{4-a}{p}\right)} = 0 \text{ if and only if } a \equiv 4 \pmod{p^2},$$

which completes the proof. □

Theorem 4.5. *Let $f(x) = x^2 + ax + a$ with $a \in \{2, 3\}$, and let $s \in \mathbb{Z}$ with $s \geq 2$. Then $f(x^{s^n})$ is monogenic for all integers $n \geq 0$ if and only if s has no prime divisors that are base-a Wieferich primes. That is, $\Lambda_f = \mathcal{S}$, where*

$$\mathcal{S} = \{s \in \mathbb{Z} : s \geq 2 \text{ and no prime divisor of } s \text{ is a base-}a \text{ Wieferich prime}\}.$$

Proof. For $a \in \{2, 3\}$, define

$$\mathcal{F}_n(x) := f(x^{s^n}) = x^{2s^n} + ax^{s^n} + a.$$

Thus, $\mathcal{F}_n(x)$ is irreducible, and

$$\Delta(\mathcal{F}_n) = (-1)^{s^n(2s^n-1)} a^{2s^n-1}(4-a)^{s^n} s^{2ns^n}$$

by Corollary 3.3. Let $n \in \mathbb{Z}$ with $n \geq 1$, and let $K = \mathbb{Q}(\theta)$, where $\mathcal{F}_n(\theta) = 0$. To show that $\mathcal{F}_n(x)$ is monogenic, we use Theorem 2.5 with $T(x) := \mathcal{F}_n(x)$, and q equal to a prime divisor of $\Delta(\mathcal{F}_n)$. That is, we need to examine the prime $q = a$ and the prime divisors q of s.

When $q = a$, we have that $\overline{T}(x) = x^{2s^n}$. So, we can let $g(x) = x$ and $h(x) = x^{2s^n-1}$. Thus,

$$F(x) = \frac{g(x)h(x) - T(x)}{q} = -x^{s^n} - 1,$$

so that $\overline{F}(0) = -1$. Hence, $\gcd(\overline{g}, \overline{F}) = 1$ and, therefore, $[\mathbb{Z}_K : \mathbb{Z}[\theta]] \not\equiv 0 \pmod{q}$ by Theorem 2.5.

Next, let $q = p$ be a prime divisor of s, where $p \neq a$ and $p^m \mid\mid s$ with $m \geq 1$. Let

$$\overline{\tau}(x) = x^{2s^n/p^{mn}} + \overline{a}x^{s^n/p^{mn}} + \overline{a} = \prod_{i=1}^{k} \overline{\tau_i}(x)^{e_i},$$

where the $\overline{\tau_i}(x)$ are irreducible. Then $\overline{T}(x) = \prod_{i=1}^{k} \overline{\tau_i}(x)^{p^{mn}e_i}$. Thus, we can let

$$g(x) = \prod_{i=1}^{k} \tau_i(x) \quad \text{and} \quad h(x) = \prod_{i=1}^{k} \tau_i(x)^{p^{mn}e_i - 1},$$

where the $\tau_i(x)$ are monic lifts of the $\overline{\tau_i}(x)$. Note also that

$$\prod_{i=1}^{k} \tau_i(x)^{e_i} = \overline{\tau}(x) + pr(x),$$

for some $r(x) \in \mathbb{Z}[x]$. Suppose that $\overline{\tau}(\alpha) = 0$.

We treat the case $a = 2$ first. Note that $p \geq 3$. Then

$$\big(\beta - (-1 + \sqrt{-1})\big)\big(\beta - (-1 - \sqrt{-1})\big) = 0,$$

where $\beta = \alpha^{s^n/p^{mn}}$. With $\beta = -1 + \sqrt{-1}$ or $\beta = -1 - \sqrt{-1}$, straightforward induction arguments reveal that

$$\alpha^{s^n} = \beta^{p^{mn}} = 2^{(p^{mn}-1)/2}\big(\epsilon_1 + \epsilon_2\sqrt{-1}\big) \tag{4.1}$$

for some $\epsilon_i \in \{-1, 1\}$. Then, the remainder when $T(x) = \mathcal{F}_n(x)$ is divided by $x - \alpha$ is

$$\begin{aligned} T(\alpha) &= 2\Big(2^{(p^{mn}-1)/2}\epsilon_1 + 1\Big) + 2^{(p^{mn}+1)/2}\epsilon_2\Big(2^{(p^{mn}-1)/2}\epsilon_1 + 1\Big)\sqrt{-1} \\ &= 2\Big(2^{(p^{mn}-1)/2}\epsilon_1 + 1\Big)\Big(2^{(p^{mn}-1)/2}\epsilon_2\sqrt{-1} + 1\Big) \equiv 0 \pmod{p}. \end{aligned}$$

Since $pF(x)=(\overline{\tau}(x)+pr(x))^{p^{mn}}-T(x)$, it follows that

$$F(\alpha)=p^{p^{mn}-1}r(\alpha)^{p^{mn}}-\frac{T(\alpha)}{p}.$$

Hence,

$$\overline{F}(\alpha)=-\frac{T(\alpha)}{p}=-\frac{2\left(2^{(p^{mn}-1)/2}\epsilon_1+1\right)\left(2^{(p^{mn}-1)/2}\epsilon_2\sqrt{-1}+1\right)}{p}.$$

If $2^{(p^{mn}-1)/2}\epsilon_2\sqrt{-1}+1\equiv 0 \pmod p$, then $-2^{p^{mn}}\equiv 2 \pmod p$, which implies that $p=2$, a contradiction. Consequently,

$$\begin{aligned}
[\mathbb{Z}_K:\mathbb{Z}[\theta]]\equiv 0 \pmod p &\iff \gcd(\overline{F},\overline{g})\neq 1\\
&\iff \overline{F}(\alpha)=0\\
&\iff 2^{(p^{mn}-1)/2}\epsilon_1+1\equiv 0 \pmod{p^2}\\
&\iff 2^{(p^{mn}-1)}\equiv 1 \pmod{p^2}\\
&\iff p \text{ is a Wieferich prime,}
\end{aligned}$$

which completes the proof when $a=2$.

Suppose now that $a=3$. Since $p\neq 3$, we have two possibilities: $p=2$ and $p\geq 5$. We first handle the situation when $p=2$. Then $\beta^3=1$, where $\beta=\alpha^{s^n/2^{mn}}\neq 1$. Thus,

$$\alpha^{s^n}=\beta^{2^{mn}}=\begin{cases}\beta & \text{if } 2^{mn}\equiv 1 \pmod 3\\ \beta^2 & \text{if } 2^{mn}\equiv 2 \pmod 3.\end{cases}$$

Hence, the remainder when $T(x)=\mathcal{F}_n(x)$ is divided by $x-\alpha$ is

$$T(\alpha)=\beta^{2^{mn}+1}+3\beta^{2^{mn}}+3=\begin{cases}2\beta+2 & \text{if } 2^{mn}\equiv 1 \pmod 3,\\ 2\beta^2+2 & \text{if } 2^{mn}\equiv 2 \pmod 3.\end{cases}$$

Since $2F(x)=(\overline{\tau}(x)+pr(x))^{2^{mn}}-T(x)$, it follows that

$$F(\alpha)=2^{2^{mn}-1}r(\alpha)^{2^{mn}}-\frac{T(\alpha)}{2}.$$

Therefore,

$$\overline{F}(\alpha)=-\frac{T(\alpha)}{2}=-\begin{cases}\beta+1\not\equiv 0 \pmod 2 & \text{if } 2^{mn}\equiv 1 \pmod 3,\\ \beta^2+1\not\equiv 0 \pmod 2 & \text{if } 2^{mn}\equiv 2 \pmod 3.\end{cases}$$

Thus, $[\mathbb{Z}_K:\mathbb{Z}[\theta]]\not\equiv 0 \pmod 2$.

We now address the situation when $p\geq 5$. In this case, we have

$$\left(\beta-\left(\frac{-3+\sqrt{-3}}{2}\right)\right)\left(\beta-\left(\frac{-3-\sqrt{-3}}{2}\right)\right)=0,$$

where $\beta = \alpha^{s^n/p^{mn}}$. With $\beta = (-3+\sqrt{-3})/2$ or $\beta = (-3-\sqrt{-3})/2$, straightforward induction arguments reveal that

$$\alpha^{s^n} = \beta^{p^{mn}} = 3^{(p^{mn}-1)/2}\left(\frac{3\epsilon_1 + \epsilon_2\sqrt{-3}}{2}\right) \tag{4.2}$$

for some $\epsilon_i \in \{-1, 1\}$. Then, the remainder when $T(x) = \mathcal{F}_n(x)$ is divided by $x - \alpha$ is

$$\begin{aligned} T(\alpha) &= \frac{3^{p^{mn}} + 3^{(p^{mn}+3)/2}\epsilon_1 + 6}{2} + \left(\frac{3^{p^{mn}}\epsilon_1\epsilon_2 + 3^{(p^{mn}+1)/2}\epsilon_2}{2}\right)\sqrt{-3} \\ &= \left(\frac{3^{(p^{mn}-1)/2} + \epsilon_1}{2}\right)(A + B), \end{aligned}$$

where

$$A = 3^{(p^{mn}+1)/2} + 6\epsilon_1 \quad \text{and} \quad B = \epsilon_1\epsilon_2 3^{(p^{mn}+1)/2}\sqrt{-3}.$$

Then

$$A^2 \equiv 45 \pm 36 \pmod{p} \quad \text{and} \quad B^2 \equiv -27 \pmod{p}.$$

Hence,

$$A^2 - B^2 \pmod{p} \in \{108, 36\}. \tag{4.3}$$

If $A + B \equiv 0 \pmod{p}$, then $A^2 - B^2 \equiv 0 \pmod{p}$, and we deduce from (4.3) that $p \in \{2, 3\}$, contradicting the fact that $p \geq 5$. Consequently, $A + B \not\equiv 0 \pmod{p}$ so that

$$\overline{F}(\alpha) = -\frac{T(\alpha)}{p} = \left(\frac{3^{(p^{mn}-1)/2} + \epsilon_1}{2p}\right)(A + B) = 0$$

if and only if 3 is a base-a Wieferich prime, which completes the proof of the theorem. □

Remark 4.6. Although the precise values of ϵ_1 and ϵ_2 in (4.1) are not essential for the proof of Theorem 4.5, it can be shown for $a = 2$ and odd $N = p^{mn}$ that

$$\beta^N = 2^{(N-1)/2}(\epsilon_1 + \epsilon_2\sqrt{-1}),$$

where $\beta = -1 + \sqrt{-1}$ and

$$(\epsilon_1, \epsilon_2) = \begin{cases} (-1, 1) & \text{if } N \equiv 1 \pmod{8} \\ (1, 1) & \text{if } N \equiv 3 \pmod{8} \\ (1, -1) & \text{if } N \equiv 5 \pmod{8} \\ (-1, -1) & \text{if } N \equiv 7 \pmod{8}. \end{cases}$$

A similar result holds for (ϵ_1, ϵ_2) in (4.2) when $a = 3$ and N is in the respective congruence classes 1,5,7,11 modulo 12.

At first encounter, Theorem 4.5 seems a bit curious, and it also raises some questions. For one, is it true that Λ_f can never contain any integers $s \geq 2$ with prime factors that are base-a Wieferich primes, where $f(x) = x^2 + ax + a$ with squarefree $a \geq 2$? The example $f(x) = x^2 + 7x + 7$ provides a negative answer to this question, since $p = 5$ is a base-7 Wieferich prime but $f(x^{5^n})$ is monogenic for all integers $n \geq 0$.

A second related question that arises is whether Λ_f must contain all primes that are not base-a Wieferich primes. The example $f(x) = x^2 + 7x + 7$ also provides a negative answer to this question since 37 is not a base-7 Wieferich prime, but $f(x^{37})$ is not monogenic.

A third question then is why is it that Theorem 4.5 cannot be extended to $a = 7$? When $a \in \{2, 3\}$, the elements $\beta^{p^{mn}}$ are well-behaved and well-understood, where $f(\beta) = 0$. This stability and clarity seem to disappear when $a \geq 5$. Could it be a result of the loss of a one-to-one correspondence between the set of possibilities for (e_1, e_2) and the congruence classes of $(\mathbb{Z}/4a\mathbb{Z})^*$? That is, we have that $\phi(4a) = 4$ if and only if $a \in \{2, 3\}$. Or could it simply be explained by the fact that $\beta \notin \mathbb{R}$ when $a \in \{2, 3\}$ and $\beta \in \mathbb{R}$ when $a \geq 5$?

A final question is how large can Λ_f be for monogenic $f(x) = x^2 + ax + a$, where $a \geq 2$ is squarefree. In particular, could Λ_f equal the set of all positive integers larger than one? We do not know the answer to this question, but we suspect the answer is negative.

One avenue of future research is to establish results for monogenic trinomials $f(x) = x^2 + ax + a$, with squarefree $a \geq 2$, that are analogous to Theorem 4.1 and Corollary 4.3. In other words, can we explicitly determine Λ_f for these trinomials in terms of conditions on a? And then, can we use this information to construct infinite collections of such trinomials $f(x)$ for which $f(x^{s^n})$ is monogenic for all integers $n \geq 0$ and $s \in \mathcal{S} \subseteq \Lambda_f$, where $\Gamma_f \subset \mathcal{S}$?

Acknowledgements. The author thanks the referee for the careful reading of the manuscript.

References

[1] H. Cohen: *A Course in Computational Algebraic Number Theory, Graduate Texts in Mathematics, 138*, Berlin: Springer-Verlag, 1993.

[2] K. Conrad, URL: www.math.uconn.edu/~kconrad/blurbs/gradnumthy/totram.pdf.

[3] J. Cullinan, URL: https://studylib.net/doc/8187082/the-discriminant-of-a-composition-of-two.

[4] J. B. Dence, T. P. Dence: *Elements of the Theory of Numbers*, San Diego, CA: Harcourt/Academic Press, 1999.

[5] F. G. Dorais, D. Klyve: *A Wieferich prime search up to* 6.7×10^{15}, J. Integer Seq. 14.9, Article 11.9.2 (2011).

[6] T. A. Gassert: *A note on the monogeneity of power maps*, Albanian J. Math. 11.1 (2017), pp. 3–12.

[7] J. Harrington, L. Jones: *Monogenic cyclotomic compositions*, Kodai Math. J. 44.1 (2021), pp. 115–125, doi: https://doi.org/10.11650/tjm/200201.

[8] J. Harrington, L. Jones: *The irreducibility and monogenicity of power-compositional trinomials*, arXiv:2204.07784v1.

[9] H. A. Helfgott: *Square-free values of $f(p)$, f cubic*, Acta Math. 213.1 (2014), pp. 107–135, doi: https://doi.org/10.1007/s11511-014-0117-2.

[10] C. Hooley: *Applications of sieve methods to the theory of numbers. Cambridge Tracts in Mathematics, No. 70*, Cambridge-New York-Melbourne: Cambridge University Press, 1976.

[11] A. Jakhar, S. K. Khanduja, N. Sangwan: *Characterization of primes dividing the index of a trinomial*, Int. J. Number Theory 13.10 (2017), pp. 2505–2514, doi: https://doi.org/10.1142/s1793042117501391.

[12] L. Jones: *Generating infinite families of monogenic polynomials using a new discriminant formula*, Albanian J. Math. 14.1 (2020), pp. 37–45, doi: https://doi.org/10.51286/albjm/1608313765.

[13] L. Jones: *Infinite families of reciprocal monogenic polynomials and their Galois groups*, New York J. Math. 27 (2021), pp. 1465–1493.

[14] J. Neukirch: *Algebraic Number Theory*, Berlin: Springer-Verlag, 1999.

[15] H. Pasten: *The ABC conjecture, arithmetic progressions of primes and squarefree values of polynomials at prime arguments*, Int. J. Number Theory 13.7 (2017), pp. 1881–1894, doi: https://doi.org/10.1142/s1793042115500396.

[16] R. G. Swan: *Factorization of polynomials over finite fields*, Pacific J. Math. 12.7 (1962), pp. 1099–1106.

Annales Mathematicae et Informaticae
55 (2022) pp. 114–124
DOI: https://doi.org/10.33039/ami.2022.07.001
URL: https://ami.uni-eszterhazy.hu

Tauberian theorems via the generalized Nörlund mean for sequences in 2-normed spaces

Valdete Loku[a], **Naim L. Braha**[b*], **Mohammad Mursaleen**[c]

[a]University of Applied Sciences, Rr. Universiteti,
p.n. 70000 Ferizaj, Kosovo
valdeteloku@gmail.com

[b]University of Prishtina, Department of Mathematics and Computer Sciences,
Avenue Mother Theresa, No-5, Prishtine, 10000, Kosovo and
Research Institute ILIRIAS, Rr-Janina, No-2, Ferizaj, 70000, Kosovo
nbraha@yahoo.com

[c]Department of Medical Research, China Medical University Hospital,
China Medical University (Taiwan), Taichung, Taiwan and
Department of Mathematics, Aligarh Muslim University,
Aligarh 202002, India
mursaleem@gmail.com

Abstract. In this paper, we will show Tauberian conditions under which ordinary convergence of the sequence (x_n) in 2-normed space X, follows from $T_n^{p,q}$-summability. In fact we give a necessary and sufficient Tauberian condition for this method of summability. Also, we prove that Tauberian Theorems for these summability methods are valid with Schmidt-type slowly oscillating condition as well as with Hardy-type "big O" condition.

Keywords: $T_n^{p,q}$-summability method; Tauberian theorems; Nörlund summability; Schmidt-type oscillating slowly sequences; Hardy-type condition; 2-normed space

AMS Subject Classification: 40E05, 40G05, 40A05

*Corresponding author

Submitted: April 1, 2021
Accepted: July 21, 2022
Published online: July 29, 2022

1. Introduction

The Tauberian theory based on the following fact. If the sequence (x_n) converges, i.e.

$$\lim_{n\to\infty} x_n = l$$

exists then it follows that the limit in sense of a regular summability method (T_n) exists and

$$\lim_{n\to\infty} T_n(x_n) = l.$$

The converse of the above fact is not true (or "does not hold") in general. Conditions under which the converse follows are known as Tauberian conditions, and the result with such conditions is known Tauberian theorem. The Tauberian theorems are investigated for many summability methods under different conditions, see for example ([2, 3, 5, 7–11]). In 1976, the well-known Hardy–Littlewood Tauberian theorem was extended to the multidimensional case by Vladimirov [14]. After that paper, the work began on a systematic investigation of the Tauberian theory of generalized functions from the standpoint of both pure mathematics and its application in theoretical and mathematical physics. In [4], some multidimensional Tauberian theorems for generalized functions were established along with their application in mathematical physics. In recent year the Tauberian theorems were proved in 2-normed spaces for the Cesàro summability method (see [12]).

The convolution $(p * q)$ of two non-negative sequences (p_n) and (q_n) is defined by

$$R_n := (p * q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

In case $(p * q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund transform $(T_n^{p,q})$ of the sequence (x_n) is given as follows

$$T_n^{p,q} = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_k q_{n-k} x_k.$$

The sequence (x_n) is generalized Nörlund summable to L (see [1]), if

$$\lim_{n\to\infty} T_n^{p,q} = L. \tag{1.1}$$

Let us define

$$A(n,t) := \{q_{\lambda_n - k} - q_{n-k} : k = 0, 1, 2, \dots, n; \lambda > 1\}$$

and

$$B(n,t) := \{q_{k-\lambda_n} - q_{n-k} : k = 0, 1, 2, \dots, \lambda_n; 0 < \lambda < 1\},$$

where $\lambda_n := [\lambda n]$ denotes the integral part of λn.

Let us suppose that the sequences $p = (p_n)$ and $q = (q_n)$ satisfies the following conditions:

$$p_n \leq q_n, \quad n \in \mathbb{N},$$
$$q_n \geq 1, \quad n \in \mathbb{N},$$
$$\sup_n A(n, \lambda) < \infty,$$

and

$$\sup_n B(n, \lambda) < \infty.$$

If

$$\lim_{n \to \infty} x_n = L$$

implies (1.1), then the method (N, p, q) is regular. The necessary and sufficient condition for the (N, p, q) method to be regular is (see [6])

$$p_{n-k} q_k = o(R_n) \;\; (n \to \infty, k \in \mathbb{N}),$$

and

$$\sum_{k=0}^{n} |p_{n-k} q_k| = O(R_n) \;\; (n \to \infty).$$

Remark 1.1. In case when $p_n \equiv q_n \equiv 1$ for $n \in \mathbb{N}$, the (N, p, q) method coincides the Cesàro $(C, 1)$ summability. For $q_n = 1$ we get the Nörlund method $(\overline{N}, p)$. In case when $p_n = \binom{n+\beta}{\beta}$, $q_n = \binom{n+\alpha-1}{\alpha}$, we get the (C, α, β) ([1]) method. Finally, for $p_n = \lambda_n$ and $q_n = 1$, we get the generalized de la Vallée-Poussin method.

In this paper we will prove Tauberian theorems for the (N, p, q) summability method in 2-normed spaces.

Definition 1.2. A sequence (x_n) converges to L in a 2-norm X, i.e.

$$x_n \overset{\|\cdot,\cdot\|_X}{\longrightarrow} L,$$

if

$$\lim_{n \to \infty} \|x_n - L, y\| = 0,$$

for all $y \in X$.

A sequence (x_n) in a 2-normed space X is $T_n^{p,q}$ summable to $L \in X$ and write, in sign: $x_n \overset{\|\cdot,\cdot\|_X}{\longrightarrow} L(T_n^{p,q})$, if

$$\lim_{n \to \infty} \|T_n^{p,q} - L, y\| = 0,$$

for all $y \in X$.

Theorem 1.3. *In a 2-normed space X, $\lim_n x_n = L \in X$, implies $\lim_n T_n^{p,q} = L$ in X. The converse statement is not true in general.*

Proof. Let us suppose that $\lim_n x_n = L$ in a 2-normed space X. It is clear that, for every $\epsilon > 0$, there exists an n_0 such that for every $n > n_0$ and any $y \in X$ we have

$$\|x_n - L, y\| < \epsilon :$$

and for any $n < n_0$, $y \in X$ there exists a $M > 0$ such that

$$\|x_n - L, y\| \leq M.$$

Now we can estimate as follows:

$$\begin{aligned}
&\|T_n^{p,q} - L, y\| \\
&= \left\| \frac{1}{(p*q)_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y \right\| = \left\| \frac{1}{(p*q)_n} \sum_{k=0}^{n} p_k q_{n-k} (x_k - L), y \right\| \\
&\leq \left\| \frac{1}{(p*q)_n} \sum_{k=0}^{n_0} p_k q_{n-k} (x_k - L), y \right\| + \left\| \frac{1}{(p*q)_n} \sum_{k \in \{n_0+1, \cdots, n\}} p_k q_{n-k} (x_k - L), y \right\| \\
&\leq M \cdot \frac{A_{n_0}^{p,q}}{(p*q)_n} + \epsilon,
\end{aligned}$$

where $A_{n_0}^{p,q} = \sum_{k=0}^{n_0} p_k q_{n-k}$. Hence, we get desired result. □

Example 1.4. Consider $X = \mathbb{R}^3$ and

$$\|x, y\| = \max\{|x_1 y_2 - x_2 y_1|, |x_1 y_3 - x_3 y_1|, |x_2 y_3 - x_3 y_2|\},$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Let

$$x_n = \left(1 + (-1)^n, 2 + (-1)^n, 3 + \frac{3(-1)^n}{2}\right),$$

and $y = (y_1, y_2, y_3) \in X$.

If we put $p_n = n$ and $q_n = 1$, then we have

$$\begin{aligned}
T_n^{p,q}(1 + (-1)^n) &= 1 + \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \\
T_n^{p,q}(2 + (-1)^n) &= 2 + \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \\
T_n^{p,q}\left(3 + \frac{3(-1)^n}{2}\right) &= 3 + \frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)}.
\end{aligned}$$

Now we will prove that $T_n^{p,q} \to (1, 2, 3)$ in the 2-normed space X.

$$\begin{aligned}
&\lim_{n\to\infty} \|T_n^{p,q} - L, y\| \\
&= \lim_{n\to\infty} \left\| \left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)} \right), y \right\|
\end{aligned}$$

$$= \lim_{n\to\infty} \max \left\{ \left| y_2\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right) - y_1\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right)\right|, \right.$$
$$\left| y_3\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right) - y_1\left(\frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)}\right)\right|,$$
$$\left.\left| y_3\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right) - y_2\left(\frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)}\right)\right| \right\} = 0.$$

So (x_n) is $T_n^{p,q}$-summable to $(1,2,3)$ in 2-normed space X. Now we will prove that (x_n) does not converge to $(1,2,3)$ in 2-normed space X. Let $y = (1,1,1) \in \mathbb{R}^3$ then

$$\lim_{n\to\infty} \|x_n - L, y\| = \lim_{n\to\infty} \left\| \left((-1)^n, (-1)^n, \frac{3(-1)^n}{2}\right), (y_1, y_2, y_3)\right\|$$
$$= \lim_{n\to\infty} \max\left\{ |(-1)^n \cdot y_2 - (-1)^n \cdot y_1|, \left|(-1)^n \cdot y_3 - \frac{3(-1)^n}{2} \cdot y_1\right|, \right.$$
$$\left. \left|(-1)^n \cdot y_3 - \frac{3(-1)^n}{2} \cdot y_2\right| \right\} \neq 0,$$

sequence (x_n) is not convergent.

2. Tauberian theorems for $T_n^{p,q}$-summability method

Theorem 2.1. *Let (p_n) and (q_n) be two sequences of real numbers defined as above and*

$$\liminf_{n\to\infty} \frac{R_{\lambda_n}}{R_n} > 1, \quad \lambda > 1 \tag{2.1}$$

where $\lambda_n = [\lambda n]$. Suppose that $\lim_n T_n^{p,q} = L$, in 2-normed space X. Then (x_n) is convergent to the same number L in 2-normed space X if and only if

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_n} p_i q_{\lambda_n - i}(x_i - x_n), y \right\| = 0 \tag{2.2}$$

and

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{i=\lambda_n+1}^{n} p_i q_{n-i}(x_n - x_i), y \right\| = 0. \tag{2.3}$$

Definition 2.2. The sequence $(x_n) \in X$ is slowly oscillating (see [13]) in a 2-normed space if

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n\le k\le\lambda_n} \|x_k - x_n, y\| = 0$$

for all $y \in X$, or equivalently

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{\lambda_n\le k\le n} \|x_n - x_k, y\| = 0$$

for all $y \in X$.

Denoting $\Delta x_n = x_n - x_{n-1}$, we can rewrite the above conditions to the form

$$\inf_{\lambda>1} \lim_{n\to\infty} \sup_{n\leq k\leq \lambda_n} \max \left\| - \sum_{i=k+1}^{n} \Delta x_i, y \right\| = 0$$

and

$$\inf_{0<\lambda<1} \lim_{n\to\infty} \sup_{\lambda_n\leq k\leq n} \max \left\| \sum_{i=k+1}^{n} \Delta x_i, y \right\| = 0,$$

for all $y \in X$.

We will need the following lemmas.

Lemma 2.3 ([3]). *For the sequences of real numbers (p_n) and (q_n), condition (2.1) is equivalent to*

$$\lim_{n\to\infty} \inf \frac{R_n}{R_{\lambda_n}} > 1, \quad 0 < \lambda < 1.$$

Lemma 2.4. *Let (p_n) and (q_n) be the sequences defined as above and relation (2.1) is satisfied. Assume that $x = (x_n)$ is $T_n^{p,q}$-convergent to L, in the 2-normed space X. Then for every $\lambda > 0$,*

$$\lim_n \| T_{\lambda_n}^{p,q} - L, y \| = 0$$

for every $y \in X$.

Proof. Case 1: $\lambda > 1$. Then from the definition of $\lambda = (\lambda_n)$, we get

$$\lim_n (n - \lambda_n) = \lim_n (R_{\lambda_n} - R_n).$$

Now from given conditions, for every $\epsilon > 0$ we have:

$$\| T_{\lambda_n}^{p,q} - L, y \| \leq \| T_{\lambda_n}^{p,q} - T_n^{p,q}, y \| + \| T_n^{p,q} - L, y \| \leq \epsilon.$$

Case 2: $0 < \lambda < 1$. For $\lambda_n = [\lambda \cdot n]$, for any natural number n, we can conclude that $(T_{\lambda_n}^{p,q})$ does not appear more than $[1 + \lambda^{-1}]$ times in the sequence $(T_n^{p,q})$. In fact if there exist integers k, l such that

$$n \leq \lambda \cdot k < \lambda(k+1) < \cdots < \lambda(k+l-1) < n+1 \leq \lambda(k+l),$$

then

$$n + \lambda(l-1) \leq \lambda(k+l-1) < n+1 \Rightarrow l < 1 + \frac{1}{\lambda}$$

and

$$\| T_{\lambda_n}^{p,q} - L, y \| \leq \left(1 + \frac{1}{\lambda}\right) \| T_n^{p,q} - L, y \| \leq \epsilon.$$

From this, $\lim_n \| T_{\lambda_n} - L, y \| = 0$ follows. ☐

Lemma 2.5. *Let (p_n) and (q_n) be the sequences defined as above and relation (2.1) be satisfied. Let $x = (x_n)$ be $T_n^{p,q}$-convergent to L, in 2-normed space X. Then for every $\lambda > 0$,*

$$\lim_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y \right\| = 0 \quad \textit{for} \quad \lambda > 1 \tag{2.4}$$

and

$$\lim_n \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} x_k - L, y \right\| = 0 \quad \textit{for} \quad 0 < \lambda < 1, \tag{2.5}$$

for every $y \in X$.

Proof. Case 1: $\lambda > 1$. We get

$$\begin{aligned}
&\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y \right\| \\
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{\lambda_n - k} x_k - L, y \right\| \\
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, \right. \\
&\qquad \left. y - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{n-k} + q_{\lambda_n - k} - q_{n-k}) x_k - L, y \right\| \\
&\leq \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{\lambda_n - k} - q_{n-k}) x_k - L, y \right\|.
\end{aligned} \tag{2.6}$$

We know that

$$\limsup_n \frac{R_n}{R_{\lambda_n} - R_n} = \left(\liminf_n \frac{R_{\lambda_n}}{R_n} - 1 \right)^{-1} < \infty. \tag{2.7}$$

Now from (2.6) and (2.7), we get (2.4).

Case 2: $0 < \lambda < 1$. Then we have

$$\begin{aligned}
&\left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} x_k - L, y \right\| \\
&= \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y - \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{n-k} x_k - L, y \right\|
\end{aligned}$$

$$= \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, \right. \tag{2.8}$$

$$\left. y - \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{\lambda_n - k} + q_{n-k} - q_{\lambda_n - k}) x_k - L, y \right\|$$

$$\leq \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y - \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y \right\|$$

$$+ \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{n-k} - q_{\lambda_n - k}) x_k - L, y \right\|. \tag{2.9}$$

In this case, we know

$$\limsup_{n\to\infty} \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} = \left(\liminf_{n\to\infty} \frac{R_n}{R_{\lambda_n}} - 1 \right)^{-1} < \infty. \tag{2.10}$$

From (2.9) and (2.10), we get (2.5). □

Proof of Theorem 2.1. Let $\lim_n x_n = L$, and $\lim_n T_n^{p,q} = L$, in 2-normed space X. Applying Lemma 2.5, we get relation (2.2) for $\lambda > 1$, and (2.3) for $0 < \lambda < 1$.

Sufficiency. Let $\lim_n T_n^{p,q} = L$ in 2-normed space X and conditions (2.1), (2.2) and (2.3) hold. We will prove that $\lim_n x_n = L$ in X. Or equivalently, $\lim_n (T_n^{p,q} - x_n) = 0$ in 2-normed space X.

For $\lambda > 1$, we have

$$x_n - T_n^{p,q} = \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} (T_{\lambda_n}^{p,q} - T_n^{p,q}) - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} (x_k - x_n).$$

From relation (2.1) and Lemma 2.4, we obtain

$$\left\| \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} (T_{\lambda_n}^{p,q} - T_n^{p,q}), y \right\| < \epsilon,$$

for every $y \in X$. From (2.2), for every $\epsilon > 0$ we get

$$\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} (x_k - x_n), y \right\| < \epsilon,$$

for every $y \in X$. From last relations we have proved that $\lim_n (T_n^{p,q} - x_n) = 0$, in 2-normed space X.

Now for the case $0 < \lambda < 1$, we get

$$x_n - T_{\lambda_n}^{p,q} = \frac{R_n}{R_n - R_{\lambda_n}} (T_n^{p,q} - T_{\lambda_n}^{p,q}) + \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} (x_n - x_k).$$

From relation (2.1) and Lemma 2.4, we have

$$\left\| \frac{R_n}{R_n - R_{\lambda_n}} (T_n^{p,q} - T_{\lambda_n}^{p,q}), y \right\| < \epsilon,$$

for every $y \in X$. From relation (2.3), for every $\epsilon > 0$ we get

$$\left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k}(x_n - x_k), y \right\| < \epsilon,$$

for every $y \in X$. Hence, we have proved that $\lim_n (T_{\lambda_n}^{p,q} - x_n) = 0$, in 2-normed space X. Now proof of the Theorem follows from Lemma 2.4. □

In what follows we will show that under the conditions that (x_n) is a slowly oscillating sequence (see [13]), the $T_n^{p,q}$-summability implies the convergence in the ordinary sense.

Theorem 2.6. *Let X be a 2-normed space and $(x_n) \in X$ be $T_n^{p,q}$-limitable to L. If (x_n) is slowly oscillating in 2-normed space X, then (x_n) converges to L in X.*

Proof. In case $\lambda > 1$ let us suppose that $T_n^{p,q}$ converges to L in X. To prove that $(x_n) \to L$ in X, it is enough to prove that

$$\lim_n \|T_n^{p,q} - x_n, y\| = 0,$$

for every $y \in X$. Let us start with

$$\begin{aligned} \|T_n^{p,q} - x_n, y\| &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - x_n, y \right\| = \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k}(x_k - x_n), y \right\| \\ &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \sum_{j=k+1}^{n} \Delta x_j, y \right\| \le \max_{0\le k\le n} \left\| \sum_{j=k+1}^{n} \Delta x_j, y \right\|. \end{aligned}$$

Taking limit superior in both sides of the above relation and then infimum, we get

$$\inf_{\lambda>1} \lim_{n\to\infty} \sup \|T_n^{p,q} - x_n, y\| = 0.$$

Hence, it is proved that (x_n) converges to L in X.

The case $0 < \lambda < 1$ is similar to the previous one and for this reason we omit it. □

The following result shows that if (x_n) satisfies Hardy ([6]) conditions, and is $T_n^{p,q}$-summable, then it converges in the ordinary sense.

Theorem 2.7. *Let $(x_n) \in X$ be $T_n^{p,q}$-summable to L in 2-normed space X. If (x_n) satisfies relation*

$$n\Delta x_n = 0(1),$$

then (x_n) converges to L in X.

Proof. It is enough to prove that

$$\lim_n \|T_n^{p,q} - x_n, y\| = 0$$

for every $y \in X$. First, suppose that $\lambda > 1$. From the condition

$$n\Delta x_n = 0(1),$$

it follows that for every $\epsilon > 0$, there exists an n_0 such that for every $n > n_0$ we have

$$|n\Delta x_n| < \epsilon.$$

A routine calculation gives

$$\begin{aligned}\|T_n^{p,q} - x_n, y\| &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - x_n, y \right\| = \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} (x_k - x_n), y \right\| \\ &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \sum_{j=k+1}^{n} \Delta x_j, y \right\| \leq \max_{0 \leq k \leq n} \left\| \sum_{j=k+1}^{n} \Delta x_j, y \right\|.\end{aligned}$$

From above relations, we get

$$\|T_n^{p,q} - x_n, y\| \leq \max_{0 \leq k \leq n} \left\| \sum_{j=k+1}^{n} \Delta x_j, y \right\| \leq \epsilon.$$

Hence, it is proved that (x_n) converges to L in X.

The second case, when $0 < \lambda < 1$, can be proved similarly. □

Competing interests. The authors declare that they have no competing interests.

Authors contributions. All the authors contributed equally and significantly in writing this paper.

References

[1] D. Borwein: *On products of sequences*, J. London Math. Soc. 33 (1958), pp. 352–357, doi: https://doi.org/10.1112/jlms/s1-33.3.352.

[2] N. L. Braha: *Tauberian conditions under which λ-statistical convergence follows from statistical summability* (V, λ), Miskolc Math. Notes 16.2 (2015), pp. 695–703, doi: https://doi.org/10.18514/MMN.2015.1254.

[3] I. Canak, N. L. Braha, U. Totur: *A Tauberian theorem for the generalized Nörlund summability method*, Georgian Math. J. 27.1 (2020), pp. 31–36, doi: https://doi.org/10.1515/gmj-2017-0062.

[4] Y. N. Drozhzhinov, B. I. Zavyalov: *Applications of Tauberian theorems in some problems in mathematical physics*, Theoretical and Mathematical Physics 157.3 (2008), pp. 1678–1693, doi: https://doi.org/10.1007/s11232-008-0140-6.

[5] O. H. H. Edely, M. Mursaleen: *Tauberian theorems for statistically double sequences, Information Sciences*, Information Sciences 176 (2006), pp. 875–886, doi: https://doi.org/10.1016/j.ins.2005.01.006.

[6] G. H. Hardy: *Divergent Series*, in: Divergent Series, New York: Chelsea, 1991.

[7] L. Leindler: *Über die de la Vallée-Pousinsche Summierbarkeit allgemeiner Orthogonalreihen*, Acta Math. Acad. Sci. Hungar. 16 (1965), pp. 375–387, doi: https://doi.org/10.1007/BF01904844.

[8] V. Loku, N. Braha, M. Et, A. Tato: *Tauberian theorems for the generalized de la Vallée-Poussin mean-convergent sequences of fuzzy numbers*, Bull. Math. Anal. Appl. 9.2 (2017), pp. 45–56.

[9] V. Loku, N. L. Braha: *Tauberian theorems by weighted summability method*, Armen. J. Math. 9.1 (2017), pp. 35–42.

[10] F. Móricz: *Necessary and sufficient Tauberian conditions for the logarithmic summability of functions and sequences*, Studia Mathematica 219.2 (2013), pp. 109–121, doi: https://doi.org/10.4064/sm219-2-2.

[11] F. Móricz, Z. Németh: *Statistical extension of classical Tauberian theorems in the case of logarithmic summability*, Analysis Mathematica 40.3 (2014), pp. 231–242, doi: https://doi.org/10.1007/s10476-014-0305-5.

[12] R. Savas, S. A. Sezer: *Tauberian theorems for sequences in 2-normed spaces*, Results Math. 72.4 (2017), pp. 1919–1931, doi: https://doi.org/10.1007/s00025-017-0747-8.

[13] R. Schmidt: *Über divergente Folgen und lineare Mittelbildungen*, Math. Z. 22.1 (1925), pp. 89–152, doi: https://doi.org/10.1007/BF01479600.

[14] V. S. Vladimirov: *Multidimensional generalization of a Tauberian theorem of Hardy and Littlewood*, Izv. Math. 10 (1976), pp. 1031–1048, doi: https://doi.org/10.1070/IM1976v010n05ABEH001824.

Annales Mathematicae et Informaticae
55 (2022) pp. 125–136
DOI: https://doi.org/10.33039/ami.2022.01.001
URL: https://ami.uni-eszterhazy.hu

Enumeration of Fuss-skew paths

Toufik Mansour[a], **José Luis Ramírez**[b]

[a]Department of Mathematics,
University of Haifa, 3498838 Haifa, Israel
tmansour@univ.haifa.ac.il

[b]Departamento de Matemáticas,
Universidad Nacional de Colombia, Bogotá, Colombia
jlramirezr@unal.edu.co

Abstract. In this paper, we introduce the concept of a Fuss-skew path and then we study the distribution of the semi-perimeter, area, peaks, and corners statistics. We use generating functions to obtain our main results.

Keywords: Skew Dyck path, Fuss-Catalan numbers, generating function

AMS Subject Classification: 05A15, 05A19

1. Introduction

A *skew Dyck path* is a lattice path in the first quadrant that starts at the origin, ends on the x-axis, and consists of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and left-steps $L = (-1, -1)$, such that up and left steps do not overlap. The definition of skew Dyck path was introduced by Deutsch, Munarini, and Rinaldi [4]. Some additional results about skew Dyck path can be found in [2, 5, 8, 14].

Let s_n denote the number of skew Dyck path of semilength n, where the semilength of a path is defined as the number its up-steps. The sequence s_n is given by the combinatorial sum $s_n = \sum_{k=1}^{n} \binom{n-1}{k-1} c_k$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. The sequence s_n appears in OEIS as A002212 [15], and its first few values are

$$1, \quad 1, \quad 3, \quad 10, \quad 36, \quad 137, \quad 543, \quad 2219, \quad 9285, \quad 39587.$$

One way to generalize the classical Dyck paths is to regard the length of an up-step U as a parameter. Given a positive number ℓ, an *ℓ-Dyck path* is a lattice path in the first quadrant from $(0, 0)$ to $((\ell + 1)n, 0)$ where $n \geq 0$ using up-steps

Submitted: December 22, 2021
Accepted: January 14, 2022
Published online: January 29, 2022

$U_\ell = (\ell, \ell)$ and down-steps $U = (1, -1)$. For $\ell = 1$, we recover the classical Dyck path. The total number of ℓ-Dyck path with length $(\ell+1)n$ is given by $c_\ell(n) = \frac{1}{tn+1}\binom{(t+1)n}{n}$ (cf. [1]). We will refer to ℓ-Dyck paths here as the "Fuss" case because the sequence $c_\ell(n)$ was first investigated by N. I. Fuss (see, for example, [7, 16] for several combinatorial interpretations for both the Catalan and Fuss-Catalan numbers).

Our focus in this paper is to introduce a Fuss analogue of the skew Dyck path. Given a positive integer ℓ, an *ℓ-Fuss-skew path* is a path in the first quadrant that starts at the origin, ends on the x-axis, and consists of up-steps $U_\ell = (\ell, \ell)$, down-steps $D = (1, -1)$, and left steps $L = (-1, -1)$, such that up and left steps do not overlap. Given an ℓ-Fuss-skew path P, we define the semilength of P, denote by $|P|$, as the number of up-steps of P. For example, Figure 1 shows a 3-Fuss-skew path of semilength 6. It is clear that the 1-Fuss-skew paths coincide with the skew Dyck paths. Let $\mathbb{S}_{n,\ell}$ denote the set of all ℓ-Fuss-skew path of semilength n, and $\mathbb{S}_\ell = \bigcup_{n\geq 0} \mathbb{S}_{n,\ell}$. For example, Figure 4 shows all the paths in $\mathbb{S}_{2,2}$.

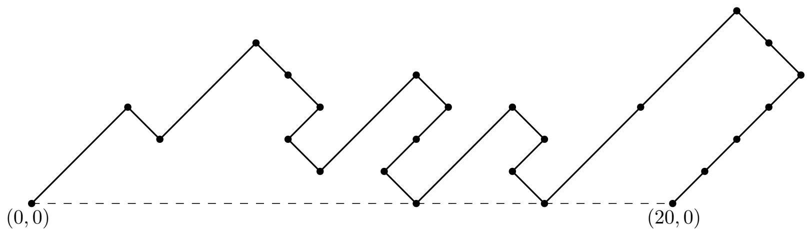


Figure 1. 3-Fuss-skew path of semilength 6.

2. Counting special steps

For a given path $P \in \mathbb{S}_\ell$, we use $u(P)$, $d(P)$, and $t(P)$ to denote the number of up-steps, down-steps, and left-steps of P, respectively. In this section, we study the distribution of these parameters over $\mathbb{S}_\ell$. Using these parameters, we define the generating function

$$F_\ell(x, p, q) := \sum_{P \in \mathbb{S}_\ell} x^{u(P)} p^{d(P)} q^{t(P)}.$$

For simplicity, we use F_ℓ to denote the generating function $F_\ell(x, p, q)$.

Theorem 2.1. *The generating function $F_\ell(x, p, q)$ satisfies the functional equation*

$$F_\ell = 1 + x(pF_\ell + q)^{\ell-1}(pF_\ell^2 + q(F_\ell - 1)). \tag{2.1}$$

Proof. Let $\mathcal{A}_i$ denote the ℓ-Fuss-skew paths whose last y-coordinate is i and let A_i denote the generating function defined by

$$A_i = \sum_{P \in \mathcal{A}_i} x^{u(P)} p^{d(P)} q^{t(P)}.$$

A non-empty ℓ-Fuss-skew path can be uniquely decomposed as either $U_\ell TDP$ or $U_\ell TL$, where $U_\ell T$ is a lattice path in $\mathcal{A}_1$ and P is an ℓ-Fuss-skew path (see Figure 2 for a graphical representation of this decomposition). From this decomposition, we obtain the functional equation (cf. [6])

$$F_\ell = 1 + x(pA_1F_\ell + qA_1). \tag{2.2}$$

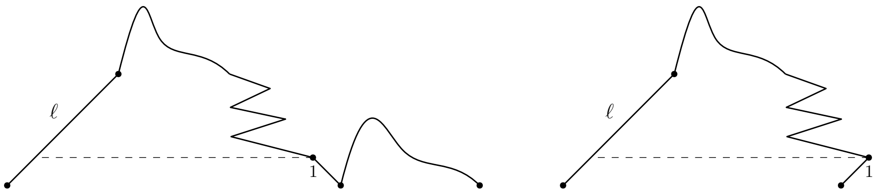


Figure 2. Decomposition of a ℓ-Fuss-skew path.

The paths of $\mathcal{A}_i$ can be decomposed as TDP or TL, where $T \in \mathcal{A}_{i+1}$ for $i = 1, \dots, \ell-2$ and $P \in \mathbb{S}_\ell$ (see Figure 3 for a graphical representation of this decomposition). Moreover, the paths of $\mathcal{A}_{\ell-1}$ are decomposed as P_1DP_2 or $P'L$, where $P_1, P_2, P' \in \mathbb{S}_\ell$ and P' is non-empty.

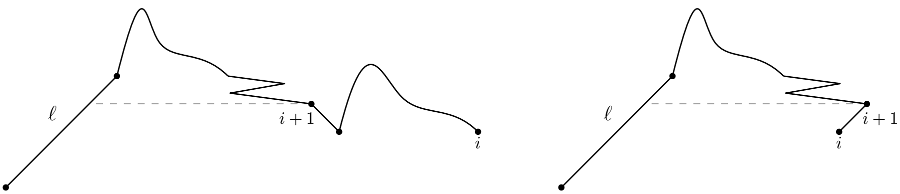


Figure 3. Decomposition of the paths in $\mathcal{A}_i$.

From the above decompositions, we obtain the functional equations

$$A_i = pA_{i+1}F_\ell + qA_{i+1}, \quad \text{for} \quad i = 1, \dots, \ell-2, \quad \text{and} \quad A_{\ell-1} = pF_\ell^2 + q(F_\ell - 1).$$

Note that in these functional equations we do not consider the first up-step because it was considered in (2.2). Therefore, we have

$$\begin{aligned} F_\ell &= 1 + x(pF_\ell + q)A_1 = 1 + x(pF_\ell + q)^2 A_2 \\ &= \cdots = 1 + x(pF_\ell + q)^{\ell-1}(pF_\ell^2 + q(F_\ell - 1)). \end{aligned}$$ □

Let $s_\ell(n, p, q)$ denote the joint distribution over $\mathbb{S}_{n,\ell}$ for the number of down and left steps, that is,

$$s_\ell(n, p, q) = \sum_{P \in \mathbb{S}_{n,\ell}} p^{d(P)} q^{t(P)}.$$

It is clear that $F_\ell = \sum_{n \geq 0} s_\ell(n, p, q)x^n$. From the Lagrange inversion theorem (see for instance [13]), we give a combinatorial expression for the sequence $s_\ell(n, p, q)$.

Theorem 2.2. *For $n \geq 1$, the sequence $s_\ell(n,p,q)$ is given by*

$$\frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}p^{2n-1-2j}(2p+q)^k(p+q)^{n(\ell-2)+2j-k+1}.$$

In particular, the total number of ℓ-Fuss-skew paths of semilength n is

$$s_\ell(n) := s_\ell(n,1,1) = \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}3^k 2^{n(\ell-2)+2j-k+1}.$$

Proof. The functional equation given in Theorem 2.1 can be written as

$$Q_\ell = x(p(Q_\ell+1)+q)^{\ell-1}(p(Q_\ell+1)^2+qQ_\ell),$$

where $Q_\ell = F_\ell - 1$. From the Lagrange inversion theorem, we deduce

$$\begin{aligned}
[x^n]H_\ell &= \frac{1}{n}[z^{n-1}](p(z+1)+q)^{(\ell-1)n}(p(z+1)^2+qz)^n\\
&= \frac{1}{n}[z^{n-1}]\sum_{s\geq 0}\binom{(\ell-1)n}{s}(pz)^s(p+q)^{(\ell-1)n-s}(pz^2+(2p+1)z+p)^n\\
&= \frac{1}{n}[z^{n-1}]\sum_{s\geq 0}\binom{(\ell-1)n}{s}(pz)^s(p+q)^{(\ell-1)n-s}\\
&\quad\times\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}p^{n-j}((2p+q)z)^k(pz^2)^{j-k}\\
&= \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}p^{2n-1-2j}(2p+q)^k(p+q)^{n(\ell-2)+2j-k+1}. \quad\square
\end{aligned}$$

For example, Figure 4 shows all 2-Fuss-skew paths of semilength 2 counted by the term $s_\ell(2,p,q) = 3p^4 + 6p^3q + 4p^2q^2 + pq^3$.

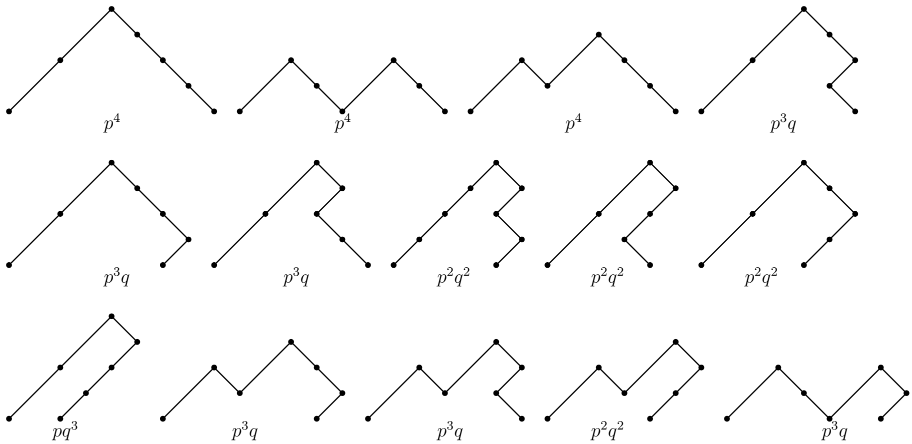


Figure 4. 2-Fuss-skew paths counted by $s_\ell(2, p, q)$.

From Theorem 2.2, we obtain that the total number of down-steps over the ℓ-Fuss-skew paths of semilength n is given by

$$\left.\frac{\partial s_\ell(n,p,1)}{\partial p}\right|_{p=1}$$
$$= \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}2^{(\ell-2)n+2j-k}3^{k-1}(3n(\ell+2)+k-6j-3).$$

Moreover, the total number of left-steps over the ℓ-Fuss-skew paths of semilength n is

$$\left.\frac{\partial s_\ell(n,1,q)}{\partial q}\right|_{q=1}$$
$$= \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}2^{(\ell-2)n+2j-k}3^{k-1}(3n(\ell-2)-k+6j+3).$$

Equation (2.1) can be explicitly solved for $\ell = 1$. In this case, we obtain the generating function

$$F_1(x,p,q) = \frac{1-qx-\sqrt{(1-qx)(1-(4p+q)x)}}{2px}.$$

Moreover, the generating functions for the total number of down-steps (A026388) and left steps (A026376) over the skew-Dyck paths are respectively

$$\frac{1-4x+3x^2-\sqrt{1-6x+5x^2}(1-x)}{2x\sqrt{1-6x+5x^2}}$$

and

$$\frac{1-3x-\sqrt{1-6x+5x^2}}{2\sqrt{1-6x+5x^2}}.$$

Notice that we recover some of the results of [5].

Finally, Table 1 shows the first few values of the total number of ℓ-Fuss-skew paths of semilength n.

Table 1. Values of $s_\ell(n,1,1)$ for $1 \le \ell \le 5$, $n = 1, \dots, 7$.

$\ell \backslash n$	1	2	3	4	5	6	7
$\ell = 1$	1	3	10	36	137	543	2219
$\ell = 2$	2	14	118	1114	11306	120534	1331374
$\ell = 3$	4	64	1296	29888	745856	19614464	535394560
$\ell = 4$	8	288	13568	734720	43202560	2681634816	172936069120

2.1. The width of a path

For a given path $P \in \mathbb{S}_\ell$, we define the *width* of P, denoted by $\nu(P)$, as the x-coordinate of the last point of P. For example, the width of the path given in Figure 1 is 20. We define the generating function

$$G_\ell(x,y) := G_\ell = \sum_{P \in \mathbb{S}_\ell} x^{u(P)} y^{\nu(P)}.$$

Note that each U_ℓ and D step of a path increases the width by ℓ units and 1 unit, respectively, while the left-step L decreases the width by 1 unit. Therefore, we have the functional equation

$$\begin{aligned} G_\ell &= 1 + xy^\ell (yG_\ell + y^{-1})^{\ell-1}(yG_\ell^2 + y^{-1}(G_\ell - 1)) \\ &= 1 + x(y^2G_\ell + 1)^{\ell-1}(y^2G_\ell^2 + (G_\ell - 1)). \end{aligned} \tag{2.3}$$

Let $g_\ell(n,y)$ denote the distribution over $\mathbb{S}_{n,\ell}$ for the width parameter, i.e.,

$$g_\ell(n,y) = \sum_{P \in \mathbb{S}_{n,\ell}} y^{\nu(P)}.$$

From the functional equation (2.3) and the Lagrange inversion theorem, we obtain the following theorem.

Theorem 2.3. *For $n \ge 1$, the sequence $g_\ell(n,y)$ is given by*

$$\frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1} y^{4(n-j)-2}(y^2+1)^{n(\ell-2)+2j-k+1}(2y^2+1)^k.$$

For example, $g_2(2, y) = y^2 + 4y^4 + 6y^6 + 3y^8$. This polynomial can be found from the paths in Figure 4. For $\ell = 1$, we obtain the explicit generating function with respect to the width of a skew Dyck path.

$$G_1(x, y) = \frac{1 - x - \sqrt{(1-x)(1-x-4xy^2)}}{2xy^2}.$$

3. Number of peaks

For a given path $P \in \mathbb{S}_\ell$, we define the *peaks* of P, denoted by $\rho(P)$, as the number of subpaths of the form $U_\ell D$ (for counting peaks in a Dyck path, for example, see [9, 11]). For example, the number of peaks of the path given in Figure 1 is 5. We define the generating function

$$P_\ell(x, y) := P_\ell = \sum_{P \in \mathbb{S}_\ell} x^{u(P)} y^{\rho(P)}.$$

Theorem 3.1. *The generating function $P_\ell(x, y)$ satisfies the functional equation*

$$P_\ell = 1 + x(P_\ell + 1)^{\ell-1}((P_\ell - 1 + y)P_\ell + (P_\ell - 1)).$$

Proof. Let C_i denote the generating function defined by $C_i = \sum_{P \in \mathcal{A}_i} x^{u(P)} y^{\rho(P)}$. From the decomposition given for the ℓ-Fuss-skew paths, we have the equation $P_\ell = 1 + x(C_1 P_\ell + C_1)$. Moreover,

$$\begin{aligned} C_i &= C_{i+1}P_\ell + C_{i+1}, \quad \text{for } i = 1, \dots, \ell - 2, \text{ and} \\ C_{\ell-1} &= (P_\ell - 1 + y)P_\ell + (P_\ell - 1). \end{aligned}$$

From these relations, we obtain the desired result. □

Let $p_\ell(n, y)$ denote the distribution over $\mathbb{S}_n$ for the peaks statistic, i.e.,

$$p_\ell(n, y) = \sum_{P \in \mathbb{S}_n} y^{\rho(P)}.$$

From the Lagrange inversion theorem, we deduce the following result.

Theorem 3.2. *For $n \geq 1$, we have*

$$p_\ell(n, y) = \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell-1)}{n-2j+k-1} 2^{n(\ell-2)+2j-k+1} y^{n-j} (y+2)^k.$$

In particular, the total number of peaks in all ℓ-Fuss-skew paths of semilength n is

$$\begin{aligned} &\left.\frac{\partial p_\ell(n, y)}{\partial y}\right|_{y=1} \\ &= \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell-1)}{n-2j+k-1} 2^{n(\ell-2)+2j-k+1} 3^{k-1} (3(n-j)+k). \end{aligned}$$

For example, $p_2(2, y) = 8y + 6y^2$. This polynomial can be found from the paths in Figure 4. For $\ell = 1$ we obtain the generating function

$$P_1(x, y) = \frac{1 - xy - \sqrt{(1 - xy)^2 - 4(1 - x)x}}{2x}.$$

Moreover, the generating function for the total number of peaks is

$$\frac{1 - x - \sqrt{1 - 6x + 5x^2}}{2\sqrt{1 - 6x + 5x^2}}.$$

Table 2 shows the first few values of the number of peaks in ℓ-Fuss-skew paths of semilength n.

Table 2. Total number of peaks in $\mathbb{S}_\ell$.

$\ell \setminus n$	1	2	3	4	5	6	7
$\ell = 1$	1	4	17	75	339	1558	7247
$\ell = 2$	2	20	226	2696	33138	415164	5270850
$\ell = 3$	4	96	2672	78848	2400896	74568704	2347934464
$\ell = 4$	8	448	29440	2054144	147986432	10878189568	810813030400

4. Number of corners

For a given path $P \in \mathbb{S}_\ell$, we define a *corner* of P as a right angle caused by two consecutive steps in the graph of P. For example, the path given in Figure 5 has 4 corners, depicted in red. This statistic has been studied in other combinatorial structures as integer partitions [3], compositions [10], and bargraphs [12].

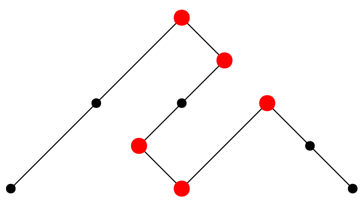


Figure 5. Corners of a path.

Let $\tau(P)$ denote the number of corners of P. We define the bivariate generating function

$$W_\ell(x, y) := W_\ell = \sum_{P \in \mathbb{S}_\ell} x^{u(P)} y^{\tau(P)}.$$

In this section, we analyze the cases $\ell = 1$ and $\ell = 2$. We leave as an open question the case $\ell \geq 3$.

Theorem 4.1. *The generating function* $W_1(x, y)$ *satisfies the functional equation*

$$xy(1+y)W_1^3 - (2 - x(2 - y^2))W_1^2 + 3(1-x)W_1 + x - 1 = 0.$$

Proof. Let $\mathcal{D}$ and $\mathcal{L}$ denote the skew Dyck paths whose last step is a down-step or a left-step, respectively. Let D and L denote the generating functions defined by

$$D = \sum_{P \in \mathcal{D}} x^{u(P)} y^{\tau(P)} \quad \text{and} \quad L = \sum_{P \in \mathcal{L}} x^{u(P)} y^{\tau(P)}.$$

A non-empty skew Dyck path can be uniquely decomposed as either UT_1L or UT_2DT_3, where T_1, T_2, and T_3 are lattice paths in $\mathbb{S}_1$ with T_1 non-empty. In the first case, T_1 has two options: the last step is a down-step or a left step, see Figure 6. Then, this case contributes to the generating function the term $x(yD + L)$.

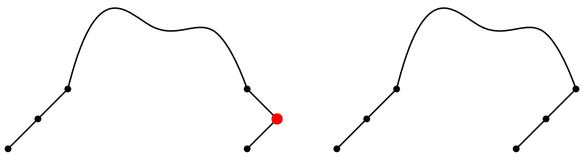


Figure 6. Decomposition of a skew Dyck path.

On the other hand, T_2 can be an empty path or a path in $\mathcal{D}$ or $\mathcal{L}$. If T_3 is empty, then this case contributes to the generating function the term $x(y + D + Ly)$. On the other hand, if the path T_3 is non-empty, then this case contributes to the generating function the term $x(y + D + yL)y(W_1 - 1)$, see Figure 7. Summarizing these cases, we obtain the functional equation

$$W_1 = 1 + x(yD + L) + x(y + D + yL)(1 + y(W_1 - 1)).$$

From a similar argument, we obtain the equations

$$D = x(y + D + yL)(1 + yD) \quad \text{and} \quad L = x(yD + L) + x(y + D + yL)(yL).$$

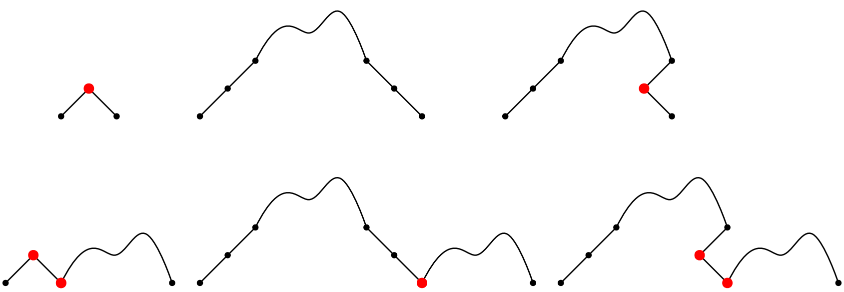


Figure 7. Decomposition of a skew Dyck path.

Using the Gröbner basis on the polynomial equations for W_1, D, and L, we obtain the desired result. □

We can use a symbolic software computation to obtain the first few terms of the formal power series of $W_1(x,y)$ as follows:

$$W_1(x,y) = 1 + xy + x^2(y + y^2 + y^3) + x^3(y + 2y^2 + 4y^3 + 2y^4 + y^5) + x^4(y + 3y^2 + 9y^3 + 9y^4 + 10y^5 + 3y^6 + y^7) + \cdots .$$

From the equation given in Theorem 4.1, we obtain

$$3xS^3(x) + 6xS^2(x)K(x) - 2xS^2(x) - 2(2-x)S(x)K(x) + 3(1-x)K(x) = 0,$$

where $K(x)$ is the generating function for the total number of corners in skew Dyck paths and $S(x) = (1 - x - \sqrt{1-6x+5x^2})/(2x)$ is the generating function for the number of the skew Dyck paths. Solving the above equation, we obtain the generating function

$$K(x) = \frac{2(1-x)(3+x)x}{(1-x)(3-2x)(1-5x) + (3-11x+4x^2)\sqrt{1-6x+5x^2}} = x + 6x^2 + 30x^3 + 145x^4 + 695x^5 + 3327x^6 + 15945x^7 + \cdots .$$

Theorem 4.2. *The generating function $W_2(x,y)$ satisfies the functional equation*

$$\begin{aligned}
&x^2y^4(1+y)^3W_2^6 - xy^2(1+y)^2(1 - x(1+6y+y^2-3y^3))W_2^5\\
&+ xy(-4-7y+3y^3 + x(1+y)^2(4+9y-11y^2-6y^3+3y^4))W_2^4\\
&+ (4 - 2x(1+y)^2(4-7y+y^2) - x^2(1+y)^2(-4+2y+21y^2-8y^3-5y^4+y^5))W_2^3\\
&+ (-12 - x^2(1+y)^2(8+4y-18y^2+4y^3+y^4) - 2x(-10-9y+6y^2+6y^3+y^4))W_2^2\\
&+ (12 + x^2(1+y)^2(5+4y-7y^2+y^3) + x(-17-16y+2y^2+4y^3+3y^4))W_2\\
&+ (-4 + x^2(1+y)^2(-1-y+y^2) + x(5+4y-y^4)) = 0.
\end{aligned}$$

Proof. Let $\mathcal{D}_2$ and $\mathcal{L}_2$ denote the 2-Fuss-skew paths whose last step is a down-step or a left-step, respectively. Let D_2 and L_2 denote the generating functions defined by

$$D_2 = \sum_{P\in\mathcal{D}_2} x^{u(P)}y^{\tau(P)} \quad \text{and} \quad L_2 = \sum_{P\in\mathcal{L}_2} x^{u(P)}y^{\tau(P)}.$$

From a similar argument as in the proof of Theorem 4.1, we obtain the system of polynomial equations

$$\begin{aligned}
W_2 &= 1 + x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)(1 + y(W_2-1)) + (D_2 + yL_2)\\
&\quad + (D_2 + yL_2)y(1 + y(W_2-1)) + (y + yD_2 + L_2)(y + yD_2 + y^2L_2)),\\
D_2 &= x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)yD_2 + (D_2 + yL_2) + (D_2 + yL_2)y(yD_2)\\
&\quad + (y + yD_2 + L_2)(y + yD_2 + y^2L_2)),\\
L_2 &= x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)(1 + yL_2) + (D_2 + yL_2)y(1 + yL_2)).
\end{aligned}$$

By using the Gröbner basis, we obtain the desired result. □

Expanding with *Mathematica* the functional equation for W_2, we find

$$W_2(x,y) = 1 + (y+y^2)x + (y+3y^2+5y^3+4y^4+y^5)x^2 \\ + (y+5y^2+16y^3+27y^4+33y^5+25y^6+9y^7+2y^8)x^3 + \cdots .$$

Moreover, the first few terms of the total number of corners in $\mathbb{S}_2$ are

$$3x+43x^2+561x^3+7209x^4+92703x^5+1197151x^6+15532917x^7+202428373x^8+\cdots .$$

From Figure 4 one can verify that there are 43 corners over all paths in $\mathbb{S}_{2,2}$.

5. Other generalization

Let $\mathbb{H}_\ell$ denote the skew Dyck paths where left steps are below the line $y = \ell$. In particular, $\mathbb{H}_0$ are the Dyck path and $\mathbb{H}_\infty$ are the skew Dyck path. We define the generating function

$$H_\ell(x,p,q) := \sum_{P\in\mathbb{H}_\ell} x^{u(P)}p^{d(P)}q^{t(P)}.$$

For simplicity, we use H_ℓ to denote the generating function $H_\ell(x,p,q)$.

Theorem 5.1. *For $\ell \geq 1$, we have*

$$H_\ell = 1 + qx(H_{\ell-1} - 1) + pxH_{\ell-1}H_\ell, \tag{5.1}$$

with the initial value $H_0 = \frac{1-\sqrt{1-4px}}{2px}$.

Proof. A non-empty skew Dyck path in $\mathbb{H}_\ell$ can be decomposed as UT_1L or UT_2DT_3, where $T_1, T_2 \in \mathbb{H}_{\ell-1}$ with T_1 a non-empty path, and $T_3 \in \mathbb{H}_\ell$. From this decomposition follows the functional equation. □

Recall that the mth Chebyshev polynomial of the second kind satisfies the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with $U_0(t) = 1$ and $U_1(t) = 2t$. Thus by induction on ℓ and Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let $t = \frac{1+qx}{2\sqrt{x(p+q-pqx)}}$ and $r = \sqrt{x(p+q-pqx)}$. The generating function H_ℓ is given by*

$$\frac{(qxU_{n-1}(t) - rU_{n-2}(t))C(px) + (1-qx)U_{n-1}(t)}{U_{n-1}(t) - rU_{n-2}(t) - pxU_{n-1}(t)C(px)},$$

where U_m is the mth Chebyshev polynomial of the second kind and $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

The generating functions for the total number of skew Dyck path in $\mathbb{H}_\ell$ for $\ell = 1, 2, 3$ are

$$H_1(x,1,1) = \frac{3-2x-\sqrt{1-4x}}{1+\sqrt{1-4x}},$$
$$H_2(x,1,1) = \frac{1+2x-2x^2-(1-2x)\sqrt{1-4x}}{1-x-2(1-x)x+(1+x)\sqrt{1-4x}},$$
$$H_3(x,1,1) = \frac{1-3x+7x^2-4x^3+(1+x-3x^2)\sqrt{1-4x}}{1-4x+2x^3+(1+2x^2)\sqrt{1-4x}}.$$

Acknowledgements. We thanks Mark Shattuck for his comments on the previous version of the paper.

References

[1] J.-C. Aval: *Multivariate Fuss-Catalan numbers*, Discrete Math. 308 (2008), pp. 4660–4669, doi: https://doi.org/10.1016/j.disc.2007.08.100.

[2] J. L. Baril, J. L. Ramírez, L. M. Simbaqueba: *Counting prefixes of skew Dyck paths*, J. Integer Seq. Article 21.8.2 (2021).

[3] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour: *Counting corners in partitions*, Ramanujan J. 39 (2016), pp. 201–224, doi: https://doi.org/10.1007/s11139-014-9666-4.

[4] E. Deutsch, E. Munarini, S. Rinaldi: *Skew Dyck paths*, J. Statist. Plann. Inference 140 (2010), pp. 2191–2203, doi: https://doi.org/10.1016/j.jspi.2010.01.015.

[5] E. Deutsch, E. Munarini, S. Rinaldi: *Skew Dyck paths, area, and superdiagonal bargraphs*, J. Statist. Plann. Inference 140 (2010), pp. 1550–1526, doi: https://doi.org/10.1016/j.jspi.2009.12.013.

[6] P. Flajolet, R. Sedgewick: *Analytic Combinatorics*, Cambridge University Press, 2009.

[7] S. Heubach, N. Y. Li, T. Mansour: *Staircase tilings and k-Catalan structures*, Discrete Math. 308 (2008), pp. 5954–5964, doi: https://doi.org/10.1016/j.disc.2007.11.012.

[8] Q. L. Lu: *Skew Motzkin paths*, Acta Math. Sin. (Engl. Ser.) 33 (2017), pp. 657–667, doi: https://doi.org/10.1007/s10114-016-5292-y.

[9] T. Mansour: *Counting peaks at height k in a Dyck path*, J. Integer Seq. Article 02.1.1 (2002).

[10] T. Mansour, A. S. Shabani, M. Shattuck: *Counting corners in compositions and set partitions presented as bargraphs*, J. Difference Equ. Appl. 24 (2018), pp. 992–1015, doi: https://doi.org/10.1080/10236198.2018.1444760.

[11] T. Mansour, M. Shattuck: *Counting Dyck paths according to the maximum distance between peaks and valleys*, J. Integer Seq. Article 12.1.1 (2012).

[12] T. Mansour, G. Yildirim: *Enumerations of bargraphs with respect to corner statistics*, Appl. Anal. Discrete Math. 14 (2020), pp. 221–238, doi: https://dx.doi.org/10.2298/AADM181101009M.

[13] D. Merlini, R. Sprugnoli, M. C. Verri: *Lagrange inversion: when and how*, Acta Appl Math. 94 (2006), pp. 233–249, doi: https://doi.org/10.1007/s10440-006-9077-7.

[14] H. Prodinger: *Partial skew Dyck paths - A kernel method approach*, arXiv:2108.09785v1 (2021), doi: https://arxiv.org/abs/2108.09785v2.

[15] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, doi: https://oeis.org/.

[16] R. Stanley: *Catalan Numbers*, Cambridge University Press, 2015.

Annales Mathematicae et Informaticae
55 (2022) pp. 137–157
DOI: https://doi.org/10.33039/ami.2022.09.002
URL: https://ami.uni-eszterhazy.hu

Two illustrating examples for comparison of uniform and proximal spaces using relators

Gergely Pataki

Department of Analysis, Institute of Mathematics,
Budapest University of Technology and Economics, Budapest, Hungary
pataki@math.bme.hu

Abstract. We introduce (generalized) proximities in the same way as (generalized) uniformities in paper of Weil. We prove the equivalence of our new definitions with classical ones.

Using these analog definitions, we compare the properties of (generalized) proximities and (generalized) uniformities. The main parts of this paper are examples of an $(X, \mathcal{R})$ relator space such that $\mathcal{R}^{\#}$ is uniformly (and proximally) transitive, but neither $\mathcal{R}$ nor $\mathcal{R}^{\Phi}$ is proximally (or uniformly) transitive.

For this, we summarize the essential properties of relators, using their theory from earlier works of Á. Száz.

Keywords: (generalized) uniformities, (generalized) proximities, relators

AMS Subject Classification: 54E05, 54E15, 54A05, 54G15, 54G20

1. Introduction

At the beginning of the 20th century some mathematicians tried to define abstract topological structures. The most relevant results are Poincaré 1895, Fréchet 1906, Hausdorff 1914, and Kuratowski 1922.

Uniform spaces in terms of relations were introduced by Weil in 1937 [13].

Proximities were first investigated by Riesz in 1909 [9], Effremovič, and Smirnov in 1952 [2] and [10].

After the works of Davis, Pervin, and Nakano [1], [8], and [4] in 1987, Száz [11] introduced the notion of relator and relator space in the following way.

Submitted: July 12, 2022
Accepted: September 13, 2022
Published online: September 24, 2022

Definition 1.1. A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a relator on X, and the ordered pair $(X, \mathcal{R})$ is called a relator space.

In the last decades, a few authors investigated the interpretation of well-known topological properties in terms of relators. In 2021, Pataki introduced the general definition for (generalized) uniformities and (generalized) proximities in [5], moreover quasi-uniformities in [6].

For more details, see, for instance [5–7, 12], but for the readers' convenience, we summarize the necessary notions and notations.

Remark 1.2. With the usual notations, $\mathcal{R}$ is a relator on X means that

$$X \neq \emptyset, \quad \emptyset \neq \mathcal{R} \subset \operatorname{Exp}(X^2),$$

where $\operatorname{Exp}(X)$ is the power set of X, and $X^2 = X \times X$.

If R is a relation on X, $x \in X$, and $A \subset X$, then the sets

$$R(x) = \{y \in X : (x, y) \in R\}, \quad \text{and} \quad R[A] = \bigcup_{x \in A} R(x)$$

are called the images of x and A under R, respectively.

2. Preliminary concepts

Definition 2.1. If R and S are relations on X, then the composition of R and S can be defined, such that $(R \circ S)(x) = R[S(x)]$ for all $x \in X$.

Moreover, let $R^{-1} = \{(y, x) : (x, y) \in R\}$, $R^0 = \Delta_X = \{(x, x) : x \in X\}$ and $R^n = R \circ R^{n-1}$, for all $n = 1, 2, \ldots$. Finally, we say that R is

- reflexive if $R^0 \subset R$,
- symmetric if $R^{-1} \subset R$,
- transitive if $R^2 \subset R$.

Lemma 2.2. *If R is a relation on X, and $A, B \subset X$, then*

$$R[A] \subset B \iff R^{-1}[X \setminus B] \subset X \setminus A.$$

Definition 2.3. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^* = \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\},$$

and

$$\mathcal{R}^\# = \{S \subset X^2 : \forall A \subset X : \exists R \in \mathcal{R} : R[A] \subset S[A]\},$$

are called the uniform and the proximal refinements of $\mathcal{R}$, respectively. For more details, see [7].

Moreover, for all $n = -1, 0, 1, 2, \ldots$, we define

$$\mathcal{R}^n = \{R^n : R \in \mathcal{R}\}.$$

Remark 2.4. $*$ and $\#$ are really refinements as we defined in [7]. That is, they are self-increasing in the sense that

$$\mathcal{R} \subset \mathcal{S}^* \iff \mathcal{R}^* \subset \mathcal{S}^* \quad \text{and} \quad \mathcal{R} \subset \mathcal{S}^\# \iff \mathcal{R}^\# \subset \mathcal{S}^\#,$$

or equivalently, they are expansive, increasing, and idempotent, in the sense that

$$\mathcal{R} \subset \mathcal{R}^*,\ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^* \subset \mathcal{S}^*,\ \mathcal{R}^{**} = \mathcal{R}^*$$

and

$$\mathcal{R} \subset \mathcal{R}^\#,\ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^\# \subset \mathcal{S}^\#,\ \mathcal{R}^{\#\#} = \mathcal{R}^\#,$$

for all $\mathcal{R}$ and $\mathcal{S}$ relators on X.

Moreover, $\#$ is $*$-dominating, $*$-invariant, $*$-absorbing, and $*$-compatible, that is

$$\mathcal{R}^* \subset \mathcal{R}^\#, \quad \mathcal{R}^\# = \mathcal{R}^{\#*}, \quad \mathcal{R}^\# = \mathcal{R}^{*\#}, \quad \mathcal{R}^{\#*} = \mathcal{R}^{*\#}.$$

For all $n = -1, 0, 1, 2, \ldots$ the mapping $\mathcal{R} \mapsto \mathcal{R}^n$ of relators on X is increasing.

Finally, $*$ and $\#$ are inversion-compatible, that is, for all $\mathcal{R}$ relators on X

$$\mathcal{R}^{*-1} = \mathcal{R}^{-1*} \quad \text{and} \quad \mathcal{R}^{\#-1} = \mathcal{R}^{-1\#}.$$

And we have that for all relators on X

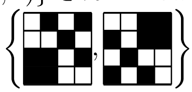
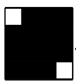
$$\mathcal{R}^{2*} = \mathcal{R}^{*2*}.$$

The following example shows that the analog assertion is not true for $\#$.

Example 2.5. Let $X = \{1, 2, 3, 4\}$, and

$$\mathcal{R} = \{\Delta_X \cup \{(1,2),(4,2),(2,1),(2,4)\}, \Delta_X \cup \{(1,3),(4,3),(3,1),(3,4)\}\}$$

is an elementwise reflexive and symmetric relator on X. Now, $\mathcal{R}^{\#2} \not\subset \mathcal{R}^{2\#}$, since $R = X^2 \setminus \{(1,4),(4,1)\} \in \mathcal{R}^{\#2}$ however $R \notin \mathcal{R}^{2\#}$.

Note, that $\mathcal{R} =$ , and $R =$ .

Definition 2.6. Let $\mathcal{R}$ be a relator on X, and $\square \in \{*, \#\}$ is a refinement for relators on X. We define the followings.

- $\mathcal{R}$ is $\square$-reflexive, if $\mathcal{R} \subset \mathcal{R}^{0\square}$;
- $\mathcal{R}$ is $\square$-symmetric, if $\mathcal{R} \subset \mathcal{R}^{-1\square}$;
- $\mathcal{R}$ is $\square$-transitive, if $\mathcal{R} \subset \mathcal{R}^{2\square}$;
- $\mathcal{R}$ is $\square$-fine, if $\mathcal{R} = \mathcal{R}^\square$.

For instance, we say that $\mathcal{R}$ is uniformly symmetric or proximally transitive instead of $*$-symmetric or $\#$-transitive.

Following Weil, we say that the relator $\mathcal{R}$ on X is a generalized uniformity/generalized proximity on X, and the ordered pair $(X,\mathcal{R})$ is a generalized uniform space/generalized proximal space if it is

- uniformly/proximally reflexive;
- uniformly/proximally symmetric;
- uniformly/proximally transitive;
- uniformly/proximally fine.

Following the notations of [3], in [6] we introduced quasi-uniformities, and now, we define generalized quasi-uniformities/generalized quasi-proximities.

We say that the relator $\mathcal{R}$ on X is a generalized quasi-uniformity/generalized quasi-proximity on X, and the ordered pair $(X,\mathcal{R})$ is a generalized quasi-uniform space/generalized quasi-proximal space if it is

- uniformly/proximally reflexive;
- uniformly/proximally transitive;
- uniformly/proximally fine.

Definition 2.7. Let $\mathcal{A}$ be a family of sets, or equivalently $\mathcal{A} \subset \operatorname{Exp}(X)$ for some set X. We call

$$\Phi(\mathcal{A}) = \left\{ \bigcap \mathcal{B} : \emptyset \neq \mathcal{B} \subset \mathcal{A}, \text{ and } \mathcal{B} \text{ is finite} \right\}$$

the filtered family of sets generated by $\mathcal{A}$.

Moreover, we say that $\mathcal{A}$ is filtered if $\Phi(\mathcal{A}) = \mathcal{A}$.

Remark 2.8. Since Φ is a refinement for relators on X, we write $\mathcal{R}^{\Phi}$ instead of $\Phi(\mathcal{R})$, if $\mathcal{R}$ is a relator on X. Note that $\mathcal{R}$ is filtered iff $\mathcal{R}^{\Phi} \subset \mathcal{R}$.

Moreover, note that Φ is an inversion-compatible refinement for relators on X. That is if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{-1\Phi} = \mathcal{R}^{\Phi-1}$.

Finally, if $\mathcal{R}$ is finite, then $\mathcal{R}^{\Phi *} = \{\bigcap \mathcal{R}\}^{*}$.

Lemma 2.9. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{*\Phi} = \mathcal{R}^{\Phi *}$, $\mathcal{R}^{2\Phi} \subset \mathcal{R}^{\Phi 2 *}$, and $\mathcal{R}^{\#2\#} \subset \mathcal{R}^{\Phi 2 \#}$.*

Definition 2.10. If $\square$ is a refinement for relators on X, then we say that the $\mathcal{R}$ relator on X is $\square$-filtered if there exists an $\mathcal{S}$ relator on X such that $\mathcal{S}^{\Phi} \subset \mathcal{S}^{\square} = \mathcal{R}^{\square}$.

We use the uniformly filtered and proximally filtered notions instead of $*$-filtered and $\#$-filtered.

Definition 2.11. If the generalized uniformity/generalized proximity (generalized quasi-uniformity/generalized quasi-proximity) $\mathcal{R}$ on X is also

- uniformly/proximally filtered,

then we say that $\mathcal{R}$ is a uniformity/proximity (quasi-uniformity/quasi-proximity) on X, and $(X,\mathcal{R})$ is a uniform space/proximal space (quasi-uniform space/quasi-proximal space).

3. Main example

Example 3.1. For all $i \in \mathbb{N}$ let $X_i = \{3i-2, 3i-1, 3i\}$ and $\pi_{i,0}$, $\pi_{i,1}$, ..., $\pi_{i,5}$ are the all bijections of $\{1,2,3\}$ to X_i such that $\pi_{i,0}$ is the only increasing one of them.

Moreover, let

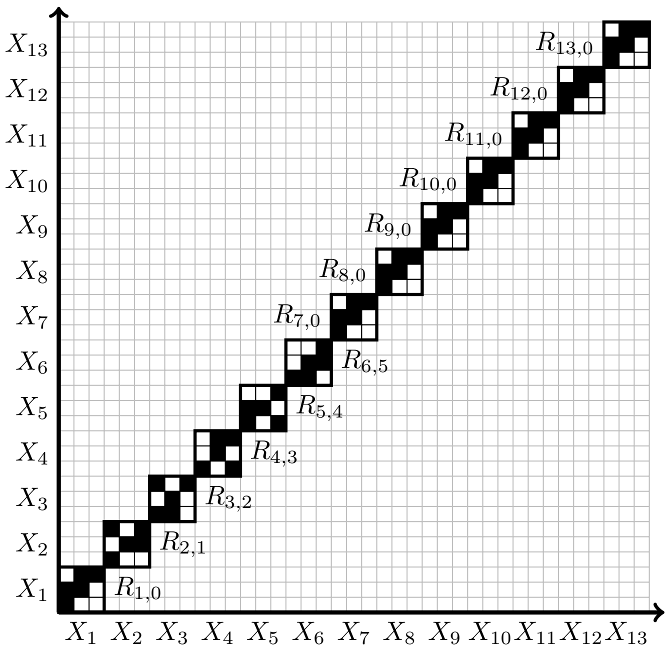
$$R_{i,k} = \{(\pi_{i,k}(1), \pi_{i,k}(2)), (\pi_{i,k}(2), \pi_{i,k}(3))\} \cup \Delta_{X_i}$$

for all $i \in \mathbb{N}$ and $k \in \{0, \dots, 5\}$.

Furthermore, for all $n \in \mathbb{N}$ let $S_n = \bigcup_{i\in\mathbb{N}} R_{i,\nu_i}$, where ν_i is the ith digits of $(n-1)$ in a positional base 6 numeral system, that is $n-1 = \sum_{i\in\mathbb{N}} \nu_i \cdot 6^{i-1}$.

The following figure shows the main part of the graph of S_{44791}, where $44790 = \sum_{i=0}^{5} i \cdot 6^i$ and

$\pi_{2,1}(1) = 4$, $\pi_{2,1}(2) = 6$, $\pi_{3,2}(1) = 8$, $\pi_{3,2}(2) = 7$, $\pi_{4,3}(1) = 11$,
$\pi_{4,3}(2) = 12$, $\pi_{5,4}(1) = 15$, $\pi_{5,4}(2) = 13$, $\pi_{6,5}(1) = 18$, $\pi_{6,5}(2) = 17$.



Finally, let $X = \mathbb{N} = \bigcup_{i\in\mathbb{N}} X_i$ and $\mathcal{R} = \{S_n : n \in \mathbb{N}\}$ is an elementwise reflexive relator on X. Then $\mathcal{R}^{\#} = \{\Delta_X\}^*$ is a uniformly transitive relator on X, but neither $\mathcal{R}$ nor $\mathcal{R}^{\Phi}$ is proximally transitive. Namely,

$$S_1 = \Delta_X \cup \bigcup_{i\in X\setminus 3X} \{(i, i+1)\} \in \mathcal{R} \subset \mathcal{R}^{\Phi},$$

but we show that if $A = \{3i-2 : i \in X\}$, then $Q^2[A] \not\subset S_1[A] = A \cup (A+1)$ for all $Q \in \mathcal{R}^{\Phi}$, that is $S_1 \notin \mathcal{R}^{\Phi 2\#}$ and hence $S_1 \notin \mathcal{R}^{2\#}$.

To prove this let $\emptyset \neq \mathcal{S} \subset \mathcal{R}$ finite, such that $Q = \bigcap \mathcal{S}$. Then there exists a greatest $m \in X$ such that $S_m \in \mathcal{S}$. If i is large enough then $\nu_i = 0$ for all elements of $\mathcal{S}$, therefore $R_{i,0} \subset \bigcap \mathcal{S} = Q$, and then $3i \in R_{i,0}^2(3i-2) \subset Q^2(3i-2) \subset Q^2[A]$.

We can change the above example to be symmetric.

Example 3.2. Namely, for all $i \in \mathbb{N}$ let $X_i = \{5i-4, 5i-3, 5i-2, 5i-1, 5i\}$ and $\pi_{i,0}$, $\pi_{i,1}$, ..., $\pi_{i,119}$ are the all bijections of $\{1,2,3,4,5\}$ to X_i such that $\pi_{i,0}$ is the only increasing one of them. Moreover, let

$$R_{i,k} = \{(\pi_{i,k}(1), \pi_{i,k}(2)), (\pi_{i,k}(2), \pi_{i,k}(1)), (\pi_{i,k}(2), \pi_{i,k}(3)), (\pi_{i,k}(3), \pi_{i,k}(2))\} \cup \Delta_{X_i}$$

for all $i \in \mathbb{N}$ and $k \in \{0, \dots, 119\}$. Furthermore for all $n \in \mathbb{N}$ let $S_n = \bigcup_{i \in \mathbb{N}} R_{i,\nu_i}$, where ν_i is the ith digits of $(n-1)$ in a positional base 120 numeral system, that is $n - 1 = \sum_{i \in \mathbb{N}} \nu_i \cdot 120^{i-1}$. Now $\mathcal{R} = \{S_n : n \in \mathbb{N}\}$ is an elementwise reflexive, elementwise symmetric relator on $X = \mathbb{N} = \bigcup_{i \in \mathbb{N}} X_i$ with the same properties.

4. A finite example

Lemma 4.1. *Let $\mathcal{R}$ be a finite relator on X. Now, $\mathcal{R}^{\Phi}$ is proximally transitive iff the relation $\bigcap \mathcal{R}$ is transitive.*

Proof. If $\mathcal{R}$ is finite, then Remark 2.4 and 2.8 yield that

$$\mathcal{R}^{\Phi 2*} = \mathcal{R}^{\Phi * 2*} = \left\{\bigcap \mathcal{R}\right\}^{*2*} = \left\{\bigcap \mathcal{R}\right\}^{2*}$$

therefore

$$\mathcal{R}^{\Phi 2\#} = \mathcal{R}^{\Phi 2*\#} = \left\{\bigcap \mathcal{R}\right\}^{2*\#} = \left\{\bigcap \mathcal{R}\right\}^{2\#} = \left\{\bigcap \mathcal{R}\right\}^{2*} = \mathcal{R}^{\Phi 2*}.$$

By using the self-increasingness of $*$ we have that

$$\left(\bigcap \mathcal{R}\right)^2 \subset \bigcap \mathcal{R} \iff \left\{\bigcap \mathcal{R}\right\} \subset \left\{\bigcap \mathcal{R}\right\}^{2*} \iff \left\{\bigcap \mathcal{R}\right\}^{*} \subset \left\{\bigcap \mathcal{R}\right\}^{2*} \iff$$
$$\iff \mathcal{R}^{\Phi *} \subset \mathcal{R}^{\Phi 2*} \iff \mathcal{R}^{\Phi} \subset \mathcal{R}^{\Phi 2*} \iff \mathcal{R}^{\Phi} \subset \mathcal{R}^{\Phi 2\#}. \qquad \square$$

Theorem 4.2. *If $\mathcal{R}$ is a finite relator on X, then uniform transitivity of $\mathcal{R}^{\#}$ implies proximal transitivity of $\mathcal{R}^{\Phi}$.*

Proof. Assume to the contrary that $\mathcal{R}^{\Phi}$ is not proximally transitive that is, by using the above Lemma, there exists an $(x, z) \in (\bigcap \mathcal{R})^2 \setminus \bigcap \mathcal{R}$. It means that there exist $x, y, z \in X$ such that

(a) $y \in (\bigcap \mathcal{R})(x)$;

(b) $z \in (\bigcap \mathcal{R})(y)$;

(c) $z \notin (\bigcap \mathcal{R})(x)$.

For all $S \in \mathcal{R}^{\#}$ there exists an $R \in \mathcal{R}$ such that $R(x) \subset S(x)$. For such an $R \in \mathcal{R}$, by using (a), it is easy to see, that $y \in (\bigcap \mathcal{R})(x) \subset R(x)$.

In a similar way (b) implies that $z \in S(y)$ for all $S \in \mathcal{R}^{\#}$ and (c) implies that $z \notin R(x)$ for some $R \in \mathcal{R}$.

In summary we have that

$$\exists R \in \mathcal{R} : \forall S \in \mathcal{R}^{\#} : S^2 \not\subset R$$

that is $\mathcal{R} \not\subset \mathcal{R}^{\#2*}$. This is a contradiction, because uniform transitivity of $\mathcal{R}^{\#}$ means that $\mathcal{R} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\#2*}$. □

The following examples show that uniform transitivity of $\mathcal{R}^{\#}$ does not imply proximal transitivity of $\mathcal{R}$ even if the space X is finite.

The appendix shows the graphs of all 24 R_π elements of the $\mathcal{R}$ relator in the following example.

Moreover, we can see the graphs of R_π^2 and S_n elements of $\mathcal{S}$, which is the smallest relator on X such that $\mathcal{S}^* = \mathcal{R}^{\#}$.

Finally, in appendix we can also see $S_k^2 \in \mathcal{S}^2$ examples for all $S_n \in \mathcal{S}$ such that $S_k^2 \subset S_n$.

Example 4.3. Let $X = \{1, 2, 3, 4\}$ and

$$R_\pi = \{(\pi(1), \pi(1)), (\pi(1), \pi(2)), (\pi(2), \pi(2)), (\pi(2), \pi(3))\}$$

for all π permutation of X. Moreover, let $\mathcal{R} = \{R_\pi : \pi \text{ is a permutation of } X\}$.

At first, we investigate $\mathcal{R}^{\#}$. For this end, let $Q \in \mathcal{R}^{\#}$ be arbitrary. By definition, there exists a π permutation of X, such that $\pi[\{1, 2, 3\}] = R_\pi[X] \subset Q[X]$, that is $Q[X]$ has at least 3 elements. Moreover, if $A \subset X$ has 3 elements, then $\pi(1) \in A$ or $\pi(2) \in A$ when π is a permutation of X such that $R_\pi[A] \subset Q[A]$, that is $Q[A]$ has at least 2 elements.

In summary, a Q relation on X is an element of $\mathcal{R}^{\#}$ iff it satisfies at least one of the following conditions.

- There exists α and β permutations of X such that $\beta(i) \in Q(\alpha(i))$ for all $i \in \{1, 2, 3\}$.
- There exists α and β permutations of X such that $\{\beta(i)\} \subsetneq Q(\alpha(i))$ for all $i \in \{1, 2\}$ and $\beta(3) \in Q(\alpha(1))$.

Now, we show that $\mathcal{R}^{\#}$ is uniformly transitive. For this end, let $Q \in \mathcal{R}^{\#}$ be arbitrary. This is possible in the following ways.

<u>$Q \subset \Delta_X$</u>:

(1) There exists an α permutation of X such that $(\alpha(1), \alpha(1))$, $(\alpha(2), \alpha(2)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(1)), (\alpha(2), \alpha(2)), (\alpha(3), \alpha(4))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

<u>$Q \subset X^2 \setminus \Delta_X$:</u>

(2) There exists an α permutation of X such that $(\alpha(1), \alpha(2))$, $(\alpha(1), \alpha(3)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(2)), (\alpha(1), \alpha(4)), (\alpha(4), \alpha(2)), (\alpha(4), \alpha(3))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Or

(3) there exists an α permutation of X such that $(\alpha(1), \alpha(2))$, $(\alpha(3), \alpha(4)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(3)), (\alpha(2), \alpha(4)), (\alpha(3), \alpha(2))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Or

(4) there exists an α permutation of X such that $(\alpha(1), \alpha(2))$, $(\alpha(2), \alpha(3))$, $(\alpha(3), \alpha(1)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(3)), (\alpha(2), \alpha(1)), (\alpha(3), \alpha(2))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

<u>$Q \not\subset \Delta_X$ and $Q \not\subset X^2 \setminus \Delta_X$:</u>

(5) There exists an α permutation of X such that $(\alpha(1), \alpha(1))$, $(\alpha(1), \alpha(2)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(1)), (\alpha(1), \alpha(3)), (\alpha(2), \alpha(4)), (\alpha(3), \alpha(4))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

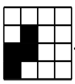
Or

(6) there exists an α permutation of X such that $(\alpha(1), \alpha(1))$, $(\alpha(2), \alpha(3)) \in Q$. In this case

$$P = \{(\alpha(1), \alpha(1)), (\alpha(2), \alpha(4)), (\alpha(4), \alpha(3))\} \in \mathcal{R}^{\#} \text{ and } P^2 \subset Q.$$

Finally, we show that $\mathcal{R}$ is not proximally transitive. For this, note that for any π permutation of X

$$R_\pi^2[\{1,3,4\}] \not\subset \{1,2\} = R_{\Delta_X}[\{1,3,4\}]$$

Note that $R_{\Delta_X} =$ .

Unfortunately, the above relator is neither reflexive nor symmetric. We can make a reflexive one.

Example 4.4. Let $X = \{1,2,3,4\}$, $Y = \{1,2,3,4,5,6,7,8\}$ and

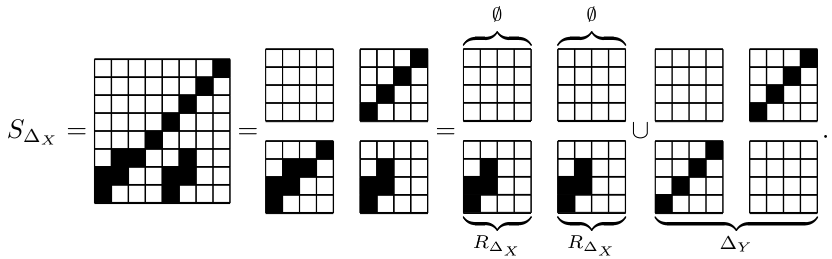
$$R_\pi = \{(\pi(1), \pi(1)), (\pi(1), \pi(2)), (\pi(2), \pi(2)), (\pi(2), \pi(3))\}$$

and

$$S_\pi = R_\pi \cup \{(\pi(1)+4, \pi(1)), (\pi(1)+4, \pi(2)), (\pi(2)+4, \pi(2)), (\pi(2)+4, \pi(3))\} \cup \Delta_Y$$

for all π permutation of X.

Moreover, let $\mathcal{S} = \{S_\pi : \pi$ is a permutation of $X\}$ be an elementwise reflexive relator on Y. Now $\mathcal{S}^\#$ is uniformly transitive, but $\mathcal{S}$ is not proximally transitive. Note that



We do not know whether if a symmetric finite $\mathcal{R}$ relator is not proximally transitive, then can $\mathcal{R}^\#$ be uniformly transitive.

References

[1] A. S. Davis: *Indexed Systems of Neighborhoods for General Topological Spaces*, The American Mathematical Monthly 68.9 (1961), pp. 886–893, ISSN: 00029890, 19300972, DOI: https://doi.org/10.2307/2311686, URL: http://www.jstor.org/stable/2311686 (visited on 09/13/2022).

[2] V. A. Efremovič: *The geometry of proximity. I.* Matematicheskii Sbornik 73.1 (1952), pp. 189–200.

[3] L. Nachbin: *Topology and order*, Princetown: D. Van Nostrand, 1965.

[4] H. Nakano, K. Nakano: *Connector theory*, Pacific J. Math. 56.1 (1975), pp. 195–213.

[5] G. Pataki: *Investigation of proximal spaces using relators*, Axioms 10.3 (2021), pp. 1–10, DOI: https://doi.org/10.3390/axioms10030143.

[6] G. Pataki: *Investigation of topological spaces using relators*, Applied General Topology 23.1 (2022), pp. 45–54, DOI: https://doi.org/10.4995/agt.2022.16128.

[7] G. Pataki, Á. Száz: *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 19 (2003), pp. 101–166.

[8] W. J. Pervin: *Quasi-uniformization of topological spaces*, Math. Ann. 147 (1962), pp. 316–317, DOI: https://doi.org/10.1007/BF01440953.

[9] F. Riesz: *Stetigkeit und abstrakte Mengenlehre*, Rom. 4. Math. Kongr. 2 (1909), pp. 18–24.

[10] Y. M. Smirnov: *On proximity spaces*, Matematicheskii Sbornik 73.3 (1952), pp. 543–574.

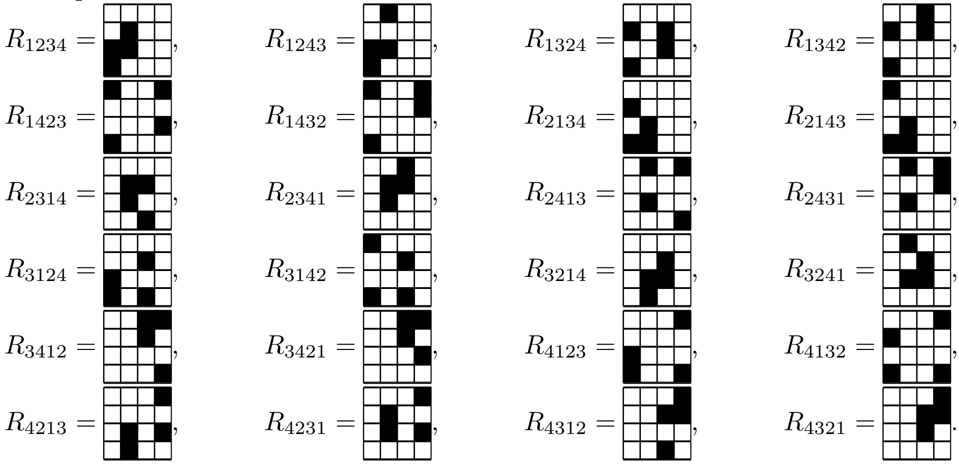
[11] Á. Száz: *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. 50.3-4 (1987), pp. 177–201, DOI: https://doi.org/10.1007/bf01903935.

[12] Á. Száz: *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat 21.1 (2007), pp. 87–97, DOI: https://doi.org/10.2298/FIL0701087S.

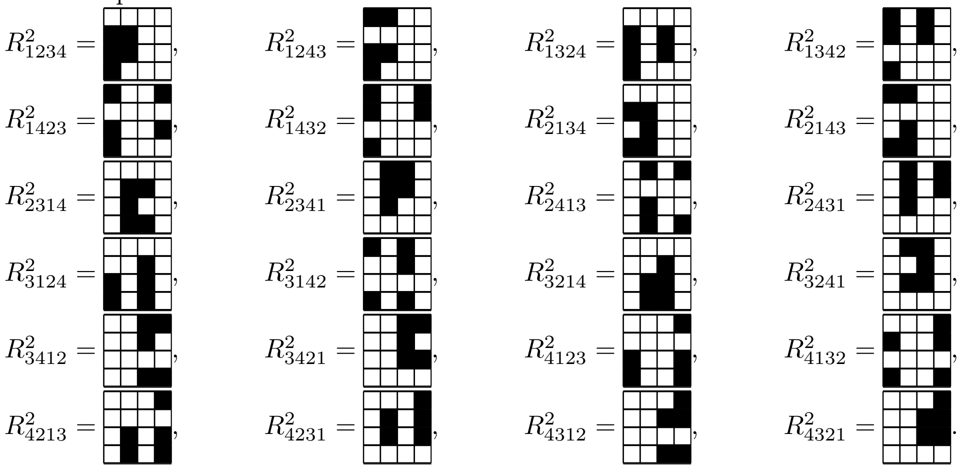
[13] A. Weil: *Sur les espaces a structure uniforme at sur la topologie générale*, Paris: Herman and Cie, 1937, DOI: https://doi.org/10.1007/978-1-4757-1705-1_27.

Appendix

List of elements of $\mathcal{R}$ of Example 4.3, with using the notation $\pi(1)\pi(2)\pi(3)\pi(4)$ for the π permutation of X.

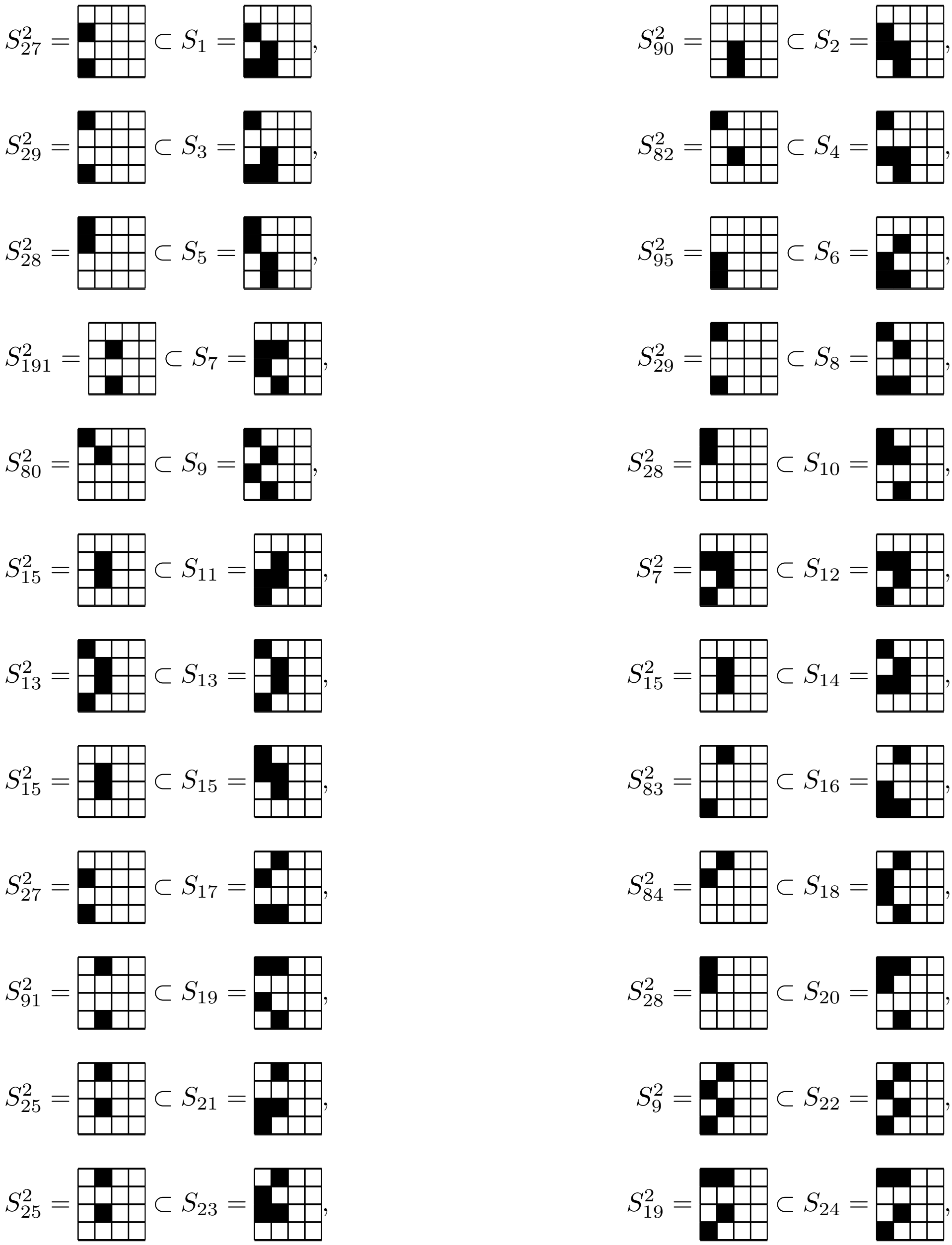


List of elements of $\mathcal{R}^2$ of Example 4.3, with using the notation $\pi(1)\pi(2)\pi(3)\pi(4)$ for the π permutation of X.



In Example 4.3, $\mathcal{S}$ is the smallest relator on X, such that $\mathcal{S}^* = \mathcal{R}^\#$

We draw the graph of elements of $\mathcal{S}$ with marked the required member of $\mathcal{S}$, which implies the $\mathcal{S} \subset \mathcal{S}^{2*}$ inclusion.



$S_{25}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{25} = \begin{array}{cccc} \blacksquare & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{array},$ $S_{10}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{26} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{array},$

$S_{10}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{27} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \blacksquare & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{array},$ $S_{10}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{28} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \blacksquare & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{array},$

$S_{10}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{29} = \begin{array}{cccc} \blacksquare & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{array},$ $S_{10}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{30} = \begin{array}{cccc} \blacksquare & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{array},$

$S_{131}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{31} = \begin{array}{cccc} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \end{array},$ $S_{82}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{32} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \end{array},$

$S_{37}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{33} = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array},$ $S_{80}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{34} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array},$

$S_{38}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{35} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array},$ $S_{38}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{36} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array},$

$S_{33}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \square & \square \end{array} \subset S_{37} = \begin{array}{cccc} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \square & \square \end{array},$ $S_{35}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{array} \subset S_{38} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \square & \square \end{array},$

$S_{147}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \square & \square \end{array} \subset S_{39} = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \square & \square \end{array},$ $S_{35}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{array} \subset S_{40} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{array},$

$S_{83}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{array} \subset S_{41} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \square & \square \end{array},$ $S_{84}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{42} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{array},$

$S_{27}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{array} \subset S_{43} = \begin{array}{cccc} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \blacksquare & \square \end{array},$ $S_{163}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{44} = \begin{array}{cccc} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \blacksquare & \square & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array},$

$S_{29}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{array} \subset S_{45} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \blacksquare & \square \end{array},$ $S_{35}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{array} \subset S_{46} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array},$

$S_{28}^2 = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{47} = \begin{array}{cccc} \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array},$ $S_{163}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{48} = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \blacksquare & \square \end{array},$

$S_{15}^2 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{49} = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \blacksquare & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array},$ $S_{38}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{array} \subset S_{50} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \blacksquare & \square \end{array},$

$S_{25}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{51} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array},$ $S_{10}^2 = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \subset S_{52} = \begin{array}{cccc} \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \blacksquare & \square \end{array},$

$$S^2_{128} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□□□□}\\\text{□■□□}\end{array} \subset S_{53} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{■□□□}\\\text{□■□□}\end{array}, \qquad S^2_{58} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{54} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{□□□□}\\\text{□■□□}\end{array},$$

$$S^2_{31} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■□□}\\\text{■□□□}\end{array} \subset S_{55} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■□□}\\\text{■□□□}\end{array}, \qquad S^2_{40} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■□□}\\\text{□□□□}\end{array} \subset S_{56} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{□■□□}\\\text{□□□□}\end{array},$$

$$S^2_{36} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□□□□}\\\text{■□□□}\end{array} \subset S_{57} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□□□□}\\\text{■□□□}\end{array}, \qquad S^2_{54} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{58} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{■□□□}\\\text{□□□□}\end{array},$$

$$S^2_{36} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□□□□}\\\text{■□□□}\end{array} \subset S_{59} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{■□□□}\\\text{■□■□}\end{array}, \qquad S^2_{66} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□□□□}\\\text{□□■□}\end{array} \subset S_{60} = \begin{array}{c}\text{□□□□}\\\text{■□■□}\\\text{■□□□}\\\text{□□■□}\end{array},$$

$$S^2_{29} = \begin{array}{c}\text{■□□□}\\\text{□□□□}\\\text{□□□□}\\\text{■□□□}\end{array} \subset S_{61} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{□□□□}\\\text{■□■□}\end{array}, \qquad S^2_{58} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{62} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{■□□□}\\\text{□□■□}\end{array},$$

$$S^2_{28} = \begin{array}{c}\text{■□□□}\\\text{■□□□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{63} = \begin{array}{c}\text{■□□□}\\\text{■□■□}\\\text{□□□□}\\\text{□□■□}\end{array}, \qquad S^2_{40} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■□□}\\\text{□□□□}\end{array} \subset S_{64} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■□□}\\\text{□■■□}\end{array},$$

$$S^2_{15} = \begin{array}{c}\text{□□□□}\\\text{□■□□}\\\text{□■□□}\\\text{□□□□}\end{array} \subset S_{65} = \begin{array}{c}\text{□□□□}\\\text{□■■□}\\\text{□■□□}\\\text{□□■□}\end{array}, \qquad S^2_{38} = \begin{array}{c}\text{□■□□}\\\text{□□□□}\\\text{□□□□}\\\text{□□■□}\end{array} \subset S_{66} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□□□□}\\\text{□■■□}\end{array},$$

$$S^2_{25} = \begin{array}{c}\text{□■□□}\\\text{□□□□}\\\text{□■□□}\\\text{□□□□}\end{array} \subset S_{67} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□■□□}\\\text{□□■□}\end{array}, \qquad S^2_{10} = \begin{array}{c}\text{□■□□}\\\text{□■□□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{68} = \begin{array}{c}\text{□■□□}\\\text{□■■□}\\\text{□□□□}\\\text{□□■□}\end{array},$$

$$S^2_{36} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□□□□}\\\text{■□□□}\end{array} \subset S_{69} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{■□■□}\\\text{■□□□}\end{array}, \qquad S^2_{27} = \begin{array}{c}\text{□□□□}\\\text{■□□□}\\\text{□□□□}\\\text{■□□□}\end{array} \subset S_{70} = \begin{array}{c}\text{□□□□}\\\text{■□■□}\\\text{□□■□}\\\text{■□□□}\end{array},$$

$$S^2_{29} = \begin{array}{c}\text{■□□□}\\\text{□□□□}\\\text{□□□□}\\\text{■□□□}\end{array} \subset S_{71} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{□□■□}\\\text{■□□□}\end{array}, \qquad S^2_{35} = \begin{array}{c}\text{■□□□}\\\text{□□□□}\\\text{□□■□}\\\text{□□□□}\end{array} \subset S_{72} = \begin{array}{c}\text{■□□□}\\\text{□□■□}\\\text{■□■□}\\\text{□□□□}\end{array},$$

$$S^2_{28} = \begin{array}{c}\text{■□□□}\\\text{■□□□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{73} = \begin{array}{c}\text{■□□□}\\\text{■□■□}\\\text{□□■□}\\\text{□□□□}\end{array}, \qquad S^2_{40} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■□□}\\\text{□□□□}\end{array} \subset S_{74} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□■■□}\\\text{□■□□}\end{array},$$

$$S^2_{72} = \begin{array}{c}\text{□□□□}\\\text{□□■□}\\\text{□□■□}\\\text{□□□□}\end{array} \subset S_{75} = \begin{array}{c}\text{□□□□}\\\text{□■■□}\\\text{□□■□}\\\text{□■□□}\end{array}, \qquad S^2_{54} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{76} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□□■□}\\\text{□■□□}\end{array},$$

$$S^2_{25} = \begin{array}{c}\text{□■□□}\\\text{□□□□}\\\text{□■□□}\\\text{□□□□}\end{array} \subset S_{77} = \begin{array}{c}\text{□■□□}\\\text{□□■□}\\\text{□■■□}\\\text{□□□□}\end{array}, \qquad S^2_{10} = \begin{array}{c}\text{□■□□}\\\text{□■□□}\\\text{□□□□}\\\text{□□□□}\end{array} \subset S_{78} = \begin{array}{c}\text{□■□□}\\\text{□■■□}\\\text{□□■□}\\\text{□□□□}\end{array},$$

$S^2_{34} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix} \subset S_{79} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix},$ $S^2_{34} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix} \subset S_{80} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix},$

$S^2_{32} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{81} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$ $S^2_{32} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{82} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix},$

$S^2_{41} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{83} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$ $S^2_{42} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{84} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix},$

$S^2_{41} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{85} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \blacksquare&\square&\blacksquare&\square \end{smallmatrix},$ $S^2_{27} = \begin{smallmatrix} \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{86} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\blacksquare&\square \end{smallmatrix},$

$S^2_{42} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{87} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{smallmatrix},$ $S^2_{42} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{88} = \begin{smallmatrix} \blacksquare&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{smallmatrix},$

$S^2_{28} = \begin{smallmatrix} \blacksquare&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{89} = \begin{smallmatrix} \blacksquare&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{smallmatrix},$ $S^2_{32} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{90} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\blacksquare&\blacksquare&\square \end{smallmatrix},$

$S^2_{34} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix} \subset S_{91} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\blacksquare&\square \end{smallmatrix},$ $S^2_{15} = \begin{smallmatrix} \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{92} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\blacksquare&\square \end{smallmatrix},$

$S^2_{25} = \begin{smallmatrix} \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{93} = \begin{smallmatrix} \square&\blacksquare&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\blacksquare&\square \end{smallmatrix},$ $S^2_{10} = \begin{smallmatrix} \square&\blacksquare&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{94} = \begin{smallmatrix} \square&\blacksquare&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{smallmatrix},$

$S^2_{41} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{95} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$ $S^2_{27} = \begin{smallmatrix} \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{96} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$

$S^2_{42} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{97} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \blacksquare&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{smallmatrix},$ $S^2_{29} = \begin{smallmatrix} \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{98} = \begin{smallmatrix} \blacksquare&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$

$S^2_{28} = \begin{smallmatrix} \blacksquare&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{99} = \begin{smallmatrix} \blacksquare&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{smallmatrix},$ $S^2_{32} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{100} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\blacksquare&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix},$

$S^2_{34} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix} \subset S_{101} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix},$ $S^2_{15} = \begin{smallmatrix} \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{102} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\blacksquare&\blacksquare&\square \\ \square&\square&\square&\square \end{smallmatrix},$

$S^2_{34} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix} \subset S_{103} = \begin{smallmatrix} \square&\blacksquare&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \end{smallmatrix},$ $S^2_{10} = \begin{smallmatrix} \square&\blacksquare&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \end{smallmatrix} \subset S_{104} = \begin{smallmatrix} \square&\blacksquare&\blacksquare&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{smallmatrix},$

$S^2_{36} = \begin{smallmatrix} \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{105} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$ $S^2_{27} = \begin{smallmatrix} \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix} \subset S_{106} = \begin{smallmatrix} \square&\square&\blacksquare&\square \\ \blacksquare&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{smallmatrix},$

$S_{42}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{107} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$

$S_{29}^2 = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \end{array} \subset S_{108} = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \blacksquare & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \end{array},$

$S_{42}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{109} = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \blacksquare & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$

$S_{32}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{110} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \end{array},$

$S_{34}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \end{array} \subset S_{111} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \end{array},$

$S_{15}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{112} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \blacksquare & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$

$S_{34}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \end{array} \subset S_{113} = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \blacksquare & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \end{array},$

$S_{25}^2 = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{114} = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \blacksquare & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$

$S_{169}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{115} = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{82}^2 = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{116} = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{133}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{117} = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{80}^2 = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{118} = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{134}^2 = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{119} = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{84}^2 = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{120} = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{149}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{121} = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{35}^2 = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{122} = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{133}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{123} = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{149}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{124} = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{144}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{125} = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{58}^2 = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{126} = \begin{array}{|c|c|c|c|}\hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{40}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{127} = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{54}^2 = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{128} = \begin{array}{|c|c|c|c|}\hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{42}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{129} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{173}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{130} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{32}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \subset S_{131} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S_{133}^2 = \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array} \subset S_{132} = \begin{array}{|c|c|c|c|}\hline \square & \square & \blacksquare & \square \\ \hline \square & \blacksquare & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \blacksquare \\ \hline \end{array},$

$S^2_{117} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array} \subset S_{133} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\square\end{array}$,

$S^2_{119} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\square\end{array} \subset S_{134} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\square\end{array}$,

$S^2_{170} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\end{array} \subset S_{135} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\blacksquare&\square&\square&\square\end{array}$,

$S^2_{80} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{136} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array}$,

$S^2_{83} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\end{array} \subset S_{137} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\blacksquare&\square&\square&\square\end{array}$,

$S^2_{84} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{138} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array}$,

$S^2_{117} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array} \subset S_{139} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\blacksquare&\square\end{array}$,

$S^2_{129} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\blacksquare&\square\end{array} \subset S_{140} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\blacksquare&\square\end{array}$,

$S^2_{129} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\blacksquare&\square\end{array} \subset S_{141} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\blacksquare&\square\end{array}$,

$S^2_{38} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\blacksquare&\square\end{array} \subset S_{142} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\blacksquare&\square\end{array}$,

$S^2_{36} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\end{array} \subset S_{143} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\blacksquare\\\blacksquare&\square&\square&\square\end{array}$,

$S^2_{58} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{144} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array}$,

$S^2_{125} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array} \subset S_{145} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\square\end{array}$,

$S^2_{54} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{146} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\blacksquare&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array}$,

$S^2_{41} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\end{array} \subset S_{147} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\blacksquare&\square&\square&\square\end{array}$,

$S^2_{117} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array} \subset S_{148} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array}$,

$S^2_{34} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\square&\blacksquare&\square&\square\end{array} \subset S_{149} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\square\end{array}$,

$S^2_{179} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array} \subset S_{150} = \begin{array}{cccc}\square&\square&\blacksquare&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array}$,

$S^2_{27} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\end{array} \subset S_{151} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\blacksquare&\square&\square&\blacksquare\end{array}$,

$S^2_{117} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\square\end{array} \subset S_{152} = \begin{array}{cccc}\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\blacksquare&\square&\square&\blacksquare\\\square&\square&\square&\blacksquare\end{array}$,

$S^2_{29} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\end{array} \subset S_{153} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\blacksquare&\square&\square&\blacksquare\end{array}$,

$S^2_{97} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{154} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\blacksquare&\square&\square&\blacksquare\\\square&\square&\square&\blacksquare\end{array}$,

$S^2_{28} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{155} = \begin{array}{cccc}\blacksquare&\square&\square&\square\\\blacksquare&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\blacksquare\end{array}$,

$S^2_{133} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\end{array} \subset S_{156} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\blacksquare\end{array}$,

$S^2_{15} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{157} = \begin{array}{cccc}\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\blacksquare&\square&\blacksquare\\\square&\square&\square&\blacksquare\end{array}$,

$S^2_{91} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\square&\blacksquare&\square&\square\end{array} \subset S_{158} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\blacksquare\\\square&\blacksquare&\square&\blacksquare\end{array}$,

$S^2_{25} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{159} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\blacksquare&\square&\blacksquare\\\square&\square&\square&\blacksquare\end{array}$,

$S^2_{10} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\end{array} \subset S_{160} = \begin{array}{cccc}\square&\blacksquare&\square&\square\\\square&\blacksquare&\square&\square\\\square&\square&\square&\blacksquare\\\square&\square&\square&\blacksquare\end{array}$,

$S^2_{66} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array} \subset S_{161} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\blacksquare \end{array},$ $S^2_{72} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{array} \subset S_{162} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\blacksquare&\blacksquare \\ \square&\square&\square&\blacksquare \end{array},$

$S^2_{50} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array} \subset S_{163} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\blacksquare \end{array},$ $S^2_{46} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{array} \subset S_{164} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\blacksquare \\ \square&\square&\square&\blacksquare \end{array},$

$S^2_{109} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{165} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\blacksquare \end{array},$ $S^2_{120} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array} \subset S_{166} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array},$

$S^2_{120} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array} \subset S_{167} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array},$ $S^2_{79} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \blacksquare&\square&\square&\square \end{array} \subset S_{168} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\blacksquare&\square&\square \\ \blacksquare&\square&\square&\square \end{array},$

$S^2_{82} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{169} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{array},$ $S^2_{83} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array} \subset S_{170} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array},$

$S^2_{121} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{171} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{array},$ $S^2_{121} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{172} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array},$

$S^2_{130} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array} \subset S_{173} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array},$ $S^2_{115} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{174} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\blacksquare&\square&\square \\ \square&\square&\blacksquare&\square \end{array},$

$S^2_{38} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array} \subset S_{175} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \end{array},$ $S^2_{135} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array} \subset S_{176} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\square \\ \blacksquare&\square&\square&\square \end{array},$

$S^2_{35} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{array} \subset S_{177} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{array},$ $S^2_{120} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array} \subset S_{178} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\square \\ \square&\blacksquare&\square&\square \end{array},$

$S^2_{150} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{array} \subset S_{179} = \begin{array}{cccc} \square&\blacksquare&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \end{array},$ $S^2_{41} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array} \subset S_{180} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array},$

$S^2_{42} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{181} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{array},$ $S^2_{34} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array} \subset S_{182} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \end{array},$

$S^2_{32} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\square \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{183} = \begin{array}{cccc} \square&\square&\blacksquare&\square \\ \square&\square&\square&\blacksquare \\ \square&\blacksquare&\square&\square \\ \square&\square&\square&\square \end{array},$ $S^2_{95} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array} \subset S_{184} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \blacksquare&\square&\square&\blacksquare \end{array},$

$S^2_{121} = \begin{array}{cccc} \square&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \end{array} \subset S_{185} = \begin{array}{cccc} \square&\square&\square&\square \\ \blacksquare&\square&\square&\blacksquare \\ \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \end{array},$ $S^2_{29} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\square \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\square \end{array} \subset S_{186} = \begin{array}{cccc} \blacksquare&\square&\square&\square \\ \square&\square&\square&\blacksquare \\ \square&\square&\square&\square \\ \blacksquare&\square&\square&\blacksquare \end{array},$

$$S^2_{97} = \begin{bmatrix}1&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{187} = \begin{bmatrix}1&0&0&0\\0&0&0&1\\1&0&0&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{28} = \begin{bmatrix}1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{188} = \begin{bmatrix}1&0&0&0\\1&0&0&1\\0&0&0&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{90} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&1&0&0\end{bmatrix} \subset S_{189} = \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&1&0&0\\0&1&0&1\end{bmatrix},$$

$$S^2_{15} = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix} \subset S_{190} = \begin{bmatrix}0&0&0&0\\0&1&0&1\\0&1&0&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{91} = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} \subset S_{191} = \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&1\end{bmatrix},$$

$$S^2_{25} = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix} \subset S_{192} = \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&1&0&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{10} = \begin{bmatrix}0&1&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{193} = \begin{bmatrix}0&1&0&0\\0&1&0&1\\0&0&0&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{149} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \subset S_{194} = \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&1&0\\0&0&1&1\end{bmatrix},$$

$$S^2_{72} = \begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \subset S_{195} = \begin{bmatrix}0&0&0&0\\0&0&1&1\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{50} = \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix} \subset S_{196} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&1&1\end{bmatrix},$$

$$S^2_{46} = \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \subset S_{197} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{109} = \begin{bmatrix}0&0&1&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{198} = \begin{bmatrix}0&0&1&0\\0&0&1&1\\0&0&0&0\\0&0&0&1\end{bmatrix},$$

$$S^2_{95} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\1&0&0&0\end{bmatrix} \subset S_{199} = \begin{bmatrix}0&0&0&0\\0&0&0&1\\1&0&0&1\\1&0&0&0\end{bmatrix},$$

$$S^2_{27} = \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix} \subset S_{200} = \begin{bmatrix}0&0&0&0\\1&0&0&1\\0&0&0&1\\1&0&0&0\end{bmatrix},$$

$$S^2_{29} = \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix} \subset S_{201} = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&0&1\\1&0&0&0\end{bmatrix},$$

$$S^2_{97} = \begin{bmatrix}1&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{202} = \begin{bmatrix}1&0&0&0\\0&0&0&1\\1&0&0&1\\0&0&0&0\end{bmatrix},$$

$$S^2_{28} = \begin{bmatrix}1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{203} = \begin{bmatrix}1&0&0&0\\1&0&0&1\\0&0&0&1\\0&0&0&0\end{bmatrix},$$

$$S^2_{90} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&1&0&0\end{bmatrix} \subset S_{204} = \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&1&0&1\\0&1&0&0\end{bmatrix},$$

$$S^2_{120} = \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{bmatrix} \subset S_{205} = \begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&0&1\\0&1&0&0\end{bmatrix},$$

$$S^2_{91} = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} \subset S_{206} = \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&0&1\\0&1&0&0\end{bmatrix},$$

$$S^2_{25} = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix} \subset S_{207} = \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&1&0&1\\0&0&0&0\end{bmatrix},$$

$$S^2_{10} = \begin{bmatrix}0&1&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{208} = \begin{bmatrix}0&1&0&0\\0&1&0&1\\0&0&0&1\\0&0&0&0\end{bmatrix},$$

$$S^2_{129} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \subset S_{209} = \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&1&1\\0&0&1&0\end{bmatrix},$$

$$S^2_{66} = \begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&1&0\end{bmatrix} \subset S_{210} = \begin{bmatrix}0&0&0&0\\0&0&1&1\\0&0&0&1\\0&0&1&0\end{bmatrix},$$

$$S^2_{50} = \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix} \subset S_{211} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&1\\0&0&1&0\end{bmatrix},$$

$$S^2_{46} = \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \subset S_{212} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix},$$

$$S^2_{109} = \begin{bmatrix}0&0&1&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \subset S_{213} = \begin{bmatrix}0&0&1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix},$$

$$S^2_{223} = \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} \subset S_{214} = \begin{bmatrix}0&0&0&1\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$S^2_{219} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{215} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \end{bmatrix},$$

$$S^2_{138} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{217} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$S^2_{215} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{219} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$S^2_{219} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{221} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix},$$

$$S^2_{215} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{223} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix},$$

$$S^2_{219} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{225} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$S^2_{215} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{227} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$S^2_{144} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{229} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$S^2_{40} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{231} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$S^2_{27} = \begin{bmatrix} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix} \subset S_{233} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \blacksquare \end{bmatrix},$$

$$S^2_{97} = \begin{bmatrix} \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{235} = \begin{bmatrix} \blacksquare & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \blacksquare \end{bmatrix},$$

$$S^2_{90} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \end{bmatrix} \subset S_{237} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \blacksquare \end{bmatrix},$$

$$S^2_{15} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{239} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \blacksquare \end{bmatrix},$$

$$S^2_{79} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix} \subset S_{216} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix},$$

$$S^2_{118} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix} \subset S_{218} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix},$$

$$S^2_{225} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{220} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix},$$

$$S^2_{131} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix} \subset S_{222} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix},$$

$$S^2_{118} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix} \subset S_{224} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \blacksquare & \square & \square & \square \end{bmatrix},$$

$$S^2_{220} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{226} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \blacksquare & \square & \square \end{bmatrix},$$

$$S^2_{36} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix} \subset S_{228} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix},$$

$$S^2_{128} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \end{bmatrix} \subset S_{230} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \blacksquare & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \end{bmatrix},$$

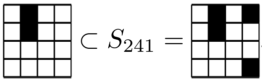
$$S^2_{95} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \end{bmatrix} \subset S_{232} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \blacksquare \end{bmatrix},$$

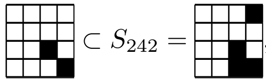
$$S^2_{202} = \begin{bmatrix} \square & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{234} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \blacksquare \end{bmatrix},$$

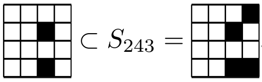
$$S^2_{28} = \begin{bmatrix} \blacksquare & \square & \square & \square \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{236} = \begin{bmatrix} \blacksquare & \square & \square & \blacksquare \\ \blacksquare & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \blacksquare \end{bmatrix},$$

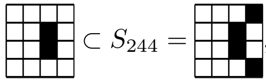
$$S^2_{133} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \blacksquare \end{bmatrix} \subset S_{238} = \begin{bmatrix} \square & \square & \square & \blacksquare \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \blacksquare \end{bmatrix},$$

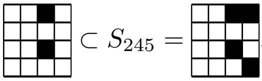
$$S^2_{25} = \begin{bmatrix} \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \subset S_{240} = \begin{bmatrix} \square & \blacksquare & \square & \blacksquare \\ \square & \square & \square & \square \\ \square & \blacksquare & \square & \square \\ \square & \square & \square & \blacksquare \end{bmatrix},$$

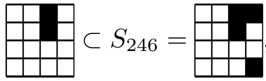
$S_{10}^2 =$ 

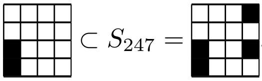
$S_{149}^2 =$ 

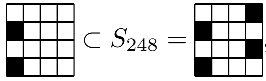
$S_{66}^2 =$ 

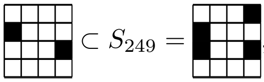
$S_{72}^2 =$ 

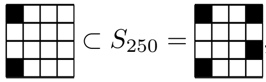
$S_{46}^2 =$ 

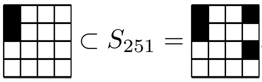
$S_{109}^2 =$ 

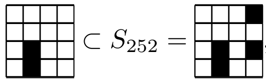
$S_{95}^2 =$ 

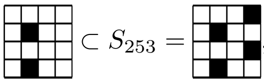
$S_{27}^2 =$ 

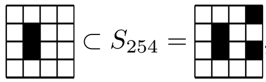
$S_{117}^2 =$ 

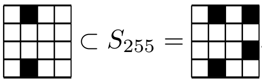
$S_{29}^2 =$ 

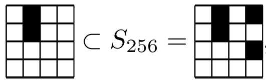
$S_{28}^2 =$ 

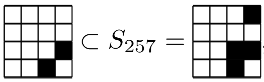
$S_{90}^2 =$ 

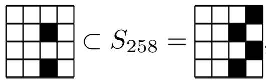
$S_{191}^2 =$ 

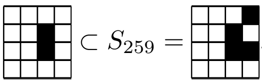
$S_{15}^2 =$ 

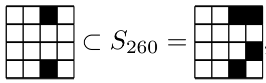
$S_{91}^2 =$ 

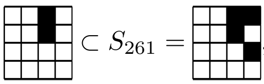
$S_{10}^2 =$ 

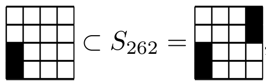
$S_{129}^2 =$ 

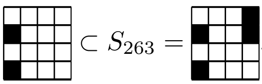
$S_{66}^2 =$ 

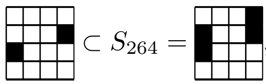
$S_{72}^2 =$ 

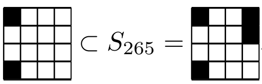
$S_{50}^2 =$ 

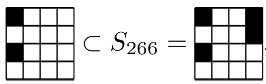
$S_{109}^2 =$ 

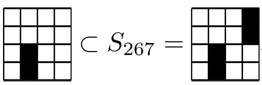
$S_{95}^2 =$ 

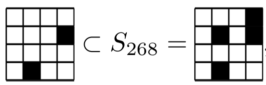
$S_{27}^2 =$ 

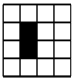
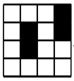
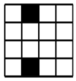
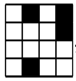
$S_{121}^2 =$ 

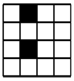
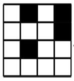
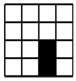
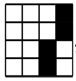
$S_{29}^2 =$ 

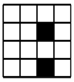
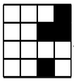
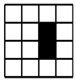
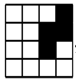
$S_{97}^2 =$ 

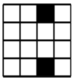
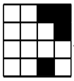
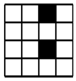
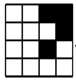
$S_{90}^2 =$ 

$S_{120}^2 =$ 

$S^2_{15} =$  $\subset S_{269} =$ , $S^2_{91} =$  $\subset S_{270} =$ ,

$S^2_{25} =$  $\subset S_{271} =$ , $S^2_{163} =$  $\subset S_{272} =$ ,

$S^2_{66} =$  $\subset S_{273} =$ , $S^2_{72} =$  $\subset S_{274} =$ ,

$S^2_{50} =$  $\subset S_{275} =$ , $S^2_{46} =$  $\subset S_{276} =$ .

Annales Mathematicae et Informaticae
55 (2022) pp. 158–171
DOI: https://doi.org/10.33039/ami.2021.12.002
URL: https://ami.uni-eszterhazy.hu

Padovan and Perrin numbers of the form $x^a \pm x^b + 1$

Salah Eddine Rihane[a], **Bir Kafle**[b*], **Alain Togbé**[b*]

[a]Department of Mathematics and Computer Sciences,
University Center of Mila, Mila, Algeria
salahrihane@hotmail.fr

[b]Department of Mathematics, Statistics, and Computer Science,
Purdue University Northwest, 1401 S U.S. 421, Westville IN 46391, USA
bkafle@pnw.edu
atogbe@pnw.edu

Abstract. Let P_k be the kth Padovan number and E_k be the kth Perrin number. In this paper, we study the Padovan and Perrin numbers of the form $x^a \pm x^b + 1$. In particular, we first find an upper bound for a, b, n as a function of x. Moreover, we determine all Padovan numbers and Perrin numbers of the form $x^a \pm x^b + 1$ such that $0 \leq b < a$ and $2 \leq x \leq 20$.

Keywords: Padovan numbers, Perrin numbers Linear form in logarithms, reduction method

AMS Subject Classification: 11B39, 11D45, 11J86

1. Introduction

Let $\mathbf{U} = \{U_n\}_{n\geq 0}$ be some *interesting sequence* of positive integers. The problem of finding U_n in a particular form has a very rich history. In 2006, Bugeaud, Mignotte and Siksek [2] proved that the only perfect power Fibonacci numbers are $0, 1, 8, 144$ and the only perfect powers among Lucas numbers are $1, 4$. Luca and Szalay [6] showed that there are only finitely many Fibonacci numbers of the form $p^a \pm p^b + 1$, where p is a number and a and b are positive integers with $\max\{a, b\} \geq 2$. In [8], Marques and Togbé determined all the Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$, where a, b and c are nonnegative integers with $c \geq \max\{a, b\}$. In [1], Bravo and Luca determined all the generalized Fibonacci numbers which are some

*The second and third authors were supported in part by Purdue University Northwest.

Submitted: July 24, 2021
Accepted: December 18, 2021
Published online: January 20, 2022

powers of two. Very recently, Qu and Zeng [10] determined all the Pell and Pell-Lucas numbers that are of the form $-2^a - 3^b + 5^c$, where a, b and c are nonnegative integers with some restrictions. For more related results, one can see [3–5, 7, 12].

In this paper, we continue this discussion to the sequences of Padovan, and Perrin numbers, which we define below. The Padovan sequence $\{P_m\}_{m\geq 0}$ is defined by

$$P_{m+3} = P_{m+1} + P_m,$$

for $m \geq 0$, where $P_0 = P_1 = P_2 = 1$. This is the sequence A000931 in the OEIS [14]. A few terms of this sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \cdots$$

Let $\{E_m\}_{m\geq 0}$ be the Perrin sequence given by

$$E_{m+3} = E_{m+1} + E_m,$$

for $m \geq 0$, where $E_0 = 3$, $E_1 = 0$ and $E_2 = 2$. Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, \cdots$$

It is the sequence A001608 in the OEIS [14].

Now, we are interested in studying the Padovan and Perrin numbers which are in the form of $x^a \pm x^b + 1$. More precisely, we consider the following equations

$$P_n = x^a \pm x^b + 1 \tag{1.1}$$

$$E_n = x^a \pm x^b + 1 \tag{1.2}$$

and prove the following results.

Theorem 1.1. *All the solutions of Diophantine equation (1.1) satisfy*

$$a < n < 2.58 \cdot 10^{31} (\log x)^4. \tag{1.3}$$

Furthermore, the only solutions of Diophantine equation (1.1) in positive integers (n, x, a, b) with $0 \leq b < a$ and $2 \leq x \leq 20$ are

$P_3 = P_4 = 2^1 - 2^0 + 1$

$P_5 = 2^2 - 2^1 + 1 = 3^1 - 3^0 + 1$

$P_6 = 2^2 - 2^0 + 1 = 4^1 - 4^0 + 1$

$P_7 = 2^3 - 2^2 + 1 = 5^1 - 5^0 + 1$

$P_8 = 2^3 - 2^1 + 1 = 3^2 - 3^1 + 1 = 7^1 - 7^0 + 1$

$P_9 = 2^4 - 2^3 + 1 = 3^2 - 3^0 + 1 = 9^1 - 9^0 + 1$

$P_{10} = 12^1 - 12^0 + 1$

$P_{11} = 2^4 - 2^0 + 1 = 4^2 - 4^0 + 1 = 16^1 - 16^0 + 1$

$P_{12} = 5^2 - 5^1 + 1$

$P_{15} = 2^6 - 2^4 + 1 = 4^3 - 4^2 + 1 = 7^2 - 7^0 + 1$

$P_{16} = 2^7 - 2^6 + 1$

and

$$P_6 = 2^1 + 2^0 + 1$$
$$P_7 = 3^1 + 3^0 + 1$$
$$P_8 = 2^2 + 2^1 + 1 = 5^1 + 5^0 + 1$$
$$P_9 = 7^1 + 7^0 + 1$$
$$P_{10} = 10^1 + 10^0 + 1$$
$$P_{12} = 2^4 + 2^2 + 1 = 4^2 + 4^1 + 1 = 19^1 + 19^0 + 1$$
$$P_{14} = 2^5 + 2^2 + 1 = 3^3 + 3^3 + 1$$
$$P_{15} = 2^5 + 2^4 + 1$$
$$P_{19} = 5^3 + 5^2 + 1.$$

Theorem 1.2. *All the solutions of Diophantine equation (1.2) satisfy*

$$n \leq 3.35 \cdot 10^{30} (\log x)^4. \tag{1.4}$$

Furthermore, the only solutions of Diophantine equation (1.2) in positive integers (n, x, a, b) with $0 \leq b < a$ and $2 \leq x \leq 20$ are

$$E_0 = 2^2 - 2^1 + 1 = 3^1 - 3^0 + 1$$
$$E_2 = 2^1 - 2^0 + 1$$
$$E_3 = 2^2 - 2^1 + 1 = 3^1 - 3^0 + 1$$
$$E_4 = 2^2 - 2^1 + 1$$
$$E_5 = E_6 = 2^3 - 2^2 + 1 = 5^1 - 5^0 + 1$$
$$E_7 = 2^2 - 2^1 + 1 = 3^2 - 3^1 + 1 = 7^1 - 7^0 + 1$$
$$E_8 = 10^1 - 10^0 + 1$$
$$E_9 = 12^1 - 12^0 + 1$$
$$E_{10} = 2^5 - 2^4 + 1 = 17^1 - 17^0 + 1$$
$$E_{12} = 2^5 - 2^2 + 1$$

and

$$E_5 = E_6 = 3^1 + 3^0 + 1$$
$$E_7 = 2^2 + 2^1 + 1 = 5^1 + 5^0 + 1$$
$$E_8 = 2^3 + 2^0 + 1 = 8^1 + 8^0 + 1$$
$$E_9 = 10^1 + 10^0 + 1$$
$$E_{10} = 2^2 + 2^1 + 1 = 15^1 + 15^0 + 1$$
$$E_{11} = 20^1 + 20^0 + 1$$
$$E_{12} = 3^1 + 3^0 + 1$$
$$E_{14} = 7^2 + 7^0 + 1.$$

The outline of this paper is as follows. In section 2, we recall some results that are useful for the proofs of Theorem 1.1 and Theorem 1.2. Particularly, we recall some of the properties of Padovan and Perrin numbers, a result of Matveev [9] that we will use to obtain lower bounds for linear forms in logarithms of algebraic numbers, de Weger reduction method [15]. In the last two sections, we will completely prove Theorem 1.1 and Theorem 1.2 using Baker method and the reduction method.

2. Auxiliary results

First, we recall some facts and properties of the Padovan and the Perrin sequences which will be used later. One can see [11]. The characteristic equation

$$\Psi(x) := x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$c_\alpha = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2 + 3\alpha + 1},$$
$$c_\beta = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2 + 3\beta + 1},$$
$$c_\gamma = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2 + 3\gamma + 1} = \overline{c_\beta}.$$

Binet's formula for P_n is

$$P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \quad \text{for all } n \geq 0 \tag{2.1}$$

and Binet's formula for E_n is

$$E_n = \alpha^n + \beta^n + \gamma^n, \quad \text{for all } n \geq 0. \tag{2.2}$$

Numerically, we have

$$1.32 < \alpha < 1.33,$$
$$0.86 < |\beta| = |\gamma| < 0.87,$$
$$0.72 < c_\alpha < 0.73,$$
$$0.24 < |c_\beta| = |c_\gamma| < 0.25.$$

It is easy to check that

$$|\beta| = |\gamma| = \alpha^{-1/2}.$$

Further, using induction on n, we can prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}, \quad \text{for all } n \geq 4 \tag{2.3}$$

and

$$\alpha^{n-2} \leq E_n \leq \alpha^{n+1}, \quad \text{for all } n \geq 2. \tag{2.4}$$

Let $\mathbb{K} := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial Ψ over $\mathbb{Q}$. Then, $[\mathbb{K} : \mathbb{Q}] = 6$. The Galois group of $\mathbb{K}$ over $\mathbb{Q}$ is given by

$$\mathrm{Gal}(\mathbb{K}/\mathbb{Q}) \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

The next tools are related to the transcendental approach to solve Diophantine equations. For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{j=1}^{d} \big(X - \gamma^{(j)}\big)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\Big(1, \big|\gamma^{(j)}\big|\Big)\right)$$

the usual absolute logarithmic height of γ.

To prove Theorems 1.1 and 1.2, we use lower bounds for linear forms in logarithms to bound the index n appearing in equations (1.1) and (1.2). We need the following general lower bound for linear forms in logarithms due to Matveev [9].

Lemma 2.1. *Let $\gamma_1, \ldots, \gamma_s$ be a real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma), |\log\gamma|, 0.16\} \quad \textit{for } j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-C(s, D)(1 + \log B)A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

After getting the upper bound of n which is generally too large, the next step is to reduce it. For this reduction, we present a variant of the reduction method of Baker and Davenport due to de Weger [15]).

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2.5}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{2.6}$$

$$Y \leq X \leq X_0. \tag{2.7}$$

When $\beta = 0$ in (2.5), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \dots]$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \dots$. We have the following results.

Lemma 2.2 ([15, Lemma 3.2]). *Let*

$$A = \max_{0 \le k \le Y_0} a_{k+1},$$

where

$$Y_0 = -1 + \frac{\log(\sqrt{5}X_0 + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (2.6) *and* (2.7) *hold for* x_1, x_2 *and* $\beta = 0$, *then*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\beta \neq 0$ in (2.5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.3 ([15, Lemma 3.3]). *Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (2.6) *and* (2.7) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2|X_0}\right).$$

We conclude this section by recalling two lemmas that we need in the sequel:

Lemma 2.4 ([13, Lemma 7]). *If* $m \ge 1$, $T > (4m^2)^m$ *and* $T > y/(\log y)^m$. *Then,*

$$y < 2^m T(\log T)^m.$$

Lemma 2.5 ([15, Lemma 2.2, page 31]). *Let* $a, x \in \mathbb{R}$ *and* $0 < a < 1$. *If* $|x| < a$, *then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x|$$

and

$$|x| < \frac{a}{1-e^{-a}}|e^x - 1|.$$

3. Padovan numbers of the form $x^a \pm x^b + 1$

In this section, we will prove Theorem 1.1.

3.1. The proof of inequality (1.3)

First of all, we find a relation between a and n. By combining (2.3) together with (1.1), we get

$$\alpha^{n-2} < P_n = x^a \pm x^b + 1 < x^a + x^b + 1 < 3x^a \tag{3.1}$$

and

$$\frac{x^a}{2} < x^a - x^b < x^a \pm x^b + 1 = P_n < \alpha^{n-1}. \tag{3.2}$$

Taking logarithms on both sides of inequalities (3.1) and (3.2) and combining them, we obtain

$$\left(\frac{\log x}{\log \alpha}\right)a - \frac{\log 2}{\log \alpha} + 1 < n < \left(\frac{\log x}{\log \alpha}\right)a + \frac{\log 3}{\log \alpha} + 2. \tag{3.3}$$

Particularly, using the fact that $x \geq 2$, we conclude that

$$a < n. \tag{3.4}$$

By using (2.1), we rewrite equation (1.1) as

$$|c_\alpha \alpha^n - x^a| \leq 2|c_\beta||\beta|^n + x^b + 1 < x^{b+1}.$$

Dividing both sides of the last inequality by x^a, we get

$$\left|c_\alpha \alpha^n x^{-a} - 1\right| < \frac{1}{x^{a-b-1}}. \tag{3.5}$$

Now, we apply Matveev's result (see Lemma 2.1) to the left-hand side of (3.5). First, the expression on the left-hand side of (3.5) is nonzero, since this expression being zero means that $c_\alpha \alpha^n = x^a \in \mathbb{Z}$, which is false since if we conjugate this relation by the automorphism of Galois $\sigma := (\alpha\beta)$ we would get $1 < x^a = |c_\beta \beta^n| < 1$. In order to apply Lemma 2.1, we take $s := 3$,

$$(\gamma_1, b_1) := (c_\alpha, 1), \quad (\gamma_2, b_2) := (\alpha, n) \quad (\gamma_3, b_3) := (x, -a).$$

For this choice we have $D = 3$, $h(\gamma_1) = (\log 23)/3$, $h(\gamma_2) = (\log \alpha)/3$, $h(\gamma_3) = \log x$ and $\max\{1, n, a\} \leq n$. In conclusion, $B := n$, $A_1 := 3.2$, $A_2 := 0.3$ and $A_3 := 3\log x$ are suitable choices. By Lemma 2.1, we obtain the following estimate

$$\left|c_\alpha \alpha^n x^{-a} - 1\right| \geq \exp(-7.79 \cdot 10^{12} \cdot \log x \cdot (1 + \log n)). \tag{3.6}$$

We combine (3.5) and (3.6) to obtain

$$a - b < 7.8 \cdot 10^{12}(1 + \log n). \tag{3.7}$$

We now use a second linear form in logarithms by rewriting equation (1.1) in a different way. Using Binet formula (2.1), we get that

$$\left|c_\alpha \alpha^n - \left(x^{a-b} \pm 1\right)x^b\right| \leq 2|c_\beta||\beta|^n + 1 < 1.5.$$

Dividing both sides of the above inequality by $x^a \pm x^b$, we obtain

$$\left|c_\alpha \left(x^{a-b} \pm 1\right)^{-1} \alpha^n x^{-b} - 1\right| < \frac{1.5}{x^a \pm x^b} < \frac{1}{\alpha^{n-10}}, \tag{3.8}$$

where we have also used the fact that $x^a \pm x^b > x^a/2$, $\alpha^{n-2} < 3x^a$ and $9 < \alpha^8$.

We observe that the left-hand side of (3.8) is nonzero, otherwise we would get

$$x^a \pm x^b = c_\alpha \alpha^n. \tag{3.9}$$

Conjugating (3.9) in $\mathbb{Q}(\alpha, \beta)$ by the automorphism $\sigma := (\alpha\beta)$, we get

$$1 < x^a \pm x^b = |c_\beta \beta^n| < 1.$$

Now we again apply Lemma 2.1 as before but with $s := 3$,

$$\gamma_1 := c_\alpha (x^{a-b} \pm 1)^{-1}, \quad \gamma_2 := \alpha, \quad \gamma_3 := x, \quad b_1 := 1, \quad b_2 := n, \quad b_3 := -b.$$

Here, we can take $D = 3$, $B := n$, $A_2 := 0.3$, $A_3 := 3 \log x$ and since (using the proprieties of absolute logarithmic height and inequality (3.7))

$$h(\alpha_1) < h(c_\alpha) + h(x^{a-b}) + \log 2 < 7.81 \cdot 10^{12} \log x(1 + \log n),$$

then we can take $A_1 := 2.35 \cdot 10^{13} \log x(1 + \log n)$. We obtain the following estimate

$$\exp(-5.72 \cdot 10^{25} \cdot (\log x)^2 (1 + \log n)^2) \leq \frac{1}{\alpha^{n-10}},$$

which leads us to

$$\frac{n}{(\log n)^2} < 8.14 \cdot 10^{26} (\log x)^2. \tag{3.10}$$

By applying Lemma 2.4 in the inequality (3.10), we obtain

$$n < 2^2 \cdot \left(8.14 \cdot 10^{26} (\log x)^2\right) \cdot \left(\log\left(8.14 \cdot 10^{26} (\log x)^2\right)\right)^2. \tag{3.11}$$

Finally, combining equations (3.4) and (3.11), and using the fact that

$$\log\left(8.14 \cdot 10^{26} (\log x)^2\right) < 89 \log x, \quad \text{for } x \geq 2,$$

we obtain

$$a < n < 2.58 \cdot 10^{31} (\log x)^4. \tag{3.12}$$

This proves the first part of Theorem 1.1. Next, we determine the solutions of equation (1.1) in the specified range.

3.2. The solutions of equation (1.1) for $2 \leq x \leq 20$

Let x be a fixed integer such that $2 \leq x \leq 20$. The inequality (3.12) gives

$$n < 2.08 \cdot 10^{33}. \tag{3.13}$$

The upper bound of n given by (3.13) is very large, so we will reduce it further. To do this, we will use several times Lemma 2.3. From inequality (3.5), we put

$$\Lambda_1 := n \log \alpha - a \log x + \log c_\alpha \quad \text{and} \quad \Gamma_1 := e^{\Lambda_1} - 1.$$

Then, for $a - b \geq 2$ and $2 \leq x \leq 20$, we have

$$|\Gamma_1| < \frac{1}{x^{a-b-1}} < \frac{1}{2^{a-b-1}} < \frac{1}{2}.$$

By Lemma 2.5 and the above inequality, we get

$$|\Lambda_1| = |\log(\Gamma_1 + 1)| < \frac{4 \log 2}{2^{a-b}} < 2.8 \exp(-0.69(a - b)).$$

Since $\max\{a, n\} = n$, then inequality (3.13) implies that we can take $X_0 := 2.08 \cdot 10^{33}$. Further, we choose

$$c := 2.8, \quad \delta := 0.69, \quad \beta := \log c_\alpha,$$
$$(\vartheta_1, \vartheta_2) := (\log \alpha, \log x), \quad \vartheta := -\log \alpha / \log x, \quad \psi := \log c_\alpha / \log x.$$

Using Maple, we find that q_{80} satisfies the hypotheses of Lemma 2.3, for all $x \in [2, 20]$. Furthermore, Lemma 2.3 implies the inequality

$$a - b \leq 237$$

in all cases.

Now, assume that $a - b \leq 237$. Let us consider

$$\Lambda_2 := n \log \alpha - b \log x - \log(c_\alpha(x^{a-b} \pm 1)) \quad \text{and} \quad \Gamma_2 := e^{\Lambda_2} - 1.$$

Then for $n \geq 13$, we have

$$|\Gamma_2| < \frac{1}{\alpha^3} < \frac{1}{2},$$

(see (3.8)). By Lemma 2.5, we get

$$|\Lambda_2| = |\log(\Gamma_2 + 1)| < \frac{2 \log 2}{\alpha^{n-10}} < 23.1 \exp(-0.28n).$$

Since $\max\{b, n\} = n$, then inequality (3.13) implies that we can take $X_0 := 2.08 \cdot 10^{33}$. Further, we can choose

$$c := 23.1, \quad \delta := 0.28, \quad \beta_m := -\log(c_\alpha(x^m \pm 1)), \quad 1 \leq m \leq 237,$$

$$(\vartheta_1, \vartheta_2) := (\log\alpha, -\log x) \quad \vartheta := -\log\alpha/\log x, \quad \psi_m = -\log(c_\alpha(x^m \pm 1))/\log x.$$

Again, we use Maple to find that q_{120} satisfies the hypotheses of Lemma 2.3 for all $x \in [2, 20]$ and $m \in [1, 237]$. Moreover, Lemma 2.3 implies that $n \leq 845$ in all cases.

We use Maple for the second time using the new upper bound of n with q_{20}, we get $n \leq 196$ which implies by (3.3) that $a \leq 82$.

Finally, we write a program in Maple to obtain P_n's which are of the form $x^a \pm x^b + 1$ with $2 \leq x \leq 20$, $1 \leq n \leq 196$, and $1 \leq a \leq 82$. One can check that the only solutions of equation (1.1) are those cited in Theorem 1.1. This completes the proof of Theorem 1.1.

4. Perrin numbers of the form $x^a \pm x^b + 1$

In this section, we will prove Theorem 1.2 using the above method to prove Theorem 1.1. For the sake of completeness, we will give almost all of the details.

4.1. The proof of inequality (1.4)

First of all, we will explore a relation between a and n. By combining (2.4) together with (1.2), we get

$$\alpha^{n-1} < E_n = x^a \pm x^b + 1 < x^a + x^b + 1 < 3x^a, \tag{4.1}$$

and

$$\frac{x^a}{2} < x^a - x^b < x^a \pm x^b + 1 = E_n < \alpha^{n+1}. \tag{4.2}$$

By taking logarithms on both sides of inequalities (4.1) and (4.2) and putting them together, we obtain

$$\left(\frac{\log x}{\log\alpha}\right)a - \frac{\log 2}{\log\alpha} - 1 < n < \left(\frac{\log x}{\log\alpha}\right)a + \frac{\log 3}{\log\alpha} + 1.$$

Particularly, using the fact that $x \geq 2$, we conclude that

$$a < n.$$

By using (2.2), we rewrite equation (1.2) into the form of

$$|\alpha^n - x^a| \leq 2|\beta|^n + x^b + 1 < x^{b+2}.$$

Dividing both sides of the last inequality by x^a, we get

$$\left|\alpha^n x^{-a} - 1\right| < x^{-(a-b-2)}. \tag{4.3}$$

Now, we are in a situation to apply Matveev's result (see Lemma 2.1) to the left-hand side of (4.3). The expression on the left-hand side of (4.3) is nonzero, since this expression being zero means that $x^a = \alpha^n$. So $\alpha^n \in \mathbb{Z}$ for some positive integer n, which is false. In order to apply Lemma 2.1, we take $s := 3$,

$$\gamma_1 := \alpha, \quad \gamma_2 := x, \quad b_1 := n, \quad b_2 := -a.$$

For this choice, we have $D = 3$, $h(\gamma_1) = (\log \alpha)/3$, $h(\gamma_2) = \log x$, and $\max\{1, n, a\} = n$. In conclusion, we take $B := n$, $A_1 := 0.3$ and $A_2 := 3 \log x$. By Lemma 2.1, we obtain the following estimate

$$\left|\alpha^n x^{-a} - 1\right| \geq \exp(-1.18 \cdot 10^{11} \cdot \log x \cdot (1 + \log n)). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$a - b < 1.19 \cdot 10^{11}(1 + \log n). \tag{4.5}$$

We now use a second linear form in logarithms by rewriting equation (1.2) in a little different way. Using Binet formula (2.2), we get

$$\left|\alpha^n - \left(x^{a-b} \pm 1\right)x^b\right| \leq 2|\beta|^n + 1 < 2.$$

Dividing both sides of the above inequality by $x^a \pm x^b$, we obtain

$$\left|\left(x^{a-b} \pm 1\right)^{-1} \alpha^n x^{-b} - 1\right| < \frac{2}{x^a \pm x^b} < \frac{1}{\alpha^{n-10}}, \tag{4.6}$$

where we have also used the fact that $x^a \pm x^b > x^a/2$, $\alpha^{n-1} < 3x^a$ and $12 < \alpha^9$.

We observe that the left-hand side of (4.6) is nonzero, otherwise we would get

$$x^a \pm x^b = \alpha^n. \tag{4.7}$$

Conjugating (4.7) in $\mathbb{Q}(\alpha, \beta)$ by using the automorphism of Galois $\sigma := (\alpha\beta)$, we get

$$1 < |x^a \pm x^b| = \beta^n < 1.$$

Now, let us apply again Lemma 2.1 as before but with $s := 3$,

$$\gamma_1 := x^{a-b} \pm 1, \quad \gamma_2 := \alpha, \quad \gamma_3 := x, \quad b_1 := -1, \quad b_2 := n, \quad b_3 := -b.$$

Again here, we take $D = 3$, $B := n$, $A_2 := 0.9$, $A_3 := 3 \log x$ and since (using the properties of absolute logarithmic height and inequality (4.5))

$$h(\alpha_1) < h(x^{a-b}) + \log 2 < 1.2 \cdot 10^{11} \log x(1 + \log n),$$

then we can take $A_1 := 3.6 \cdot 10^{11} \log x(1 + \log n)$. We obtain the following estimate

$$\exp(-7.89 \cdot 10^{24} \cdot (\log x)^2(1 + \log n)^2) \leq \frac{1}{\alpha^{n-10}},$$

which leads us to

$$\frac{n}{(\log n)^2} < 1.13 \cdot 10^{26}(\log x)^2, \tag{4.8}$$

where we used $1 + \log n < 2 \log n$. Finally, by Lemma 2.4 and using fact that $\log\left(1.13 \cdot 10^{26}(\log x)^2\right) < 86 \log x$, for $x \geq 2$, inequality (4.8) gives us

$$n \leq 3.35 \cdot 10^{30}(\log x)^4. \tag{4.9}$$

This proves the first part of Theorem 1.2.

4.2. The solutions of equation (1.2) for $2 \le x \le 20$

Let x be a fixed integer such that $2 \le x \le 20$. The inequality (4.9) becomes

$$n < 2.7 \cdot 10^{32}. \tag{4.10}$$

Now, we will reduce the upper bound of n given by (4.10) as this bound is very large. To do this, we will use several times Lemma 2.2. From inequality (4.3), we put

$$\Lambda_3 := n \log \alpha - a \log x \quad \text{and} \quad \Gamma_3 := e^{\Lambda_3} - 1.$$

Then, for $a - b \ge 3$ and $2 \le x \le 20$, we have

$$|\Gamma_3| < \frac{1}{x^{a-b-1}} < \frac{1}{2^{a-b-2}} < \frac{1}{2}.$$

By Lemma 2.5 and the above inequality, we get

$$|\Lambda_3| = |\log(\Gamma_3 + 1)| < \frac{4 \log 2}{2^{a-b}} < 2.8 \exp(-0.69(a-b)).$$

As $\max\{a, n\} = n$, then inequality (4.10) implies that we take $X_0 := 2.7 \cdot 10^{32}$ and

$$Y_0 := 155.85544\ldots, \qquad c := 2.8, \qquad \delta := 0.69,$$
$$(\vartheta_1, \vartheta_2) := (\log \alpha, -\log x), \qquad \vartheta := -\log \alpha / \log x.$$

Using Maple, we find that

$$A = 1584.$$

So from Lemma 2.2, we deduce that

$$a - b \le 121$$

in all the cases.

Suppose now that $a - b \le 121$. Let us consider

$$\Lambda_4 := n \log \alpha - b \log x - \log(x^{a-b} \pm 1) \quad \text{and} \quad \Gamma_4 := e^{\Lambda_4} - 1.$$

Then for $n \ge 13$, inequality (4.6)) give

$$|\Gamma_4| < \frac{1}{\alpha^3} < \frac{1}{2}.$$

By Lemma 2.5, we get

$$|\Lambda_4| = |\log(\Gamma_4 + 1)| < \frac{2 \log 2}{\alpha^{n-10}} < 23.1 \exp(-0.28n).$$

We know that $\max\{b, n\} = n$, then inequality (3.13) implies that we can take $X_0 := 2.7 \cdot 10^{32}$. Further, we choose

$$c := 23.1, \quad \delta := 0.28, \quad \beta_m := -\log(x^m \pm 1), \quad 1 \le m \le 101,$$

$$(\vartheta_1, \vartheta_2) := (\log \alpha, -\log x), \quad \vartheta := -\log \alpha / \log x, \quad \psi_m := \log(x^m \pm 1)/\log x.$$

With Maple, we find that q_{125} satisfies the hypotheses of Lemma 2.3 for all $x \in [2, 20]$ and $m \in [1, 121]$ except in the cases

$$(a, x) \in \{(1, 1), (1, 3), (1, 9), (2, 5), (2, 3), (2, 9), (3, 3), (3, 9), (3, 7), (4, 15), (4, 17)\}.$$

Furthermore, Lemma 2.3 gives us $n \leq 890$ in all the cases.

In the cases when

$$(a, x) \in \{(1, 1), (1, 3), (1, 9), (2, 5), (2, 3), (2, 9), (3, 3), (3, 9), (3, 7), (4, 15), (4, 17)\},$$

we use Lemma 2.2 and get $n \leq 309$. So, in all the cases we have $n \leq 890$.

Finally, we write a program in Maple to determine E_n's which are of the form of $x^a \pm x^b + 1$ with $2 \leq x \leq 20$, $1 \leq n \leq 890$, $1 \leq b < a \leq 340$. We find that the only solutions of the equation (1.2) are the ones cited in Theorem 1.2. Hence, Theorem 1.2 is completely proved.

Acknowledgements. The authors are grateful to the referee for useful comments to improve this paper.

References

[1] J. Bravo, F. Luca: *Powers of two in generalized Fibonacci sequence*, Rev. Columbiana Math. 46.1 (2012), pp. 67–79.

[2] Y. Bugeaud, M. Mignotte, S. Siksek: *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers*, Annals of Mathematics 163.3 (2006), pp. 969–1018.

[3] A. P. Chaves, D. Marques, A. Togbé: *On the sum of powers of terms of a linear recurrence sequence*, Bull. Braz. Math. Soc. New Series 43.3 (2012), pp. 397–406, doi: https://doi.org/10.1007/s00574-012-0018-y.

[4] N. Irmak, M. Alp: *k-Generalized Fibonacci numbers close to the form* $2^a + 3^b + 5^c$, Acta Math. Univ. Comenianae 86.2 (2017), pp. 279–286.

[5] N. Irmak, L. Szalay: *Tribonacci numbers close to the sum* $2^a + 3^b + 5^c$, Math. Scand. 118.1 (2016), pp. 27–32, doi: https://doi.org/10.7146/math.scand.a-23293.

[6] F. Luca, L. Szalay: *Fibonacci numbers of the form* $p^a \pm p^b + 1$, Fibonacci Quart. 45.2 (2007), pp. 98–103.

[7] F. Luca: *Fibonacci numbers of the form* $k^2 + k + 2$, in: Applications of Fibonacci numbers, Springer, 1999, pp. 241–249.

[8] D. Marques, A. Togbé: *Fibonacci and Lucas numbers of the form* $2^a + 3^b + 5^c$, Proc. Japan Acad. 89.3 (2013), pp. 47–50.

[9] E. M. Matveev: *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbres II*, Izv. Math. 64 (2000), pp. 1217–1269.

[10] Y. Qu, J. Zeng: *Pell and Pell-Lucas numbers of the form* $-2^a - 3^b + 5^c$, Chechoslovak Math. J. 70 (2020), pp. 281–289.

[11] S. E. Rihane, C. A. Adegbindin, A. Togbé: *Fermat Padovan and Perrin numbers*, J. Integer Sequences 23 (2020).

[12] N. Robbins: *Fibonacci numbers of the forms* $px^2 \pm 1$, $px^3 \pm 1$, *where* p *is prime*, in: Applications of Fibonacci numbers, Springer, 1988, pp. 77–88.

[13] S. G. Sánchez, F. Luca: *Linear combinations of factorials and* S*-units in a binary recurrence sequences*, Ann. Math. Québec 38 (2014), pp. 169–188.

[14] N. J. A. Sloane: *The Online Encyclopedia of Integer Sequences*, url: https://oeis.org.

[15] B. M. M. de Weger: *Algorithms for Diophantine equations*, PhD thesis, University of Leiden, 1987. Also appeared as CWI Tract 65, Centre for Mathematics and Computer Science, Amsterdam 1989.

Annales Mathematicae et Informaticae
55 (2022) pp. 172–183
DOI: https://doi.org/10.33039/ami.2022.05.003
URL: https://ami.uni-eszterhazy.hu

On distance signless Laplacian spectral radius of power graphs of cyclic and dihedral groups*

Rezwan Ul Shaban[a], Bilal A. Rather[a], S. Pirzada[a], A. Somasundaram[b]

[a]Department of Mathematics, University of Kashmir, Srinagar, India
rezwanbhat21@gmail.com
bilalahmadrr@gmail.com
pirzadasd@kashmiruniversity.ac.in

[b]Department of General Sciences, Birla Institute of Technology and Science, Pilani, Dubai Campus
asomasundaram@dubai.bits-pilani.ac.in

Abstract. For a finite group $\mathcal{G}$, the power graph $\mathcal{P}(\mathcal{G})$ is a connected simple graph, whose vertex set is the set of elements of $\mathcal{G}$ and two vertices are connected by an edge if and only if one is the power of the other. In this article, we obtain sharp bounds for the distance signless Laplacian spectral radius of the power graphs of cyclic groups, dihedral and dicyclic groups. Furthermore, we characterize the extremal power graphs attaining such bounds and give some open problems.

Keywords: Distance signless Laplacian matrix, spectral radius, cyclic groups, dihedral group, power graphs

AMS Subject Classification: 05C50, 05C12, 15A18

1. Introduction

We follow the text [23] for graph theory terminology and basic definitions. A graph $G = (V(G), E(G))$ (simply written as G) consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the set of unordered pairs of elements of $V(G)$ is the edge set

*This research was supported by the SERB-DST, New Delhi under the research project number MTR/2017/000084.

Submitted: December 11, 2021
Accepted: May 11, 2022
Published online: May 18, 2022

$E(G)$. The *order* n of G is the number of elements in the set $V(G)$ and the *size* m is the number of elements in the set $E(G)$. The *neighbourhood* of $v \in V(G)$, denoted by $N(v)$, is the set of vertices incident on v. The *degree* of v, denoted by d_v, is the number of elements in $N(v)$. A graph G is said to be r regular if the degree of each vertex is r. We assume all our graphs are simple, connected and undirected. An alternating sequence of vertices and edges, beginning and ending with vertices such that no edge is traversed or covered more than once. The walk is said to be open if the initial and terminal vertices are distinct, otherwise closed. An open walk in which no vertex (and therefore no edge) is repeated is called a path and is denoted by P_n. A graph is said to be complete if it contains all possible edges and a complete graph with n vertices is denoted K_n. A graph $G(V, E)$ is said to be bipartite (or 2-partite) if its vertex set can be partitioned into two different sets V_1 and V_2 with $V = V_1 \cup V_2$ such that $uv \in E$ if and only if $u \in V_1$ and $v \in V_2$. A bipartite graph is said to be complete if $uv \in E$ for all $u \in V_1$ and $v \in V_2$. The complete bipartite graph $K_{1,n-1}$ is called a star.

The adjacency matrix of G, denoted by $A(G) = (a_{ij})$ is the matrix of order $n \times n$, defined as

$$A(G) = (a_{ij})_n = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacenct,} \\ 0 & \text{otherwise.} \end{cases}$$

We denote the determinant of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ by $\det(M)$. The characteristic polynomial of the matrix $A(G)$ is $\det(A(G) - xI)$, where I is the identity matrix. Since $A(G)$ is a real symmetric matrix, so the zeros of the polynomial $\det(A(G) - xI)$ are all real and can be ordered. The set of all the eigenvalues including multiplicity is known as the spectrum of $A(G)$ (or simply spectrum of G). The largest eigenvalue of $A(G)$ is called the spectral radius of G. More about the adjacency matrix can be seen in [13].

In a graph G, the *distance* between the two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is defined as the length of the smallest path between them. The *distance matrix* indexed by the vertices of a connected graph G, denoted by $\mathcal{D}(G)$, is defined as

$$\mathcal{D}(G) = (d_{uv})_n = \begin{cases} 0 & \text{if } u = v, \\ d(u, v) & \text{otherwise.} \end{cases}$$

A complete survey of the matrix $\mathcal{D}(G)$ is given in [8]. The *transmission* of the vertex v (or transmission degree), denoted by $\mathrm{Tr}(v)$ (or Tr_v), is defined to be the sum of the distances from v to all other vertices in G, that is, $\mathrm{Tr}(v) = \sum_{u \in V(\mathcal{G})} d(u, v)$. We observe that the transmission of v_i is same as the i^{th} row sum of the matrix $\mathcal{D}(G)$.

Let $\mathrm{Tr}(G) = \mathrm{diag}(\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n)$ be the diagonal matrix of vertex transmissions of G. The authors in [10] introduced the distance Laplacian

$$\mathcal{L}(G) = \mathrm{Tr}(G) - \mathcal{D}(G)$$

and the distance signless Laplacian

$$\mathcal{Q}(G) = \mathrm{Tr}(G) + \mathcal{D}(G)$$

for the distance matrix of a connected graph G. These matrices are real symmetric and positive semi-definite (definite), so the spectrum is real and non negative. In this article, we focus on the matrix $\mathcal{Q}(G)$, and we denote its eigenvalues by ρ_i's. We order them as $\rho_n \leq \rho_{n-1} \leq \cdots \leq \rho_1$, where ρ_1 is known as the distance signless Laplacian spectral radius of G. Since $\mathcal{Q}(G)$ is irreducible, so by *Perron–Frobenius theorem*, ρ_1 is a simple eigenvalue and the entries of its corresponding eigenvector are positive. Further information about the matrix $\mathcal{Q}(G)$ can be seen in [2–7, 9–11, 24–27].

Kelarev and Quinn [19] defined the directed power graph of a semigroup S as a directed graph with vertex set S in which two distinct vertices $x, y \in S$ are joined by an arc from x to y if and only if $x \neq y$ and $y^i = x$, for some positive integer i. Chakrabarty et al. [15] defined the undirected power graph $\mathcal{P}(G)$ of a group G as an undirected graph with vertex set as G and two vertices $x, y \in G$ are adjacent if and only if $x^i = y$ or $y^j = x$, for some $2 \leq i, j \leq n$. Such graphs have valuable applications and are related to the automata theory [20], besides being useful in characterizing the finite groups. More on power graphs can be seen in [1, 14, 15]. Laplacian spectrum of power graphs of finite cyclic and dihedral groups have been investigated in [16], where it is shown that the Laplacian spectral radius of the power graph of any finite group coincides with the order of group $\mathcal{G}$. Panda [22] studied the Laplacian spectral properties including vertex connectivity, Laplacian integrability and others. Spectral properties of the adjacency matrix of $\mathcal{P}(\mathcal{G})$ were investigated in [21]. Other spectral results of the power graphs can be seen in [12, 17].

The identity of the group G is denoted by e. The proper power graph of $\mathcal{P}(G)$, denoted by $\mathcal{P}(G^*) = \mathcal{P}(G \setminus \{e\})$, is obtained by removing the vertex e. Let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ be the cyclic group of integers modulo n. Then by U_n, we denote the set

$$\{\overline{a} \in \mathbb{Z}_n \mid 1 \leq \overline{a} < n,\ \gcd(\overline{a}, n) = 1\}$$

and $U_n^* = U_n \cup \{\overline{0}\}$. $\mathbb{M}_n(\mathbb{F})$ denotes the set of $n \times n$ matrices with entries from the field $\mathbb{F}$. For other undefined notations and terminology, the readers are referred to [13, 18, 23].

The rest of the paper is organized as follows. In Section 2, we give the sharp bounds for the distance signless Laplacian spectral radius of $\mathcal{P}(\mathbb{Z}_n)$ and characterize the power graphs attaining such bounds. In Section 3, we find the distance signless Laplacian spectrum of the power graphs of the dihedral and the dicyclic groups for some special cases. We also obtain the bounds for the distance signless Laplacian spectral radius for these graphs.

2. Distance Laplacian spectral radius of the power graphs of finite cyclic group $\mathbb{Z}_n$

The first result gives the bounds for the largest distance signless Laplacian eigenvalue of the power graph of the finite cyclic group $\mathbb{Z}_n$.

Theorem 2.1. *Let $\mathcal{P}(\mathbb{Z}_n)$ be the power graph of order $n \geq 3$. Then the distance signless Laplacian spectral radius ρ_1 of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the following*

$$\frac{n-2+r_{\min}+\sqrt{D}}{2} \leq \rho_1 \leq \frac{n-2+r_{\max}+\sqrt{D'}}{2},$$

where $D = r_{\min}^2 - (2n-\phi(n))r_{\min} + n^2 + 8n\phi(n) + 4n - 8\phi(n) - 4$, $D' = r_{\max}^2 - (2n-\phi(n))r_{\max} + n^2 + 8n\phi(n) + 4n - 8\phi(n) - 4$, $r_{\min}$ and $r_{\max}$ are the minimum and maximum row sums of $\mathcal{A}$, which is the block matrix of (2.1). Equality occurs if and only if n is a prime power. (Note that D and D' are positive, since they are the roots of the spectral eigenequation of a real symmetric matrix).

Proof. We list the vertices of $\mathcal{P}(\mathbb{Z}_n)$ first by those vertices which are adjacent to every vertex and then by others. Under this labelling, the distance signless Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$ can be partitioned as

$$\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)_{n\times n} = \begin{pmatrix} \big((n-2)I+J\big)_{\phi(n)+1} & J_{(\phi(n)+1)\times(n-\phi(n)-1)} \\ J_{(n-\phi(n)-1)\times(\phi(n)+1)} & \mathcal{A}_{n-\phi(n)-1} \end{pmatrix}, \tag{2.1}$$

where I is the identity matrix and J is the matrix of all ones. Clearly, the constant row sum of $\big((n-2)I+J\big)_{\phi(n)+1}$, $J_{(\phi(n)+1)\times(n-\phi(n)-1)}$ and $J_{(n-\phi(n)-1)\times(\phi(n)+1)}$ are $n+\phi(n)+1$, $n-\phi(n)-1$ and $\phi(n)+1$, respectively. Let $r_{\min}$ and $r_{\max}$ be the minimum and the maximum row sums of the matrix $\mathcal{A}_{n-\phi(n)-1}$. Then, we know that they are bounded below by the constant row sum of $J_{(n-\phi(n)-1)\times(\phi(n)+1)}$ and we take $r_{\min}-\phi(n)-1$ and $r_{\max}-\phi(n)-1$ as the minimum and the maximum row sums of $\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)$. As $\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)_{n-\phi(n)-1}$ is an irreducible matrix, so by Perron–Frobenius theorem, the signless Laplacian spectral radius is simple and its corresponding eigenvector, say X, has positive entries. Let $X = (x_1, x_2, \dots, x_n)^\top$ and assume that $x_i = \min_{1\leq k\leq \phi(n)+1} x_k$ and $x_j = \min_{\phi(n)+1<k\leq n} x_k$. Therefore, taking the i^{th} eigenvalue equation of $\mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n)\big)X = \rho_1 X$ and using the fact that

$$q_{ik} = \begin{cases} n-1 & \text{if } i=k, \\ 1 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}\rho_1 x_i &= q_{i1}x_1 + q_{i2}x_2 + \cdots + q_{i(\phi(n)+1)}x_{(\phi(n)+1)} \\ &\quad + q_{i(\phi(n)+2)}x_{(\phi(n)+2)} + \cdots + q_{in}x_n \\ &\geq \phi(n)x_i + (n-1)x_i + (n-\phi(n)-1)x_j,\end{aligned}$$

which is equivalent to

$$(\rho_1 - \phi(n) - n + 1)x_i \geq (n - \phi(n) - 1)x_j. \tag{2.2}$$

Also, taking the j^{th} eigenvalue equation, we have

$$\begin{aligned}\rho_1 x_j &= q_{j1}x_1 + q_{j2}x_2 + \cdots + q_{j(\phi(n)+1)}x_{(\phi(n)+1)} \\ &\quad + q_{j(\phi(n)+2)}x_{(\phi(n)+2)} + \cdots + q_{jn}x_n \\ &\geq (\phi(n)+1)x_i + (r_{\min} - \phi(n) - 1)x_j,\end{aligned}$$

which implies that

$$(\rho_1 - r_{\min} + \phi(n) + 1)x_j \geq (\phi(n) + 1)x_i. \tag{2.3}$$

Thus, from Equations (2.2) and (2.3), we obtain

$$\rho_1^2 - (n + r_{\min} - 2)\rho_1 + r_{\min}(n + \phi(n) - 1) - 2n\phi(n) - 2n + 2\phi(n) + 2 \geq 0.$$

So, the lower bound follows

$$\rho_1 \geq \frac{n - 2 + r_{\min} + \sqrt{r_{\min}^2 - (2n - \phi(n))r_{\min} + n^2 + 8n\phi(n) + 4n - 8\phi(n) - 4}}{2}.$$

Again, letting $x_i = \max_{1 \leq k \leq \phi(n)+1} x_k$ and $x_j = \max_{\phi(n)+1 < k \leq n} x_k$, and proceeding as above, we have

$$\rho_1^2 - (n + r_{\max} - 2)\rho_1 + r_{\max}(n + \phi(n) - 1) - 2n\phi(n) - 2n + 2\phi(n) + 2 \leq 0$$

and the upper bound for ρ_1 of $\mathcal{Q}(\mathcal{P}(\mathbb{Z}_n))$ follows.

Now, equality occurs in both the cases if and only $r_{\min} = r_{\max}$, which is possible if and only if $G \cong K_n$ and hence the equality holds if and only if $n = p^{n_1}$, where p is prime and n_1 is a positive integer. □

The following result [13] gives a relation between the eigenvalues of a symmetric matrix and its principal submatrix.

Theorem 2.2 (Interlacing Theorem)**.** *Let $M \in \mathbb{M}_n(\mathbb{R})$ be the real symmetric matrix and A be its principal submatrix of order m, $(m \leq n)$, respectively. Then the eigenvalues of M and A satisfy the following relation*

$$\lambda_{i+n-m}(M) \leq \lambda_i(A) \leq \lambda_i(M), \quad \textit{with } 1 \leq i \leq m.$$

The next result gives the lower bounds for the largest and the second largest distance signless Laplacain eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ in terms of the maximum transmission degree and the second maximum transmission degree.

Theorem 2.3. *Let $\mathcal{P}(\mathbb{Z}_n)$ be the power graph of $\mathbb{Z}_n$ having the maximum transmission degree $\mathrm{Tr}_{\max}$ and the second maximum transmission degree $\mathrm{Tr}^2_{\max}$. Then*

$$\rho_1 \geq \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 4}\right)$$

and

$$\rho_1 \geq \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 16}\right),$$

according as the two vertices of maximum and second maximum transmission degree are adjacent or non-adjacent.

Proof. Assume that $n \geq 3$ and let v_1 and v_2 be the vertices having the maximum transmission degree $\mathrm{Tr}_{\max}$ and the second maximum transmission degree $\mathrm{Tr}^2_{\max}$, respectively. We have the following two possibilities.
(i). Suppose that v_1 and v_2 are adjacent. Then it is clear that $d(v_1, v_2) = 1$. Now, consider the principal 2×2 submatrix

$$A = \begin{pmatrix} \mathrm{Tr}_{\max} & 1 \\ 1 & \mathrm{Tr}^2_{\max} \end{pmatrix}.$$

By using Theorem 2.2, we have

$$\rho_1(\mathcal{P}(\mathbb{Z}_n)) \geq \rho_1(A) = \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 4}\right).$$

(ii). If v_1 and v_2 are not adjacent, then as power graphs of finite groups are of diameter at most two, so $d(v_1, v_2) = 2$. Again, consider the principal 2×2 submatrix

$$B = \begin{pmatrix} \mathrm{Tr}_{\max} & 2 \\ 2 & \mathrm{Tr}^2_{\max} \end{pmatrix}.$$

Thus, by Theorem 2.2, we obtain

$$\rho_1(\mathcal{P}(\mathbb{Z}_n)) \geq \rho_1(B) = \frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} + \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 16}\right).$$

With the same notations and procedure as in Theorem 2.3, we see that the second largest distance signless Laplacian eigenvalues are bounded below by

$$\frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} - \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 4}\right)$$

and

$$\frac{1}{2}\left(\mathrm{Tr}_{\max} + \mathrm{Tr}^2_{\max} - \sqrt{(\mathrm{Tr}_{\max} - \mathrm{Tr}^2_{\max})^2 + 16}\right). \qquad \square$$

3. Distance signless Laplacian eigenvalues of dihedral and dicyclic groups

Let $M \in \mathbb{M}_n(\mathbb{R})$ be partitioned in the blocks matrices B_j and let Q be the new matrix whose ij^{th} entry is the average row sum of B_i block. Then Q is called the quotient matrix, and the eigenvalues of M interlace the eigenvalues of Q. In case row sums of each block are some constants, the partition is said to be equitable, and in such a situation, each eigenvalue of Q is an eigenvalue of M.

Let G be any graph of order n and let $G_i(V_i, E_i)$ be graphs of order m_i, where $i = 1, \dots, n$. The *joined union* of graphs $G_1, G_2, \dots, G_n$, denoted by $G[G_1, G_2, \dots, G_n]$, is the union of graphs $G_1, G_2, \dots, G_n$ together with the edges from every vertex of G_i to each vertex of G_j whenever v_i and v_j are adjacent in G.

The next result gives the distance signless Laplacian spectrum of $G[G_1, \dots, G_n]$ together with the eigenvalues of the quotient matrix, where G_i is an r_i regular graph.

Theorem 3.1 ([25]). *Let G be a graph of order n having vertex set $V(G) = \{v_1, \dots, v_n\}$. Let G_i be r_i regular graphs of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{in_i}$, where $i = 1, 2, \dots, n$. The distance signless Laplacian spectrum of the joined union graph $G[G_1, \dots, G_n]$ of order $N = \sum_{i=1}^{n} n_i$ consists of the eigenvalues $2n_i + n'_i - r_i - \lambda_{ik} - 4$ for $i = 1, \dots, n$ and $k = 2, 3, \dots, n_i$, where $n'_i = \sum_{k=1, k\neq i}^{n} n_k d_G(v_i, v_k)$. The remaining n eigenvalues are given by the equitable quotient matrix*

$$Q = \begin{pmatrix} 4n_1 + n'_1 - 2r_1 - 4 & n_2 d_G(v_1, v_2) & \dots & n_n d_G(v_1, v_n) \\ n_1 d_G(v_2, v_1) & 4n_2 + n'_2 - 2r_2 - 4 & \dots & n_n d_G(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ n_1 d_G(v_n, v_1) & n_2 d_G(v_n, v_2) & \dots & 4n_n + n'_n - 2r_n - 4 \end{pmatrix}.$$

Next, we find the distance Laplacian spectrum of the dihedral group and the dicyclic group for some particular values of n. The dihedral group of order $2n$ and the dicyclic group of order $4n$ are denoted and presented as follows:

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle,$$
$$Q_n = \langle a, b \mid a^{2n} = e, b^2 = a^n, ab = ba^{-1} \rangle.$$

If n is a power of 2, then Q_n is called the *generalized quaternion group* of order $4n$.

Now, we obtain the distance signless Laplacian spectrum of the power graph of the dihedral and the dicyclic group for some special cases and obtain bounds for the spectral radius.

Proposition 3.2. *If n is a prime power, then the distance signless Laplacian spectrum of $\mathcal{P}(D_{2n})$ is*

$$\Big\{(3n-2)^{[n-2]}, (4n-5)^{[n-1]}, x_1 \geq x_2 \geq x_3\Big\},$$

where x_i, for $i = 1, 2, 3$ are the zeros of the following polynomial

$$x^3 - (12n-9)x^2 + \big(44n^2 - 106n + 24\big)x - 48n^3 + 188n^2 - 140n + 20.$$

Proof. As $\langle a\rangle$ generates the cyclic group of order n, its power graph behaves as that of $\mathcal{P}(\mathbb{Z}_n)$. The other n elements of D_{2n} in $\mathcal{P}(D_{2n})$ form a star graph with identity as the vertex of maximum degree. Therefore, the power graph of D_{2n} can be obtained from the power graph $\mathcal{P}(\mathbb{Z}_n)$ by adding the n pendent vertices at the identity vertex e. If $n = p^{m_1}$, where m_1 is a positive integer, then

$$\mathcal{P}(D_{2n}) = P_3[K_{n-1}, K_1, \overline{K}_n],$$

that is, $\mathcal{P}(D_{2p^{m_1}})$ is the *pineapple graph*, the graph obtained from K_n by appending vertices of degree 1 at some vertex of K_n. Now, the value of n_i''s are given by $n_1' = 2n+1$, $n_2' = 2n-1$ and $n_3' = 2n-1$. Thus, by Theorem 3.1, the distance signless Laplacian spectrum of $\mathcal{P}(D_{2n})$ consists of the eigenvalue

$$2n_i + n_i' + r_i + \lambda_{1k} - 4 = 2(n-1) + 2n + 1 - n + 2 + 1 - 4 = 3n - 2,$$

with multiplicity $n-2$. Similarly, the other distance signless Laplacian eigenvalue is $4n-5$ with multiplicity $n-1$ and the remaining three distance signless Laplacian eigenvalues are the eigenvalues of the following matrix

$$\begin{pmatrix} 4n-3 & 1 & 2n \\ n-1 & 2n-11 & n \\ 2n-2 & 1 & 6n-5 \end{pmatrix}. \qquad \square$$

The following lemma gives an equivalent method for finding determinant (det) of a matrix.

Lemma 3.3 ([18])**.** *Let M_1, M_2, M_3 and M_4 be respectively $p\times p$, $p \times q$, $q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible. Then*

$$\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3M_1^{-1}M_2) = \det(M_4)\det(M_1 - M_2M_4^{-1}M_3),$$

where $M_4 - M_3M_1^{-1}M_2$ and $M_1 - M_2M_4^{-1}M_3$ are called Schur complement of M_1 and M_4, respectively.

The next result gives the distance signless Laplacian spectrum of the generalized quaternions.

Proposition 3.4. *Let $n = 2^{m_1}$, where m_1 is a positive integer. Then the distance signless Laplacian eigenvalues of $\mathcal{P}(Q_n)$ are the simple eigenvalue $4n-2$, the eigenvalue $6n-2$ with multiplicity $2n-3$, the eigenvalue $8n-4$ with multiplicity n, the eigenvalue $8n-6$ and the two zeros of the polynomial*

$$\det(M_4)\det(M_1 - M_2M_4^{-1}M_3).$$

Proof. The identity element is always adjacent to every other vertex of $\mathcal{P}(Q_n)$. In particular, if n is a power of 2, then it can be seen that a^n is also adjacent to all other vertices of $\mathcal{P}(Q_n)$. By using these observations, the power graph $\mathcal{P}(Q_n)$ can be written as

$$\mathcal{P}(Q_n) = S[K_2, K_{2n-2}, \underbrace{K_2, K_2, \dots, K_2}_{n}],$$

where $S = K_{1,n+1}$. Using Theorem 3.1, we see that $2n_1 + n_1' - n_1 + 1 + 1 - 4 = n_1 + n_1' - 2 = 4n - 2 + 2 - 2 = 4n - 2$ is the simple distance signless Laplacian eigenvalue of $\mathcal{P}(Q_n)$. Similarly, $6n-2$ and $8n-6$ are the distance signless Laplacian eigenvalues with multiplicity $2n - 3$ and n, respectively. The remaining distance signless Laplacian eigenvalues of $\mathcal{P}(Q_n)$ are the eigenvalues of following matrix

$$\begin{pmatrix} 4n & 2n-2 & 2 & \dots & 2 & 2 \\ 2 & 8n-4 & 2 & \dots & 2 & 2 \\ 2 & 4n-4 & 8n-4 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4n-4 & 2 & \dots & 8n-4 & 2 \\ 2 & 4n-4 & 2 & \dots & 2 & 8n-4 \end{pmatrix}.$$

Let

$$M_1 = \begin{pmatrix} 4n & 2n-2 \\ 2 & 8n-2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2 & \dots & 2 & 2 \\ 2 & \dots & 2 & 2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 2 & \dots & 2 & 2 \\ 4n-4 & \dots & 4n-4 & 4n-4 \end{pmatrix}^{\top},$$

$$M_4 = \begin{pmatrix} 8n-4 & \dots & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots \\ 2 & \dots & 8n-4 & 2 \\ 2 & \dots & 2 & 8n-4 \end{pmatrix}.$$

Now, by Lemma 3.3, it is easy to verify that the polynomial $\det(M_4 - xI)$ has a zero $8n - 6$ with multiplicity n. The remaining two distance signless Laplacian eigenvalues are the zeros of the following polynomial

$$\det(M_4)\det(M_1 - M_2 M_4^{-1} M_3). \qquad \square$$

The distance signless Laplacian matrix of $\mathcal{P}(D_{2n})$ can be written as

$$\mathcal{Q}\big(\mathcal{P}(D_{2n})\big) = \begin{pmatrix} \mathcal{Q}\big(\mathcal{P}(\mathbb{Z}_n) + \mathcal{A}\big) & B_n \\ B_n' & C_n \end{pmatrix},$$

where

$$\mathcal{A} = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 2n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2n \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 4n-3 & 2 & \cdots & 2 \\ 2 & 4n-3 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 4n-3 \end{pmatrix}$$

As C_n is invertible, so by Schur's Lemma 3.3,

$$\det(C - xI)\det((\mathcal{Q}(\mathcal{P}(\mathbb{Z}_n) + \mathcal{A}) - xI) - (B - xI)\det(C - xI)^{-1}(B - xI)$$

gives the characteristic polynomial of the matrix $\mathcal{Q}(\mathcal{P}(Q_{2n}))$. Clearly, $x = 4n - 5$ is a zero of the characteristic polynomial $\det(C - xI)$ with multiplicity n.

4. Conclusion

In general, to find all the distance signless Laplacian eigenvalues of a power graph of any group is difficult. So in this regard, we have obtained the bounds on the largest distance signless Laplacian eigenvalue of the power graph of the finite cyclic group $\mathbb{Z}_n$. However to find the bounds for other eigenvalues of such power graphs remains open. Also, we find some distance signless Laplacian eigenvalues (including bounds) of the power graphs of D_{2n} and Q_n, for some special cases. Though in general, the distance signless Laplacian eigenvalues of these graphs remain challenging, we need to devise more techniques and information about the structure of the power graphs, so that more distance signless Laplacian eigenvalues (if not all) need to be obtained. For the remaining distance signless Laplacian eigenvalues, best possible bounds need to be established. All other distance signless Laplacian spectral parameters like distance signless Laplacian energy, distance signless Laplacian spread, distance signless Laplacian Estrada index and others can be discussed for power graphs of finite groups.

Acknowledgements. We are highly grateful to the anonymous referee for his useful comments. This research is supported by SERB-DST, New Delhi under the research project number CRG/2020/000109.

References

[1] J. Abawajy, A. Kelarev, M. Chowdhury: *Power graphs: A survey*, Electronic J. Graph Theory Appl. 1.2 (2013), pp. 125–147.

[2] A. Alhevaz, M. Baghipur, Y. Alizadeh, S. Pirzada: *On eigenvalues of the reciprocal distance signless Laplacian matrix of graphs*, Asian European J. Mathematics 14.10 (2021), p. 2150176.

[3] A. Alhevaz, M. Baghipur, H. A. Ganie, Y. Shang: *Bounds for the generalized distance eigenvalues of a graph*, Symmetry 11.12 (2019), p. 1529, doi: https://doi.org/10.3390/sym11121529.

[4] A. Alhevaz, M. Baghipur, S. Pirzada: *On distance signless Laplacian Estrada index and energy of graphs*, Kragujevac J. Mathematics 45.6 (2021), pp. 837–858.

[5] A. Alhevaz, M. Baghipur, S. Pirzada, Y. Shang: *Some inequalities involving the distance signless Laplacian eigenvalues of graphs*, Transactions on Combinatorics 10.1 (2021), pp. 9–29.

[6] A. Alhevaz, M. Baghipur, Y. Shang: *Merging the spectral theories of distance Estrada and distance signless Laplacian Estrada indices of graphs*, Mathematics 7.10 (2019), p. 995, doi: https://doi.org/10.3390/math7100995.

[7] A. Alhevaz, M. Baghipur, Y. Shang: *On generalized distance Gaussian Estrada index of graphs*, Symmetry 11.10 (2019), p. 1276, doi: https://doi.org/10.3390/sym11101276.

[8] M. Aouchiche, P. Hansen: *Distance spectra of graphs: a survey,* Linear Algebra Appl. 458 (2014), pp. 301–386.

[9] M. Aouchiche, P. Hansen: *On the distance signless Laplacian of a graph*, Linear Multilinear Algebra 64 (2016), pp. 1113–1123.

[10] M. Aouchiche, P. Hansen: *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl. 439 (2013), pp. 21–33.

[11] M. Baghipur, M. Ghorbani, H. A. Ganie, Y. Shang: *On the second-largest reciprocal distance signless Laplacian eigenvalue*, Mathematics 9.5 (2021), pp. 1113–1123, doi: https://doi.org/10.3390/math9050512.

[12] S. Banerjee, A. Adhikari: *Signless Laplacian spectra of power graphs of certain finite groups*, AKCE Int. J. Graphs Combinatorics 17.1 (2020), pp. 356–366.

[13] A. E. Brouwer, W. H. Haemers: *Spectra of Graphs*, New York: Springer, 2010.

[14] P. J. Cameron, S. Ghosh: *The power graphs of a finite group*, Discrete Math. 311.1 (2011), pp. 1220–1222.

[15] I. Chakrabarty, M. Ghosh, M. K. Sen: *Undirected power graph of semigroups*, Semigroup Forum 78 (2009), pp. 410–426.

[16] S. Chattopadhyay, P. Panigrahi: *On Laplacian spectrum of power graphs of finite cyclic and dihedral groups*, Linear Multilinear Algebra 63.7 (2015), pp. 1345–1355.

[17] A. Hamzeh: *Signless and normalized Laplacian spectrums of the power graph and its supergraphs of certain finite groups*, J. Indonesian Math. Soc. 24.1 (2018), pp. 61–69.

[18] R. Horn, C. Johnson: *Matrix Analysis*, London: Cambridge University Press, 2013.

[19] A. V. Kelarev, S. J. Quinn: *Directed graphs and combinatorial properties of semigroups*, J. Algebra 251 (2002), pp. 16–26.

[20] A. V. Kelarev, S. J. Quinn: *Graph algebras and automata*, New York: Marcel Dekker, 2003.

[21] Z. Mehranian, A. Gholami, A. R. Ashrafi: *The spectra of power graphs of certain finite groups*, Linear Multilinear Algebra 65.5 (2016), pp. 1003–1010.

[22] R. P. Panda: *Laplacian spectra of power graphs of certain finite groups*, Graphs Combinatorics 35.5 (2019), pp. 1209–1223.

[23] S. Pirzada: *An Introduction to Graph Theory*, Hyderabad: Universities Press, Orient Blackswan, 2012.

[24] S. Pirzada, B. A. Ganie, A. Alhevaz, M. Baghipur: *On sum of the powers of distance signless Laplacian eigenvalues of graphs*, J. Pure and Applied Mathematics 51.3 (2020), pp. 1143–1163.

[25] S. Pirzada, B. A. Rather, M. Aijaz, T. A. Chishti: *Distance signless Laplacian spectrum of graphs and spectrum of zero divisor graphs of* $\mathbb{Z}_n$, Linear MultiLinear Algebra (2020), doi: https://doi.org/10.1080/03081087.2020.1838425.

[26] B. A. Rather, S. Pirzada, Z. Guofei: *On distance Laplacian spectra of power graphs of finite groups*, Acta Math. Sinica (2022), to appear.

[27] B. A. Rather, S. Pirzada, T. A. Naikoo: *On distance signless Laplacian spectra of power graphs of the integer modulo group*, Art Discrete Appl. Math. (2022), doi: https://doi.org/10.26493/2590-9770.1393.2be.

Annales Mathematicae et Informaticae
55 (2022) pp. 184–188
DOI: https://doi.org/10.33039/ami.2022.11.003
URL: https://ami.uni-eszterhazy.hu

A note on the trace of Frobenius for curves of the form $y^2 = x^3 + dx$

P. G. Walsh

University of Ottawa
gwalsh@uottawa.ca

Abstract. An explicit description of the trace of Frobenius is given for any elliptic curve over $\mathbb{Q}$ of the form $y^2 = x^3 + dx$. This description leads to an algorithm which computes the trace at a cost of one modular exponentiation.

Keywords: Elliptic curve, trace of Frobenius

AMS Subject Classification: 11G05

1. Introduction

One of the more notable problems currently being pursued in Number Theory is the conjecture attributed to Lang and Trotter [7] on the distribution of primes with given trace of Frobenius. Since the appearance of their paper, considerable effort has been made to quantify the distribution, with many notable achievements in this regard. Results in the literature on this topic vary from averaging results, stemming from the ground-breaking work of David and Pappalardi [2], quantitative bounds dependent upon GRH-type assumptions due to Cocojaru and Murty [1], connections between the distribution of the trace for CM curves and the Hardy–Littlewood conjecture by Ji and Qin [4], and numerous other fascinating lines of research. The reader may wish to consult the survey paper by Katz [5] for more on the Lang–Trotter conjecture.

Although the literature on this topic has grown substantially, with many results *about* the Lang–Trotter conjecture, it was a curiosity of this author as to the depth of the conjecture, which we take a moment to elaborate on now. Let us consider what could be considered a simplest possible case, namely, the elliptic curve E given by $y^2 = x^3 + x$ and trace equal to 2. E has complex multiplication, meaning that its endomorphism ring is $End(E) = \mathbb{Z}[i]$, and the characteristic polynomial of the Frobenius endomorphism $c_E(X) = X^2 - 2X + p$ therefore splits in $End(E)$. It

Submitted: October 20, 2021
Accepted: November 5, 2022
Published online: November 14, 2022

follows that the discriminant $2^2 - 4p$ of c_E is a square $(a+bi)^2$ in $\mathbb{Z}[i]$, from which it follows that $a = 0$, b is even, and $p = (b/2)^2 + 1$. We now see that the distribution of primes for which this curve has trace equal to 2, i.e. the Lang-Trotter conjecture for this instance, is tantamount to the distribution of primes of the form $x^2 + 1$, a notoriously and profoundly difficult problem in analytic number theory. We remark that the considerations made here were alluded to in the opening remarks in a paper by Murty [8].

As a consequence of this observation, our interest in this research area moved swiftly to simply understanding the trace for curves of the form $y^2 = x^3 + dx$. The primary goal of this paper is to give an exact description of the trace, and show that for a given coefficient d and prime p not dividing d, one can compute the trace very efficiently using this description.

2. The main result

As noted above, we are interested in the family of curves $y^2 = x^3 + dx$, with $d \in \mathbb{Z}$, and we wish to determine the trace of the curve, denoted a_p, at a prime p. Note that we need to restrict to those p not dividing d, for otherwise the curve is singular. If $p = 2$, then we need only consider $d = 1$, and in this case $a_p = 0$. Similarly, if p is any prime satisfying $p \equiv 3 \pmod 4$, then by Deuring's reduction theorem (for example, see Theorem 12 in Ch. 13 of [6]), or the method given in Example 4.5 on p. 144 of [9], the curve in question is supersingular, that is, $a_p = 0$. Therefore, we may restrict our attention to primes p satisfying $p \equiv 1 \pmod 4$.

In what follows, p will represent a prime which is 1 modulo 4. We will denote by a and b integers such that $p = a^2 + b^2$, a odd, and $b > 0$ even. However, a will not necessarily be positive, as it will be specified throughout by the congruence $a \equiv 1 \pmod 4$.

Let G denote the multiplicative group $\mathbb{Z}/p\mathbb{Z}^*$. Then G is a cyclic group whose order is a multiple of 4. Let H denote the cyclic subgroup of G consisting of the 4-th powers of all elements in G. Then H has order $(p-1)/4$, and G/H is a cyclic group of order 4. Because of the congruence $a^2 \equiv -b^2 \pmod p$, the 4-th roots of unity in G are 1, -1, a/b, and b/a. Therefore, if u is an element in G satisfying $u^{(p-1)/4} \equiv a/b \pmod p$ or $u^{(p-1)/4} \equiv b/a \pmod p$, then uH generates G/H. In what follows, a non-square element $u \in G$ will be chosen specifically by the congruence

$$u^{(p-1)/4} \equiv a/b \pmod p. \tag{2.1}$$

Theorem 2.1. *Let $d \in \mathbb{Z}$, p a prime not dividing d, $p \equiv 1 \pmod 4$, and a and b integers for which $p = a^2 + b^2$ as specified above. Let $u \in G$ be an element for which (2.1) holds, and H as above. Let E_d be the elliptic curve given by $y^2 = x^3 + dx$ and a_p the trace of E_d at p. Then $a_p \in \{2a, -2a, 2b, -2b\}$.*

More precisely,

$$a_p = \begin{cases} 2a & \textit{if } d \pmod p \in H \\ -2a & \textit{if } d \pmod p \in u^2H \\ 2b & \textit{if } d \pmod p \in uH \\ -2b & \textit{if } d \pmod p \in u^3H. \end{cases}$$

Remark. From a computational perspective, one can compute the trace of E_d at a prime p very quickly by evaluating the modular exponentiation $d^{(p-1)/4} \pmod p$, as the value of this expression will be one of 1, -1, $a/b \pmod p$ or $b/a \pmod p$, explicitly determining the value of the trace as $2a$, $-2a$, $2b$ or $-2b$ respectively.

Proof. The proof of the assertion concerning the set of possible values of the trace is basically identical to the argument given in the introduction, and so we leave that for the reader to verify.

We will now proceed to each of the possible values of the trace, starting with $2a$, and for the sake of pedagogy, we will describe two different ways to arrive at this result.

Let d be any integer for which $d \pmod p$ is in H. The map from E_d to E_1 given by $(x, y) \to (d^2x, d^3y)$ evidently shows that these two curves are isomorphic over $GF(p)$, hence have the same order modulo p. Thus we focus on computing the trace of E_1 at p.

What would be considered a more standard approach to this is to appeal once again to Example 4.5 on p.144 of [9], wherein Silverman tersely points out that the trace is given by the binomial coefficient $\binom{(p-1)/2}{(p-1)/4}$, from which the result follows from a congruence of Gauss, which is given explicitly in Theorem 7.1 in the seminal paper by Hudson and Williams [3].

Another somewhat more long-winded way to arrive at this result is as follows. Firstly, notice that since $2a \equiv 2 \pmod 8$, the desired result is a straightforward deduction from the equation

$$|E_1 \pmod p| = p + 1 - a_p,$$

provided that we can prove $|E_1 \pmod p| \equiv 0 \pmod 8$ for $p \equiv 1 \pmod 8$ and $|E_1 \pmod p| \equiv 4 \pmod 8$ for $p \equiv 5 \pmod 8$.

In order to prove these two congruences, we combine certain facts involving the points of order two on $E_1 \pmod p$. Firstly, as is well known, the group structure of this group is of the form $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ with n_1 a divisor of n_2. The desired result will follow from the observation that for $p \equiv 1 \pmod 8$, $n_1 \equiv n_2 \equiv 0 \pmod 4$, whereas for $p \equiv 5 \pmod 8$, $n_1 \equiv n_2 \equiv 2 \pmod 4$.

Briefly, it is evident that the polynomial $x^3 + x$ has three distinct roots, say r_1, r_2, r_3, in $GF(p)$, and the resulting points $(r_i, 0)$ on E_1 are points of order 2. Using the doubling formula on E_1, we can compute precisely when these points are in $2E_1$. In fact, if $2(x, y) = (r_i, 0)$, then x is a root of the polynomial $(x - 1)(x + 1)(x^4 + 6x^2 + 1)$. However, for $x = 1$ or $x = -1$ to give rise to a point on $E_1 \pmod p$, the value of $x^3 + x$ must be a square in $GF(p)$, from which it follows that $p \equiv 1 \pmod 8$. In summary then, $E_1 \pmod p$ has points of order 4 only for $p \equiv 1 \pmod 8$ and not for $p \equiv 5 \pmod 8$. The remark above concerning the group structure now proves the desired $\pmod 8$ congruences above.

We now consider the second case, namely the set of curves with trace $-2a$. We will show that if u is a non-square modulo p, and $d \pmod p \in u^2H$, then the trace of E_d at p is $-2a$. As argued in the previous case, we need only consider the curve E_{u^2}. Our approach will be to compare points on $E_1 \pmod p$ and $E_{u^2} \pmod p$. Let C_1, respectively C_2, denote the number of $x \in GF(p)$ for which x^3+x, respectively x^3+u^2x, is a non-zero square in $GF(p)$. Then $|E_1 \pmod p| = 4+2C_1$ and $|E_{u^2} \pmod p| = 4+2C_2$. We forego displaying the computations, but it is straightforward to verify that because u is a non-square, $\left(\frac{x^3+x}{p}\right) = -\left(\frac{(ux)^3+u^2(ux)}{p}\right)$. Finally, a simple counting exercise gives the relation $C_1+C_2 = p-3$, from which it follows that $|E_{u^2} \pmod p| = p+1+2a$.

We wish to remark that in the last step of the proof above, multiplication by u can be thought of as flipping x, like a light switch. It is an illuminating way to think of the proof.

As the fourth case follows from the third case in exactly the same way that the second case followed from the first case, we are left only to deal with the third case. For this, we will use the observation made by Silverman in Example 4.5 on p.144 of [9], but provide the reader with a little more to go on.

By the remark used earlier concerning the fact that all curves in the same class mod H are isomorphic over $GF(p)$, we may restrict our attention to the curve E_u given by $y^2 = x^3+ux$, where u is a fixed non-square in $GF(p)$ satisfying (2.1). We note that for for a fixed non-zero $x \in GF(p)$, the value of $1+\left(\frac{x^3+ux}{p}\right)$ is either 0 if x does not give rise to a point on the curve, 1 if x is a root of the cubic giving rise to 1 point, or 2 is x gives rise to 2 points with y coordinates of opposite sign. Therefore, counting 1 for the point at infinity, we have that

$$|E_u \pmod p| = 1+\sum_{x=0}^{p-1} 1+\left(\frac{x^3+ux}{p}\right) = p+1+\sum_{x=0}^{p-1}(x^3+ux)^{(p-1)/2}.$$

Therefore, the trace of interest a_p is explicitly given by this last summand but with opposite parity. Continuing from above by expanding the polynomials, switching order of summation, and pulling out common factors, we see that

$$\begin{aligned} a_p &= -\sum_{x=0}^{p-1}\sum_{i=0}^{(p-1)/2}\binom{(p-1)/2}{i}x^{3i}(ux)^{(p-1)/2-i} \\ &= -\sum_{i=0}^{(p-1)/2}\binom{(p-1)/2}{i}u^{(p-1)/2-i}\left(\sum_{x=0}^{p-1}(x)^{(p-1)/2+2i}\right). \end{aligned}$$

A closer look at the far right term in this last expression shows that for $i \neq (p-1)/4$, the sum represents possibly multiple copies of a complete sum over a non-trivial subgroup of $\mathbb{Z}/p\mathbb{Z}^*$, and hence must sum to 0 modulo p. We now use the congruence quoted above from [3], together with our assumption on the choice of u, and the fact that $a/b \equiv -b/a \pmod p$, to deduce finally that

$$a_p = -\binom{(p-1)/2}{(p-1)/4} u^{(p-1)/4} \equiv -2a(a/b) \equiv -2a(-b/a) \equiv 2b \pmod p. \qquad \square$$

Acknowledgements. The author would like to thank Chantal David, Ram Murty and Kenneth Williams for their valuable insights during the course of this work.

References

[1] A. Cojocaru, M. Murty: *Cyclicity of elliptic curves mod p and elliptic curve analogues of Linnik's problem*, Math. Annalen 330 (2005), pp. 601–625, doi: https://doi.org/10.1007/s00208-004-0562-x.

[2] C. David, F. Pappalardi: *Average Frobenius distributions of elliptic curves*, Internat. Math. Res. Notices 4 (1999), pp. 165–183, doi: https://doi.org/10.1155/S1073792899000082.

[3] R. Hudson, K. Williams: *Binomial coefficients and Jacobi sums*, Trans. Amer. Math. Soc. 281 (1984), pp. 431–505, doi: https://doi.org/10.1090/S0002-9947-1984-0722761-X.

[4] Q. Ji, H. Qin: *CM elliptic curves and primes captured by quadratic polynomials*, Asian J. Math. 18 (2014), pp. 707–726, doi: https://doi.org/10.4310/AJM.2014.v18.n4.a7.

[5] N. Katz: *Lang-Trotter revisited*, Bull. American Math. Soc. 46 (2009), pp. 413–457, doi: https://doi.org/10.1090/S0273-0979-09-01257-9.

[6] S. Lang: *Elliptic Functions*, New York: Springer-Verlag, 1987, doi: https://doi.org/10.1007/978-1-4612-4752-4.

[7] S. Lang, H. Trotter: *Frobenius Distributions in GL2-extensions*, Berlin: Springer-Verlag, 1976, doi: https://doi.org/10.1007/BFb0082087.

[8] M. Murty: *Recent developments in elliptic curves*, Proceedings of the Ramanujan International Conference 1 (1988), pp. 45–53.

[9] J. Silverman: *The Arithmetic of Elliptic Curves*, Springer, 1986, doi: https://doi.org/10.1007/978-1-4757-1920-8.

Annales Mathematicae et Informaticae
55 (2022) pp. 189–195
DOI: https://doi.org/10.33039/ami.2022.12.009
URL: https://ami.uni-eszterhazy.hu

Passing the exam and not mastering the material in geometry*

Csilla Bereczky-Zámbó[bc†], **Csaba Szabó**[ac], **Anna Muzsnay**[dc], **Janka Szeibert**[bc]

[a]Eötvös Loránd University
szabo.csaba.mathdid@ttk.elte.hu

[b]Eötvös Loránd University, Faculty of Primary and Pre-School Education
csilla95@gmail.com
szeibert.janka@gmail.com

[c]MTA-ELTE Theory of Learning Mathematics Research Group

[d]University of Debrecen, Doctoral School of Mathematical and Computational Sciences
annamuzsnay@gmail.com

Abstract. It is a common assumption that taking a mathematics course and passing the exam means that one has mastered the course requirements and gained a sufficiently deep understanding of the course material. According to the communication part of the Van Hiele Theory, if someone does not reach the expected entry-level, they won't be able to develop during the course.

In our research, we investigated this contradiction in the field of geometry. We examined this phenomenon with mathematics major and pre-service mathematics teacher students during their first geometry course.

Keywords: van Hiele levels, understanding geometry, development

1. Introduction and problem statement

The role of communication is crucial in the teaching and the learning process [3]. According to Vygotsky's zone of proximal development (ZPD), during the teaching procedure, one should take into consideration the pupils' level of understanding and their knowledge of the terms [21, p. 86]. The theory about the zone of proximal

*This research was supported by the ÚNKP-19-2 New National Excellence Program of the Ministry for Innovation and Technology and by the ELTE Tehetséggondozási Tanács.
†The second author thanks the fund Mészáros Alapítvány for their support.

Submitted: May 31, 2022
Accepted: December 19, 2022
Published online: December 21, 2022

development was translated into the field of geometry by the van Hiele couple [17]. The van Hieles suggest a possible way of structuring and describing people's understanding of geometry: focusing on understanding geometrical shapes and structures. They distinguish five levels of geometrical understanding characterized as visual, descriptive, relational, formal deduction, and rigor. Their theory says that a student advances sequentially from the initial level (Visualization) to the highest level (Rigor). Students cannot achieve one level of thinking successfully without having passed through the previous levels. To move from a level to the next one, the teaching process has to start at the proper Van Hiele level.

It is a common assumption that taking a mathematics course and passing the exam means that one has mastered the course requirements and gained a sufficiently deep understanding of the course material. Parallelly, according to the communication part of the van Hiele theory, if someone does not reach the expected entry-level of a course, they will not be able to develop during the course in terms of understanding. However, it often happens that someone doesn't fulfill the prerequisites of a course and passes the exam. This is a contradiction arising the question which statement is true: "If one has completed the subject and passed the exam, one understands the material." or, "If one has arrived underprepared, one cannot gain a real understanding of the material and cannot pass the exam.". To investigate this question, we need to find some students who took a course underprepared (based on measurement at the beginning of the course), passed the exam, and we need to determine their level of understanding of the subject. In our research we measured mathematics major and pre-service mathematics teacher students' van Hiele level before and after taking a geometry course. The tool of measurement was the van Hiele Geometry Test [17]. The following research question guided our research: The van Hiele theory states that if during the teaching process the teacher's communication is not adequate considering students' actual geometric level, no real development can be achieved. Does this statement hold at higher van Hiele levels (levels 3, 4, and 5)?

2. Description of geometrical understanding in the National Core Curriculum

In order to investigate pupils' understanding of geometry van Hiele's framework have been used in over 40 countries [1, 2, 4, 6, 8, 9, 14, 16, 18–20, 24]. In these studies the test developed by Usiskin [17] was used as a measure. Investigations were typically carried out in primary schools and high schools, focusing mostly on van Hiele levels 1–3. A few studies have examined the level of pre-service teachers, where, surprisingly, in almost all cases, researchers have reported low performance [11, 12]. Pre-service teachers scored at level 3 or below. When it comes to the case of people with a higher level of geometric understanding, the number of studies is limited. The van Hiele theory is probably the best and most well-known theory for students' levels of thinking in the field of geometry, it is not obvious, whether or not the theory works efficiently at higher levels [4], especially on the fifth level [22].

This study explores the van Hiele level of Hungarian mathematics major students and pre-service mathematics teachers. The Hungarian National Core Curriculum is parallel to the van Hiele levels [15], here we present only the correspondence for levels 3–5. For lower levels see.

Level 3: Abstraction At level 2 students perceive relationships between properties and between figures, they are able to establish the interrelationships of properties both within figures (e.g., in a quadrilateral, opposite angels being equal necessitates opposite sides being equal) and among figures (a rectangle is a parallelogram because it has all the properties of a parallelogram). So, at this level, class inclusion is understood, and definitions are meaningful. They are also able to give informal arguments to justify their reasoning. However, a student at this level does not understand the role and significance of formal deduction.

Level 4: Deduction The 4th level is the level of deduction: students can construct smaller proofs (not just memorize them), understand the role of axioms, theorems, postulates and definitions, and recognize the meaning of necessary and sufficient conditions. The possibility of developing a proof in more than one way is also seen and distinctions between a statement and its converse can be made at this level.

Level 5: Rigor This level is the most abstract of all. A person at this stage can think and construct proofs in different kind of geometric axiomatic systems. So, students at this level can understand the use of indirect proof and proof by contra-positive and can understand non-Euclidean systems.

The logic of this structure is also confirmed by the observation that the Van Hiele levels can be recognized in the Hungarian National Core Curriculum [23] step by step. The following sentences and requirements connecting to different grades are from the NCC.

- Grade 5–8: "Triangles and their categories. Quadrilaterals, special quadrilateral (trapezoids, parallelograms, kites, rhombuses). Polygons, regular polygons. The circle and its parts. Sets of points that meet given criteria."
- Grade 9–12: "The classification of triangles and quadrilaterals. Altitudes, centroid, incircle and circumcircle of triangles. The incircle and circumcircle of regular polygons. Thales' theorem."
  "Remembering argumentation, refutations, deductions, trains of thought; applying them in new situations, remembering proof methods is important."
  "Generalization, concretization, finding examples and counterexamples (confirming general statements by deduction; proving, disproving: demonstrating errors by supplying a counterexample); declaring theorems and proving them (directly and indirectly) is also necessary."

The levels correspond to age group, an 8th grader (14 years old) should be on at least level 3, and at grade 12 students (18 years old) have to reach level 4, which means they have to reach the level of deductions – students have to be able to construct smaller proofs, understand the role of axioms, theorems, postulates and definitions.

Our reseach was carried out at Eötvös Loránd University, Budapest. We chose the sampling procedure by convinience [10], 65 mathematics students and 46 mathematics pre-service teachers were involved in the study, all from Eötvös Loránd University. All 111 participants were starting their first geometry course in their second semester at the university and had had passed several mathematics exams before. According to the National Core Curriculum all students were on at least on the 4th van Hiele level. Although the curriculums of pre-service teachers and of math majors differ, both courses require logical reasoning ability, understanding, and the ability of constructing proofs. We measured the Van Hiele levels of all students right before their first geometry courses, and two weeks after accomplishing the courses, as well. Mathematics students completed the van Hiele tests in paper, while pre-service teachers completed the test electronically. In this study, the 1–5 scheme was used for the levels, which is consistent with Pierre van Hiele's numbering of the levels. All references and all results from studies using the 0–4 scale have been translated to the 1–5 scheme.

3. Results and discussion

The results of the test can be seen in Table 1. Altogether 28 math major and all 66 presevice teachers students filled in the post-test. All preservice teachers filled in the post-test. They had a follow-up class with the researchers, hence they felt more oblidged to fill in the second round. Although at least level 4 is a prerequisite for both courses, in both of them more then 40% of students filled in the test at level 3. This is not a surprise, as earlier findings show that there is a gap between the knowledge of students entering the university and the expectations and prequisites of the universities' curriculum [5]. All other students filled in the test on level 5. The exam was an oral exam, where students had to explain a topic of the course with full proofs and had to answer the questions of the examiner. In both cases the examiner was the lecturer, different for the two courses. Hence, on the exam the student had to convince the professor about their understanding of the material. Such an exam lasts usually 20–40 minutes and is quite rigorous. On one hand, one would expect that passing this exam is a sign of understanding and mastering the material. And on the day of the exam it seemed to be true for all students. On the other hand we would expect that those who did not fulfill the course prequisites, namely who were on level 3, will not be able to develope on the course, and will remain on level 3. After two weeks of the exams it seemed to be true. At the same time, the expectation is that passing such an exam means being on level 4 or 5. So, it is not easy to decide, whether or not it is a surprise that all students who filled in the test on level 3 passed the exam and remained on level 3.

Table 1. Cumulative results.

	math stud.		pre. teach.	
test	pre	post	pre	post
level 3	27	11	24	24
level 4	0	0	0	0
level 5	38	17	22	22
3 → 5	0		0	
5 → 3	0	0	0	0

This result strengthens the theory of Vigotsky and its van Hiele version for higher level mathematics. Accordingly, the teacher has to be aware of the student's understanding and has to correctly determine their ZPD. As it is known, the ZPD is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers" [21]. The theory suggests that every act of teaching should start from the actual level of the student, taking into consideration that there is a maximum which they can achieve in one step. Moreover, if the teacher speaks in a too sophisticated language, then they permanently remain beyond students' ZPD and this way they do not provide scaffolding for proper development. In the geometry courses none of the two initial conditions were fulfilled. Both professors assumed level 4 form the students and presented the course in that manner.

Parallelly, by the van Hiele theory students cannot achieve one level of thinking successfully without having passed through the previous levels. The advancement of students from one van Hiele level to the next depends more on teaching than, for example, on the age of the student. To move from a level to the next one, the teaching process has to start at the proper Van Hiele level. The model also states that people reasoning at different levels may not understand each other. It means that a student on level n will not understand the thinking of level $n+1$ or higher. It follows that a student at level 3 cannot understand the reasoning of a teacher who speaks in a way that is adequate for students at level 4 or level 5. The teacher should evaluate how the student is interpreting a topic to communicate effectively. Probably, in both cases the course was adjusted to a standard level 4 ZPD.

It sounds surprising that a big proportion of students are on level 3. The van Hiele level of Hungarian high school students is fairly well investigated. It is shown that the average van Hiele level in Hungary is between 2 and 3, independently of age [15]. Talented students and special math classes are exceptions. They reach level 4 as early as grade 10, as expected by the NCC [7]. Pre-service math teacher and math major students are supposed to be over the average in mathematics. In Hungary high school studies are finished with a final exam, and the score of this exam counts to the tertiary entrance points. A thorough analysis of geometry problems on the final exams show that the level and topic of geometry problems are predictable and require level 3 [13]. It is an easy conclusion that math teachers prepare their students to the final exam, and do not teach the full curriculum. Thus, students on level 3 enter universities on exactly that level that they were taught to.

It seems that students enter the university with a geometry knowledge that does not meet the expectations of the university curriculum. One cannot learn a subject without being ready for it, without having the prequisites. And if the teacher or professor explains on a higher level where the student is, the student cannot learn the material. Still, the result contradicts the fact that these students passed the exam. This suggests that the so called "exam memory" exists in case of higher mathematics, where not only lexical knowledge is needed. Unfortunately this knowledge is just a short term knowledge and it is not accompanied by a higher level of understanding geometry.

References

[1] A. H. Abdullah, E. Zakaria: *Enhancing Students' Level of Geometric Thinking through Van Hiele's Phase-based Learning*, Indian Journal of Science and Technology 6.5 (2013), pp. 4432–4446.

[2] R. Astuti, D. Suryadi, T. Turmudi: *Analysis on geometry skills of junior high school students on the concept congruence based on Van Hiele's geometric thinking level*, Journal of Physics Conference Series (2018), pp. 1–5, doi: https://doi.org/10.1088/1742-6596/1132/1/012036.

[3] J. Brendefur, J. Frykhlom: *Promoting Mathematical Communication in the Classroom: Two Preservice Teachers' Conceptions and Practices.* Journal of Mathematics Teacher Education 3 (2000), pp. 125–153, doi: https://doi.org/10.1023/A:1009947032694.

[4] W. F. Burger, J. M. Shaughnessy: *Characterizing the van Hiele Levels of Development in Geometry*, Journal for Research in Mathematics Education 17.1 (1986), pp. 31–48, doi: https://doi.org/10.2307/749317.

[5] É. Erdélyi, A. Dukán, C. Szabó: *The transition problem in Hungary: curricular approach*, Teaching Mathematics and Computer Science 17.1 (2019), pp. 1–16, doi: https://doi.org/10.5485/TMCS.2019.0454.

[6] T. Erdogan, S. Durmus: *The effect of the instruction based on Van Hiele model on the geometrical thinking levels of preservice elementary school teachers*, Procedia - Social and Behavioral Sciences 1.1 (2009), pp. 154–159, doi: https://doi.org/10.1016/j.sbspro.2009.01.029.

[7] Á. Győry, E. Kónya: *Development of high school students' geometric thinking with particular emphasis on mathematically talented students*, Teaching Mathematics and Computer Science (2018), pp. 93–110, doi: https://doi.org/10.5485/TMCS.2018.0448.

[8] K. Jones: *Issues in the teaching and learning of geometry*, in: Aspects of Teaching Secondary Mathematics: perspectives on practice. London, GB: Routledge, 2002, pp. 121–139, doi: https://doi.org/10.4324/9780203165874.

[9] G. Kospentaris, P. Spyrou: *Assessing the development of geometrical thinking from the visual towards the analytic-descriptive level*, Annales de didactique et de sciences cognitives 13.5 (2008), pp. 133–157.

[10] J. H. McMillan: *Educational Research. Fundamentals for the consumers (3rd ed.)* New York: Addison Wesley, 2000.

[11] T. O. Moyer: *Raising the Van Hiele Level of College Students,* PRIMUS (2021), pp. 1–15, doi: https://doi.org/10.1080/10511970.2021.1978603.

[12] D. Patkin: *Global van Hiele (GVH) Questionnaire as a Tool for Mapping Knowledge and Understanding of Plane and Solid Geometry,* Research in Mathematical Education 18(2) (2014), pp. 103–128, doi: https://doi.org/10.7468/jksmed.2014.18.2.103.

[13] A. Rékasi, C. Szabó: *Modification of the geometry curriculum in relation to the curriculum reform in the light of the Van Hiele levels*, in: Münster: WTM Verlag für wissenschaftliche Texte und Medien, 2021, pp. 93–110, doi: https://doi.org/10.37626/GA9783959872003.0.15.

[14] S. Senk: *Van Hiele levels and achievement in writing geometry proofs*, Journal for Research in Mathematics Education 20.3 (1989), pp. 309–321, doi: https://doi.org/10.2307/749519.

[15] C. Szabó, C. Zámbó, A. Muzsnay, J. Szeibert: *Students' non-development in high school geometry*, Annales Mathematicae et Informaticae (2020), pp. 309–319, doi: https://doi.org/10.33039/ami.2020.12.004.

[16] A. Ural: *Investigating 11th Grade Students' Van-Hiele Level 2 Geometrical Thinking*, Journal Of Humanities And Social Science 21.12 (2016), pp. 13–19, doi: https://doi.org/10.9790/0837-2112061319.

[17] Z. Usiskin: *Van Hiele Levels and Achievement in Secondary School Geometry. CDASSG Project.* Chicago: Chicago Univ, IL., 1982.

[18] M. D. de Villers: *Some reflections on the Van Hiele theory*, in: June 2010.

[19] M. D. de Villers: *The role and function of a hierarchical classification of the quadrilaterals*, For the Learning of Mathematics 14.1 (1994), pp. 11–18.

[20] I. Vojkuvkova, J. Haviger: *The van Hiele Levels at Czech Secondary Schools*, Procedia - Social and Behavioral Sciences 171 (2015), pp. 912–918, doi: https://doi.org/10.1016/j.sbspro.2015.01.209.

[21] L. Vygotsky: *Mind in Society: Development of Higher Psychological Processes*, Cambridge: Harvard University Press, Cambridge, 1978, doi: https://doi.org/10.2307/2955559.

[22] M. Wilson: *Measuring a van Hiele Geometry Sequence: A Reanalysis,* Research in Mathematical Education 21(3) (1990), pp. 230–237, doi: https://doi.org/10.2307/749376.

[23] www.ofi.hu: *The Hungarian National Core Curriculum*, Teaching Mathematics and Computer Science (2012).

[24] I. Zachos: *Register of Educational Research in the United Kingdom – Problem Solving in Euclidean Geometry in Greek Schools*, in: vol. 10, 0615, London and New York: Routledge, 1995.

Annales Mathematicae et Informaticae
55 (2022) pp. 196–207
DOI: https://doi.org/10.33039/ami.2022.02.002
URL: https://ami.uni-eszterhazy.hu

Solving selected problems on the Chinese remainder theorem

Viliam Ďuriš[a], Veronika Bojdová[a], Timotej Šumný[b]

[a]Department of Mathematics, Faculty of Natural Sciences, Constantine The Philosopher University in Nitra, Tr. A. Hlinku 1, Nitra, Slovakia
vduris@ukf.sk, vbojdova@ukf.sk

[b]Štefan Moyses Primary School, Školská 608, Tesárske Mlyňany, Slovakia
sumnyt@gmail.com

Abstract. The Chinese remainder theorem provides the solvability conditions for the system of linear congruences. In section 2 we present the construction of the solution of such a system. Focusing on the Chinese remainder theorem usage in the field of number theory, we looked for some problems. The main contribution is in section 3, consisting of Problems 3.1, 3.2 and 3.3 from number theory leading to the Chinese remainder theorem. Finally, we present a different view of the solution of the system of linear congruences by its geometric interpretation, applying lattice points.

Keywords: Chinese remainder theorem, proof, construction of a solution, geometric interpretation

AMS Subject Classification: 11A07, 01A99

1. Introduction

One of the first systematic knowledge of the discipline we now call number theory came from ancient China [10, 11], where queries leading to linear indeterminate equations and systems of linear congruences occurred. Indeterminate linear equations of two unknowns occurred mainly in commercial tasks, e.g. by selling several kinds of goods at integer prices. Apart from mathematics (e.g. in astronomy) appeared more complicated problems leading to systems of linear indeterminate equations with multiple unknowns, which we classify within the congruence domain nowadays. A very significant knowledge from this period, in particular, is the so-called Chinese remainder theorem, which determines the necessary and sufficient

Submitted: November 16, 2021
Accepted: February 16, 2022
Published online: February 17, 2022

conditions for the solvability of a system of linear congruences. In the mathematical treatise, which comes from the ancient Chinese mathematician Sun-c' (4th century AD), we can find this problem [7]: "An unknown number of things is given. If they are counted by three, two remain, if they are counted by five, three remain, if they are counted by seven, there remain two. Determine the number of these things."

Solving this problem is easy. The least common multiple of the numbers 3, 5, and 7 is 105, so one solution of the problem will occur in the set $\{1, 2, \ldots, 105\}$. From the second condition, it follows, that the solution is a number in the form $5n+3$ from the given set. Therefore it suffices to check the numbers $3, 8, 13, \ldots, 103$, if the remainder after division by three (resp. seven) meets the problem conditions. The smallest suitable solution is $x = 23$. There are still other solutions, which are "repeated by 105", and are in the form $y \equiv 23 \mod 105$. The problem of the Chinese mathematician Sun-c' reached both India and Europe [6], and especially in the 18th and 19th centuries engaged the attention of mathematicians L. Euler and C. F. Gauss. The Chinese used a lunar calendar, in which small and large months changed by 29 and 30 days, so the year had 354 days. However, such a calendar brought problems, because due to the different length of the solar year, which the Chinese set at $365\frac{1}{4}$ days, it happened that the beginnings of the years were not in fixed dates. The Chinese inserted after 19 lunar years another 7 lunar months (around 600 BC) [2]. At the end of the first millennium AD, Chinese mathematicians and astronomers devoted great effort to calculate the so-called Great period, i.e. the question in how many years the three periods will meet – the tropical year with $365\frac{1}{4}$ days, the lunar month with $29\frac{499}{940}$ days and a sixty-day cycle [13]. The problem led to a system of congruences with large numbers. The Chinese mathematician Qin Jiushao [12] came up with a solution to the system of congruences $x \equiv 193440 \pmod{1014000}$, $x \equiv 16377 \pmod{499067}$, where $x = 6172608n$, (n is the number of years elapsed since the Great Period) [8].

There are several ways to formulate the Chinese remainder theorem.

Theorem 1.1. *Let there be a system of solvable linear congruences*

$$\begin{aligned} a_1 x &\equiv b_1 \pmod{m_1} \\ a_2 x &\equiv b_2 \pmod{m_2} \\ &\vdots \\ a_k x &\equiv b_k \pmod{m_k}, \end{aligned} \tag{1.1}$$

where $a_i, b_i, m_i (i = 1, 2, \ldots, k)$ are given integers. If m_i are pairwise coprime, then the system is solvable; or more precisely, elements of the congruence class modulo $m = m_1 m_2 \cdots m_k$ satisfy all given congruences. This statement is called the Chinese remainder theorem [9].

Proof. By mathematical induction. First, consider a system with two congruences:

$$\begin{aligned} a_1 x &\equiv b_1 \pmod{m_1} \\ a_2 x &\equiv b_2 \pmod{m_2}. \end{aligned}$$

The first congruence is solvable from assumption, hence there exists $x \equiv c_1($ $\pmod{m_1})$ which satisfies the first congruence. Substitute this solution $x = c_1 + tm_1, t \in Z$, into the second congruence:

$$a_2(c_1 + m_1 t) \equiv b_2 \pmod{m_2}$$
$$(a_2 m_1)t \equiv (b_2 - a_2 c_1) \pmod{m_2}.$$

Since m_1 and m_2 are coprime, then $\gcd(a_2 m_1, m_2) = \gcd(a_2, m_2)$. From the theorem assumptions the second congruence is solvable too, therefore $\gcd(a_2, m_2) \mid b_2$. However, this already results in $\gcd(a_2 m_1, m_2) \mid b_2$, which is the condition for solvability of the congruence $(a_2 m_1)t \equiv (b_2 - a_2 c_1) \pmod{m_2}$. So we have $t \equiv c_2$ $\pmod{m_2}$, which satisfies the second congruence. Then we can rewrite x as:

$$x = c_1 + m_1(c_2 + sm_2) = (c_1 + c_2 m_1) + s(m_1 m_2),$$

where $c_1, c_2, s \in Z$. Thus, the solution of the system of the two linear congruences is the whole congruence class

$$x \equiv e_1 \pmod{m_1 m_2},$$

where $e_1 = c_1 + c_2 m_1$.

Now suppose the statement holds true for $k = \nu$. Consider the system of $\nu + 1$ solvable linear congruences with pairwise coprime moduli $m_1, m_2, \ldots, m_{\nu+1}$. The system of first ν congruences is solvable from the induction assumption, so we have

$$x \equiv e_\nu \pmod{m_1 m_2 \ldots m_\nu}$$

satisfying the first ν congruences. We have to find out if any element of this congruence class is also the solution of the last congruence. We solve the system of congruences:

$$x \equiv e_\nu \pmod{m_1 m_2 \ldots m_\nu}$$
$$a_{\nu+1} x \equiv b_{\nu+1} \pmod{m_{\nu+1}}.$$

Since $\gcd(m_{\nu+1}, m_1, \ldots, m_\nu) = 1$, then there exists the solution of this system of two congruences (by analogy to the first step of the proof). □

The Chinese remainder theorem says nothing about a case of the congruence system (1.1) with non-coprime moduli. In this case, the system can be unsolvable, although individual congruences are solvable. But the system also can be solvable.

2. The construction of a solution of a system of linear congruences

First, we present the applicable construction method for a solution of the system (1.1). We show, that u in the following form is a solution of the system (1.1).

Theorem 2.1. *Consider the solvable system of linear congruences (1.1). Then*

$$u = \sum_{i=1}^{k} \frac{m}{m_i} c_i r^{(i)} = \frac{m}{m_1} c_1 r^{(1)} + \dots + \frac{m}{m_k} c_k r^{(k)}$$

is a common solution of given system, where $r^{(i)}$ is a solution of $a_i x \equiv b_i \pmod{m_i}$ and c_i is a solution of

$$\frac{m}{m_i} y \equiv 1 \pmod{m_i}, \quad m = m_1 m_2 \dots m_k, \ i = 1, \dots, k, \quad \gcd\left(\frac{m}{m_i}, m_i\right) = 1.$$

Proof. First, let us solve the congruences

$$\frac{m}{m_i} y \equiv 1 \pmod{m_i}, \quad i = 1, \dots, k, \quad \gcd\left(\frac{m}{m_i}, m_i\right) = 1,$$

where c_i is the appropriate solution. Let $r^{(i)}$ be a solution satisfying

$$a_i x \equiv b_i \pmod{m_i}, \quad i = 1, \dots, k.$$

We show that

$$u = \sum_{i=1}^{k} \frac{m}{m_i} c_i r^{(i)} = \frac{m}{m_1} c_1 r^{(1)} + \dots + \frac{m}{m_k} c_k r^{(k)}$$

satisfies any of the congruences $a_i x \equiv b_i \pmod{m_i}$.

We express

$$a_i x = a_i u = a_i \sum_{i=1}^{k} \frac{m}{m_i} c_i r^{(i)} = a_i \left(\frac{m}{m_1} c_1 r^{(1)} + \dots + \frac{m}{m_i} c_i r^{(i)} + \dots + \frac{m}{m_k} c_k r^{(k)} \right).$$

Since all members $\frac{m}{m_1}, \dots, \frac{m}{m_k}$ except member $\frac{m}{m_i}$ are divisible by the number m_i, we get

$$a_i u \equiv a_i \frac{m}{m_i} c_i r^{(i)} \pmod{m_i}.$$

Since c_i is a solution of $\frac{m}{m_i} y \equiv 1 \pmod{m_i}$, then $\frac{m}{m_i} c_i \equiv 1 \pmod{m_i}$, and thus

$$a_i u \equiv a_i r^{(i)} \pmod{m_i}.$$

And finally from $a_i r^{(i)} \equiv b_i \pmod{m_i}$ we have $a_i u \equiv b_i \pmod{m_i}$.

Now we show that any $x = u + tm, t \in Z$ satisfies the congruence $a_i x \equiv b_i \pmod{m_i}$. We have

$$a_i(u + tm) = a_i u + a_i tm,$$

where $a_i u \equiv b_i \pmod{m_i}$ and $a_i tm \equiv 0 \pmod{m_i}$, while $\exists h \in Z : m = hm_i$. Then

$$a_i(u + tm) \equiv b_i \pmod{m_i}.$$

If the congruence $a_i x \equiv b_i \pmod{m_i}$ has n_i incongruent solutions $r^{(i)}$, then we have together $n_1 n_2 \cdots n_k$ incongruent solutions $u = \frac{m}{m_1} c_1 r^{(1)} + \cdots + \frac{m}{m_i} c_i r^{(i)} + \cdots + \frac{m}{m_k} c_k r^{(k)}$ of the system (1.1). We show, that all are incongruent by modulo m.

If we changed any of the solutions $r^{(i)}$ of the congruence $a_i x \equiv b_i \pmod{m_i}$ of the common solution u of the system to an incongruent one by modulo m_i, we would get an incongruent solution u. Let's change, e.g., h solutions $r^{(i)}$ $(h \leq k)$ to incongruent ones by modulo m_i and arrange the expressions $\frac{m}{m_i} c_i r^{(i)}$ in u by placing forward those, which contain an incongruent solution $r^{(i)}$. Then, after re-indexing members in u and re-denoting incongruent solutions, we can write

$$u_2 = \frac{m}{m_1} c_1 r_2^{(1)} + \cdots + \frac{m}{m_i} c_i r_2^{(i)} + \cdots + \frac{m}{m_h} c_h r_2^{(h)} + \frac{m}{m_{h+1}} c_{h+1} r^{(h+1)} + \cdots + \frac{m}{m_k} c_k r^{(k)}.$$

We show, that u_2 is not congruent with u by modulo m. By contradiction, if $u \equiv u_2 \pmod{m}$, then $m \mid u - u_2 \wedge m_i \mid m \Rightarrow m_i \mid u - u_2$, hence $u \equiv u_2 \pmod{m_i}$. Then

$$\frac{m}{m_1} c_1 r^{(1)} + \cdots + \frac{m}{m_i} c_i r^{(i)} + \cdots + \frac{m}{m_h} c_h r^{(h)} - \left(\frac{m}{m_1} c_1 r_2^{(1)} + \cdots + \frac{m}{m_i} c_i r_2^{(i)} + \cdots + \frac{m}{m_h} c_h r_2^{(h)} \right) \equiv 0 \pmod{m_i}.$$

From the last congruence we have

$$\frac{m}{m_i} c_i r^{(i)} - \frac{m}{m_i} c_i r_2^{(i)} \equiv 0 \pmod{m_i} \quad \Leftrightarrow \quad \frac{m}{m_i} c_i r^{(i)} \equiv \frac{m}{m_i} c_i r_2^{(i)} \pmod{m_i}.$$

Since $\frac{m}{m_i} c_i \equiv 1 \pmod{m_i}$, then $r^{(i)} \equiv r_2^{(i)} \pmod{m_i}$, what is a contradiction. Hence solution u_2 can not be congruent with u by modulo m. This means we have incongruent solutions u and u_2. □

3. Selected problems from number theory leading to use of Chinese remainder theorem

Focusing on the use of the Chinese remainder theorem, we present the proofs of selected problems from number theory. We also present simple codes in R language to demonstrate the solutions to these problems.

Problem 3.1. There are at most two n-digit numbers with the property $x^2 = k10^n + x$. Such numbers x, whose squares end in themselves, are called 1-automorphic numbers (see e.g. [5]).

Solution. We are searching for natural numbers x, among n-digit numbers $0 \leq x < 10^n$, with the property:

$$x^2 = k10^n + x.$$

Hence

$$x^2 - x = k10^n = k2^n5^n,$$

which leads to congruence $x^2 \equiv x \pmod{10^n}$, or

$$x^2 - x = x(x-1) \equiv 0 \pmod{10^n}. \tag{3.1}$$

Since $x \in N$, then it is true that if x is even then $x-1$ is odd, or if x is odd then $x-1$ is even. Then from (3.1) we get, that x satisfies either system of congruences

$$\begin{aligned} x &\equiv 0 \pmod{2^n} \\ x-1 &\equiv 0 \pmod{5^n} \quad \Leftrightarrow \quad x \equiv 1 \pmod{5^n} \end{aligned} \tag{3.2}$$

or system of congruences

$$\begin{aligned} x &\equiv 0 \pmod{5^n} \\ x-1 &\equiv 0 \pmod{2^n} \quad \Leftrightarrow \quad x \equiv 1 \pmod{2^n}. \end{aligned} \tag{3.3}$$

Since $\gcd(2^n, 5^n) = 1$ and $0 \equiv 1 \pmod 1$ holds true, then the system (3.2) and also the system (3.3) has a unique solution modulo 2^n5^n. Consequently there are at most two n-digit numbers with the property $x^2 = k10^n + x$.

We present a code in R language to demonstrate solutions for $n \in \{1, \ldots, 8\}$:

```
library(numbers)
n=8
a1=c(1,0)
a2=c(0,1)
for (i in 1:n) {
 m=c(2^i, 5^i)
  print(chinese(a1,m))
  print(chinese(a2,m)) }
> 5
  6
  25
  76
  625
  376
  9376
  90625
  890625
  109376
  2890625
  7109376
  12890625
  87109376
```

Problem 3.2 (inspired by [1]). For every positive integer n, there exist n consecutive positive integers such that none of them is a power of a prime.

Solution. We show that for any n there exists $x \in N$ such that none of the numbers $x+1, x+2, \ldots, x+n \in N$ is a power of a prime. The number $x+i$ $(i = 1, 2, \ldots, n)$ is not a power of a prime if there are two different primes p, q, that divide $x+i$.

Let $n \in N$, $i = 1, 2, \ldots, n$, and let all $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$ be distinct primes. We look for $x \in N$, which satisfies $p_i q_i \mid x + i$ for each $i = 1, 2, \ldots, n$. Written as the system of congruences we have

$$x + i \equiv 0 \pmod{p_i q_i},$$

or

$$x \equiv p_i q_i - i \pmod{p_i q_i} \tag{3.4}$$

for $i = 1, 2, \ldots, n$.

Since p_i, q_i $(i = 1, 2, \ldots, n)$ were distinct, then $\gcd(p_i q_i, p_j q_j) = 1$. Hence there exists one solution x of the system (3.4). Thus, for any $n \in N$ we found (constructed) $x \in N$ such that numbers $x+1, x+2, \ldots, x+n \in N$ have two different prime divisors.

A simple code in R language allows us to demonstrate the solution for $n = 3$: the three consecutive integers are 18458, 18459 and 18460. With help of prime factorization it's easy to see, that none of them is a power of a prime.

```
library(numbers)
n = 3
p = c(11,7,5)
q = c(2,3,13)
i = 1:n
x = chinese(p*q-i,p*q)
print(x)
> 18457

library(gmp)
factorize(x+1)
> 2   11  839

factorize(x+2)
> 3   3   7   293

factorize(x+3)
> 2  2  5  13 71
```

Problem 3.3 (inspired by [1]). There exists a set S of three positive integers such that for any two distinct $a, b \in S$ $a - b$ divides a and b but none of the other elements of S.

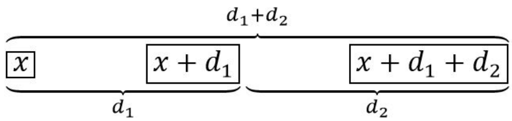


Figure 1. Elements of S.

Solution. Denote three positive integers from S by $x, x+d_1, x+d_1+d_2$, where d_1, d_2 denote the differences between consecutive elements of S (Figure 1).
We have 3 pairs of distinct elements and we write down the divisibility conditions for the first element x:

$$\begin{aligned} d_1 \mid x \quad &\Leftrightarrow \quad x \equiv 0 \pmod{d_1} \\ d_1+d_2 \mid x \quad &\Leftrightarrow \quad x \equiv 0 \pmod{d_1+d_2} \\ d_2 \nmid x \quad &\Leftrightarrow \quad x \equiv a_1 \pmod{d_2}, \end{aligned} \tag{3.5}$$

where $a_1 \in \{1, 2, \dots, d_2-1\}$ is the non-zero remainder. We show, that it suffices to choose any coprime positive integers $d_1, d_2, d_1 < d_2$, and then the existence of x follows from the Chinese remainder theorem.

Let $d_1, d_2, d_1 < d_2$, be any coprime positive integers, hence also d_1, d_2, d_1+d_2 are pairwise coprime. Remainder $a_1 \in \{1, 2, \dots, d_2-1\}$ depends on the choice of d_1, d_2 following way. From the condition $x+d_1 \equiv 0 \pmod{d_2}$ we have $x \equiv -d_1 \pmod{d_2}$, which together with congruence

$$x \equiv a_1 \pmod{d_2}$$

gives the result for a_1: $a_1 \equiv -d_1 \pmod{d_2}$, so we can put $a_1 = d_2 - d_1$ (since $d_1 < d_2$). Since d_1, d_2, d_1+d_2 are pairwise coprime moduli, then there is a unique solution of the system (3.5):

$$\begin{aligned} x &\equiv 0 \pmod{d_1} \\ x &\equiv 0 \pmod{d_1+d_2} \\ x &\equiv d_2 - d_1 \pmod{d_2}. \end{aligned}$$

We can get some solutions of this example by using the following code in R language:

```
library(numbers)
d1 = 8
d2 = 15
a = c(0,0,d2-d1)
m = c(d1,d1+d2,d2)
x = chinese(a,m)
print(c(x,x+d1,x+d1+d2))
```

Table 1. Some solutions.

d_1	d_2	x	$x + d_1$	$x + d_1 + d_2$
2	3	10	12	15
2	7	54	56	63
8	15	2392	2400	2415

4. Geometric interpretation of the solution of system of linear congruences

Finally, we present a different view of the solution of the system of linear congruences by its geometric interpretation, applying lattice points. For some basic knowledge of the lattice points, see, e.g. [4].

Consider a congruence $ax \equiv b \pmod{m}, a, b, x, m \in Z, m > 1$. There is a direct connection between this congruence relation and the Diophantine equation [3], while $ax - b = ym, y \in Z$, represents the linear Diophantine equation

$$ax - my = b$$

(where $x, y \in Z$ are the unknowns and $a, b, m \in Z, a, b \neq 0$, are given constants).

On the other hand, the equation

$$ax - my - b = 0 \tag{4.1}$$

represents a straight line (in Euclidean plane). So for given $a, b, m \in Z$ the solution of the congruence $ax \equiv b \pmod{m}$ geometrically represents all intersection points $[x_0, y_0], x_0, y_0 \in Z$, of the straight line (4.1) and the lattice of integral coordinates.

Example 4.1. Consider system of congruences

$$\begin{aligned} 3x &\equiv 4 \pmod{8} \\ 4x &\equiv 2 \pmod{5}. \end{aligned}$$

Both congruences are solvable ($\gcd(3, 8) = 1 \mid 4$ and $\gcd(4, 5) = 1 \mid 2$). The solution of the second congruence $4x \equiv 2 \pmod{5}$ is $x \equiv 3 \pmod{5}$. Substitute $x = 3 + 5t,\ t \in Z$ into the first congruence, then

$$3(3 + 5t) \equiv 4 \pmod{8},$$

hence

$$15t \equiv -5 \pmod{8}$$

with a solution $t \equiv 5 \pmod{8}$. Finally, after substitution $t = 5 + 8y,\ y \in Z$ into x:

$$x = 3 + 5(5 + 8y) = 28 + 40y,$$

we get the solution $x \equiv 28 \pmod{40}$ of the system.

Figure 2 shows the geometric representation of the congruence $x \equiv 28 \pmod{40}$. That means, there are infinitely many points $[x_0, y_0]$ with integer coordinates on the green straight line $x - 28 - 40y = 0$. See, that e.g. the lattice point $[68, 1]$ is one of the solution points.

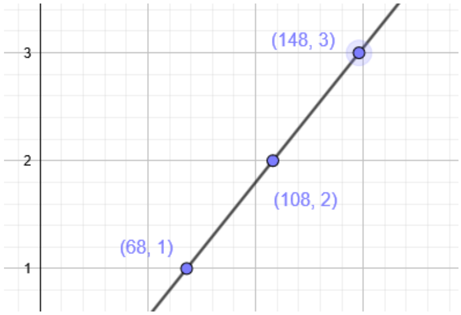


Figure 2. Example of the intersection of straight line and a lattice of the integer coordinates.

In our geometric interpretation of the Diophantine equation, we consider the solvability conditions based on the lattice points, through which the line represented by equation (4.1) passes.

Now consider a system of linear congruences (1.1), where $\gcd(m_i, m_j) = 1$ for all $i, j,\ i \neq j,\ i, j = 1, \ldots, k$. Such a system of congruences can be converted to Diophantine equations with the same consideration as mentioned above. Since we are looking for a common solution for these Diophantine equations, geometrically this means that we are looking for a line that passes through all the lattice, which is characteristic for concrete Diophantine equations. The solution of such a system of equations is a congruence

$$x \equiv u \pmod{\prod m_i},$$

which we can interpret as a straight line in the form

$$x - u - y \prod m_i = 0.$$

In other words, considered congruences give us information about a line in various specific scales, and we're looking for its formula. For an illustration of this representation, an example follows.

Example 4.2. Consider system of congruences

$$\begin{aligned} 2x - 4 &\equiv 0 \pmod{8} \\ 2x - 1 &\equiv 0 \pmod{3} \\ 13x - 4 &\equiv 0 \pmod{5}. \end{aligned} \tag{4.2}$$

We will construct the solution of the system of congruences according to the Theorem 2.1. We see, that $\gcd(8,3) = \gcd(3,5) = \gcd(8,5) = 1$, so we can apply the Chinese remainder theorem. Denote $m = 2^3 \cdot 3 \cdot 5 = 120$. Then the number of system solutions is $n = n_1 n_2 n_3 = 2 \cdot 1 \cdot 1 = 2$.

Congruence $2x - 4 \equiv 0 \pmod 8$ has solutions $r_1^{(1)} = 2, r_2^{(1)} = 6$, congruence $2x - 1 \equiv 0 \pmod 3$ has a solution of $r^{(2)} = 2$ and congruence $13x - 4 \equiv 0 \pmod 5$ has a solution of $r^{(3)} = 3$.

Solutions of congruences $\frac{120}{8} y \equiv 1 \pmod 8$, $\frac{120}{3} y \equiv 1 \pmod 3$ and $\frac{120}{5} y \equiv 1 \pmod 5$ are $c_1 = 7, c_2 = 1$ and $c_3 = 4$, respectively.

Finally $u = 15 \cdot 7r^{(1)} + 40 \cdot 1r^{(2)} + 24 \cdot 4r^{(3)} = 105r^{(1)} + 40r^{(2)} + 96r^{(3)}$.

Table 2. Summary of the resulting two solutions.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	u
1.	2	2	3	$578 \equiv 98 \pmod{120}$
2.	6	2	3	$998 \equiv 38 \pmod{120}$

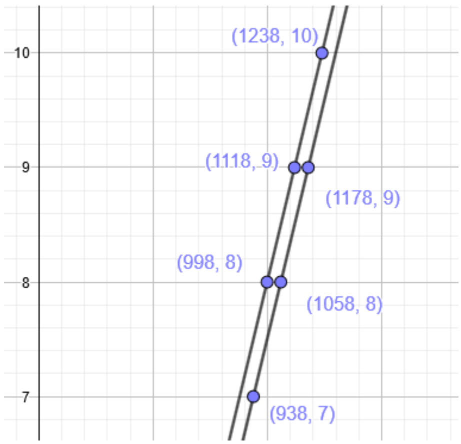


Figure 3. Geometric interpretation of the solution.

The solutions from Table 2 are represented in the Figure 3 by straight lines with equations

$$x - 120y - 98 = 0,$$
$$x - 120y - 38 = 0.$$

Finally we mention, that there exists one residue class containing all solutions in form $x \equiv 38 \pmod{60}$, represented by a straight line with equation $x - 60y - 38 = 0$.

5. Conclusion

The paper introduces the historical background of the Chinese remainder theorem, focusing on one of its proofs. Section 2 presents the construction of a solution of a system of linear congruences, which gives the applicable solving method of the system (1.1). The main contribution is in section 3, consisting of three problems from number theory, leading to the Chinese remainder theorem. The article also deals with the geometric interpretation of the solution of the system of linear congruences. It introduces a different perspective of the solution, applying lattice points and the relationship between the congruence and the Diophantine equation. Illustrating examples supplement all of the theoretical results.

References

[1] E. Chen: *The Chinese Remainder Theorem, February 3, 2015, article for olympiad students*, 2015.

[2] C. Cullen: *Astronomy and Mathematics in Ancient China: The 'Zhou Bi Suan Jing'*, UK: Cambridge University Press, 2007.

[3] V. Ďuriš, D. Gonda, A. Tirpáková, G. Pavlovičová: *Teaching Congruences in Connection with Diophantine Equations*, Education Sciences 11.9 (2021), pp. 1–14, doi: https://doi.org/10.3390/educsci11090538.

[4] V. Ďuriš, T. Šumný: *Diophantine Geometry in Space E2 and E3*, TEM Journal 8.1 (2019), pp. 78–81, doi: https://doi.org/10.18421/TEM81-10.

[5] V. de Guerre, R. A. Fairbairn: *Automorphic Numbers*, J. Recr. Math. 1 (1968), pp. 173–179.

[6] A. P. Juškevič: *History of mathematics in the Middle Ages*, Prague: Academia, 1977.

[7] S. Kangsheng: *Historical Development of the Chinese Remainder Theorem*, Arch. Hist. Exact Sci. 38.4 (1988), pp. 285–305, doi: https://doi.org/10.1007/bf00357063.

[8] J. C. Martzloff: *Astronomy and Calendars – The Other Chinese Mathematics, 1st ed.* Switzerland: Springer Nature, 2016.

[9] P. Ribenboim: *The Little Book of Bigger Primes, 2nd ed.* New York: Springer-Verlag, 2004.

[10] D. E. Smith: *History of Mathematics*, vol. I, US: Dover Publications, 1958.

[11] D. E. Smith: *History of Mathematics*, vol. II, US: Dover Publications, 1958.

[12] C. Smoryński: *Logical Number Theory I*, Germany: Springer Verlag Berlin Heidelberg, 1991.

[13] D. Yinke: *Ancient Chinese Inventions*, China: China Intercontinental Press, 2005.

Annales Mathematicae et Informaticae
55 (2022) pp. 208–221
DOI: https://doi.org/10.33039/ami.2022.10.001
URL: https://ami.uni-eszterhazy.hu

On a new LaTeX package for automatic Hungarian definite article

Tibor Tómács

Eszterházy Károly Catholic University,
Institute of Mathematics and Informatics, Eger, Hungary
tomacs.tibor@uni-eszterhazy.hu

Abstract. In the LaTeX document preparation system (see [3]) it is possible to insert automatically the appropriate definite article for cross-references and other commands containing texts in Hungarian language documents. Thus, if these change, the definite articles will also change accordingly.

Such a tool is the `magyar.ldf`, which sets the Hungarian typography and also handles the automatic definite articles. Another one is the `nevelok` package, created specifically for this task. Both packages work with numerous errors and have shortcomings. This motivated the author of this paper to develop a new package, that corrects these errors and fills the gaps in. The new package is called *huaz*.

Keywords: Hungarian definite article, LaTeX, babel package, magyar.ldf, utf8, latin2, pdflatex, xelatex, lualatex

AMS Subject Classification: 68U01, 68U15, 68U99

1. Introduction

In Hungarian there are two definite articles, *"a"* and *"az"*, which are determined by the pronunciation of the subsequent word. The definite article is *"az"*, if the first phoneme of the pronounced word is a vowel, otherwise it is *"a"*. This seems simple, but consider the following cases:

- When you refer to a page number in a LaTeX document, you have to use the `\pageref` command. If you want to put a definite article in front of it, it depends on what the page number is. For example *"az 1. oldal"* (the page 1), *"a 2. oldal"*, *"az 5. oldal"*, *"a 10. oldal"*. So "`a \pageref{`⟨key⟩`}`" may not

Submitted: August 24, 2022
Accepted: October 14, 2022
Published online: October 14, 2022

give the correct result. The problem is similar for all cross-references (`\ref`, `\pageref`, `\eqref`, `\cite`).

- It is also important whether a word is a Roman numeral or not. For example *"az V. fejezet"* (the Chapter V), if *V* is a Roman numeral, but *"a V. fejezet"*, if *V* represents a letter or the 22 alphanumeric number (*V* is the 22$^{\text{nd}}$ letter in the English alphabet).

- Some Hungarian consonants have special properties from the aspect of the definite article. For example *"az M betű"*, but *"a Magyar Közlöny"*; *"az Ny betű"*, but *"a Nyugdíjfolyósító Igazgatóság"*; etc.

First, we mention two existing LaTeX solutions for the automatic handling of Hungarian definite articles, highlighting their errors and shortcomings.

1.1. The magyar.ldf

A possible way is to use the `magyar.ldf` file, that sets the `magyar` (or `hungarian`) option of the `babel` package (see full documentation in [6, 10]).

The commands defined in `magyar.ldf` (`\az`, `\aref`, `\apageref`, `\acite`, etc.) work properly on a basic level, but major errors can occur. This motivated the writing of a new package:

- The `\eqref` command has no definite article version. (The `\eqref` is defined in the `amsmath` package to reference equations. See in [1].) Instead, `\aref({⟨key⟩})` can be used, but in italics font style environment, it does not give upright result like the `\eqref`. Another option solving this problem is the `\az{\eqref{⟨key⟩}}` command. However, neither solution handles Roman numerals, or `\tag` commands labeled equations.

- The `\az{\ref{⟨key⟩}}` and the `\az{\pageref{⟨key⟩}}` do not handle Roman numeral references.

- The previous error also exists when the `\az` command contains a text starting with a Roman numeral. For example the result of the `\az{V.~osztály}` is *"a V. osztály"* (the Class V). The correct form would be *"az V. osztály"*.

- If you need an automatic definite article for a none cross-reference, but for a text or a command that stores text, then accented letters are detected incorrectly in case of UTF-8 encoding. The basic cause of this error is that UTF-8 use more than one byte to encode the characters, which `magyar.ldf` does not take into account. Therefore, for example, the result of the `\az{ágy}` is *"a ágy"* (the bed), because it does not perceive the letter *"á"* as a letter, so it does not recognize that it is a vowel. Surprisingly, the result of the `\az{száz}` is *"az száz"* (the hundred), which is also incorrect. The reason is that since *"á"* is not a letter for the `magyar.ldf`, it considers the two-digit consonant *"sz"* separate, which requires to be preceded by *"az"*.

- Although it cannot be considered as a mistake, it is uncomfortable that, as an example, in the `\az{\textbf{N betű}}` the `\textbf` command interferes with the detection of the space after N, so the definite article will incorrectly become *"a"* ("a **N betű**" = the letter N).

1.2. The nevelok package

The `nevelok` LaTeX package was created in 2015, and it is also designed to handle automatic Hungarian definite articles (see in [2]). The author probably didn't know the opportunities of the `magyar.ldf`, that already existed at that time, that is why he has created this package.

Tested on TeX Live 2022, I have experienced that the `nevelok` package does not handle any form of accented letters, and in a lot of cases gives wrong definite article for cross-references. The reason of the latter problem is that it examines the `\ref` command, that is not expandable. It is not recommended to use this package in the current version (1.03).

2. Purpose and operation of the huaz package

The mistakes and shortcomings of the `magyar.ldf` and the `nevelok.sty` motivated me to design a completely new package called *huaz*.

The purpose of the `huaz` package is to help the Hungarian LaTeX users inserting automatically the correct definite articles for cross-references and commands containing text. Thus, if these change, the definite articles will also change accordingly.

The package is still under testing, but the Reader can try it on the Overleaf before publishing it on The Comprehensive TeX Archive Network.

The `huaz` package adds the definite article "az" to the given text in the following cases:

1. The first letter is a vowel (lowercase or uppercase, accented or not, Hungarian or not, specified with UTF-8, ISO-8859-2 characters or accent commands).

2. The first letter is a consonant, whose pronunciation begins with a vowel (for example F, L, M, etc.) while the second character (if any) is not a letter, but a number, punctuation mark or space. For example, "M-10". We also listed some non-Hungarian accented consonants here. For example "Ň.1".

3. The block of the first two characters is a two-digit consonant, whose pronunciation begins with a vowel (for example Ny, Ly, Sz, etc.) while the third character (if any) is not a letter, but a number, punctuation mark or space. For example, "NY betű".

4. The first character is 5.

5. It starts with a 1, 4, 7 or 10 digit number and the first digit is 1 (egy = one, ezer = thousand, egymillió = one million, egymilliárd = one billion).

If the characters, at the beginning of the word, can also be interpreted as Roman numerals, you can choose to convert them to Arabic numerals and define the definite article for that or not. For example, in the case of "X/A".

3. Using the huaz package

The `huaz` package must be loaded in the usual way:

```
\usepackage{huaz}
```

There are no package options. It works for UTF-8 (`utf8`) and ISO-8859-2 (`latin2`) encoded source files, but it also handles accent commands correctly. It is also compatible with `pdflatex`, `xelatex` and `lualatex` compilers. As an example, with the `pdflatex` compiler, the following loading is suitable:

```
\documentclass{article}
\usepackage[T1]{fontenc}
\usepackage{huaz}
\PassOptionsToPackage{defaults=hu-min}{magyar.ldf}
\usepackage[magyar]{babel}
\begin{document}
 ...
\end{document}
```

It is also compatible with the `hyperref` package (see [5]).

The `defaults=hu-min` option of the `magyar.ldf` makes its own automatic definite article commands available. It is not required, but useful to turn it off when using the `huaz` package. To do this, replace the line

```
\PassOptionsToPackage{defaults=hu-min}{magyar.ldf}
```

with

```
\PassOptionsToPackage{defaults=hu-min,az=no,
                          shortrefcmds=no,hunnewlabel=no}{magyar.ldf}
```

The `huaz` package uses the services of the `xstring`, `refcount` and `iftex` packages (see [4, 7, 9]), so they are also loaded in. On the other hand, some changes in the LaTeX kernel introduced on October 10, 2021 are used (see in [8]), so the package only works correctly on systems installed after that.

3.1. The commands

`\az{⟨text⟩}` • The ⟨text⟩ is preceded by the appropriate definite article in a lowercase form. If at the beginning of the ⟨text⟩, there are characters that can be interpreted as Roman numerals, then the definite article is adjusted to the Arabic equivalent. For example

```
Idén \az{V.B}~osztály rendezi a farsangot.
```

„Idén az V.B osztály rendezi a farsangot.”.

The ⟨text⟩ can also be a command that stores text. For example

```
\newcommand{\osztaly}{V.B}
Idén \az{\osztaly}~osztály rendezi a farsangot.
```

„Idén az V.B osztály rendezi a farsangot.”.

The ⟨text⟩ can also contain text formatting commands (see the Subsection 3.3 for more information). For example

```
\newcommand{\osztaly}{V.B}
Idén \az{\textbf{\osztaly}}~osztály rendezi a farsangot.
```

„Idén az **V.B** osztály rendezi a farsangot.”.

The ⟨text⟩ can also be a standard cross-reference (`\ref`, `\pageref`, `\eqref`, `\cite`). For example

```
\section{Cím}\label{seca}
\section{Cím}\label{secb}
\az{\ref{seca}}.~szakaszban, \az{\textbf{\ref{secb}}}.~szakaszban
```

„az 1. szakaszban, a **2**. szakaszban”.

```
\renewcommand{\thesection}{\Roman{section}}
\section{Cím}\label{seca}
\section{Cím}\label{secb}
\az{\ref{seca}}.~szakaszban, \az{\textbf{\ref{secb}}}.~szakaszban
```

„az I. szakaszban, a **II**. szakaszban”.

The ⟨text⟩ and the `\az` command have the following limitations:

1. At the beginning of the ⟨text⟩, only `\ref`, `\pageref`, `\eqref`, `\cite` cross-reference commands work correctly. For example the `\ref*` and `\pageref*` commands of the `hyperref` package do not work directly as ⟨text⟩, but it can be solved with the `\az*` command (see later).

2. At the beginning of the ⟨text⟩, the `\cite` command works fine in default case, with `natbib`, and with `bibtex`. When using the `biblatex` package, it works well if the `style` or `citestyle` options are `numeric`, `numeric-verb`, `alphabetic`, `alphabetic-verb` or `authoryear`. It also works well if we do not specify any of the `style` or `citestyle` options.

3. You cannot insert text into the pdf bookmark with the `\az` command. So, for example, the following code will not give correct bookmark, if you use `hyperref` or `bookmark` package:

   ```
   \section{...\az{\ref{sec}}...}
   ```

   However, the title will appear fine in the text, headers and the table of contents. The problem can be solved with the `\azsaved` command (see later).

`\az*{`⟨text⟩`}` • The same case without `*`, but only the definite article is written out, the ⟨text⟩ is not. For example, using the `hyperref` package

```
\section{Cím}\label{sec}
\az*{\ref{sec}}~\ref*{sec}.~szakaszban
```

`\azv{`⟨text⟩`}` • Same as `\az{`⟨text⟩`}`, but if at the beginning of the ⟨text⟩ there are characters that can be interpreted as Roman numerals, then the definite article is not aligned with the Arabic equivalent, but as simple characters. For example

```
\renewcommand{\thesection}{\Alph{section}}
\setcounter{section}{21}
\section{Cím}\label{sec}
\az{\ref{sec}}.~szakaszban, \azv{\ref{sec}}.~szakaszban
```

The result is “az V. szakaszban, a V. szakaszban”, as in the first case the letter V was interpreted as a Roman numeral, but not in the second case. Since the `section` counter is now set to an alphanumeric number, the second case is the right one.

`\azv*{`⟨text⟩`}` • The same case without `*`, but only the definite article is written out, the ⟨text⟩ is not.

`\Az{`⟨text⟩`}` `\Az*{`⟨text⟩`}` `\Azv{`⟨text⟩`}` `\Azv*{`⟨text⟩`}` • In the names of the commands, the letter “a” can be replaced by the letter “A”. Then the definite article will begin with capital letter, which is necessary at the beginning of sentences.

`\azsaved` • When you use any of the previous commands, an `\azsaved` expandable command is generated, too. The expansion of it is the definite article that must precede the ⟨text⟩.

Using `hyperref` or `bookmark` package, as mentioned before, the title, header, table of contents will be fine with the following code, but the bookmark of the pdf will not:

```
\section{\Az{\ref{sec}}...}
```

The `\azsaved` command can solve this problem:

```
\section{\texorpdfstring{\Az{\ref{sec}}...}{\azsaved~\ref{sec}...}}
```

Then the code

```
\azsaved~\ref{sec}...
```

is added to the bookmark, that finally gives correct result.

`\aznotshow` • The previous problem can be solved with the command `\aznotshow` instead of `\texorpdfstring`. Because by placing `\aznotshow` before `\az` (or any version of it), the result is not shown, only `\azsaved` is generated with the appropriate definite article. Therefore

```
\aznotshow\Az{\ref{sec}}
\section{\azsaved~\ref{sec}...}
```

also gives correct result in the pdf bookmark.

3.2. Abbreviations

`\aref{⟨key⟩}` ≡ `\az{\ref{⟨key⟩}}`
`\aref*{⟨key⟩}` ≡ `\az*{\ref{⟨key⟩}}`
`\avref{⟨key⟩}` ≡ `\azv{\ref{⟨key⟩}}`
`\avref*{⟨key⟩}` ≡ `\azv*{\ref{⟨key⟩}}`

`\aeqref{⟨key⟩}` ≡ `\az{\eqref{⟨key⟩}}`
`\aeqref*{⟨key⟩}` ≡ `\az*{\eqref{⟨key⟩}}`
`\aveqref{⟨key⟩}` ≡ `\azv{\eqref{⟨key⟩}}`
`\aveqref*{⟨key⟩}` ≡ `\azv*{\eqref{⟨key⟩}}`

`\apageref{⟨key⟩}` ≡ `\az{\pageref{⟨key⟩}}`
`\apageref*{⟨key⟩}` ≡ `\az*{\pageref{⟨key⟩}}`
`\avpageref{⟨key⟩}` ≡ `\azv{\pageref{⟨key⟩}}`
`\avpageref*{⟨key⟩}` ≡ `\azv*{\pageref{⟨key⟩}}`

`\acite[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}` ≡ `\az{\cite[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}}`
`\acite*[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}` ≡ `\az*{\cite[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}}`
`\avcite[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}` ≡ `\azv{\cite[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}}`
`\avcite*[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}` ≡ `\azv*{\cite[⟨text⟩]{⟨key1⟩,⟨key2⟩,...}}`

In the names of the commands, the first letter "a" can be replaced by the letter "A". Then the definite article will begin with a capital letter, which is necessary at the beginning of sentences:

\Aref \Avref \Aeqref \Aveqref \Apageref \Avpageref \Acite \Avcite

For example

```
\section{Cím}\label{seca}
\section{Cím}\label{secb}
\Aref{seca}.~és \aref{secb}.~szakaszban
```

„Az 1. és a 2. szakaszban”.

3.3. The huaz hook

During the process of determining the definite article, the `huaz` package replaces the cross-reference commands with their expandable versions, and the formatting commands (`\emph`, `\textbf`, `\small`, etc.) are ignored. Because of this, it is possible that the following example codes work:

```
\newcommand{\osztaly}{V.B}
Idén \az{\textbf{\osztaly}}~osztály rendezi a farsangot.
```

```
\section{Cím}\label{sec}
\az{\textbf{\ref{sec}}}
```

\AddToHook{huaz}{⟨code⟩} • The `huaz` package ignores all the text formatting commands, that is listed in the `huaz` hook. If a formatting command is not in this hook, it can be added by the user. For example

```
\newcommand{\myfont}[1]{{\usefont{T1}{yv1d}{m}{n}#1}}
\newcommand{\mytext}{X.A~osztály}
\az{\myfont{\mytext}}
```

This code returns with an error, since the `\myfont` is not listed in the `huaz` hook. This can be supplemented with the following code:

```
\AddToHook{huaz}{\def\myfont{}}
```

This results that while determining the definite article, the `\myfont` command means nothing. So the following code works perfectly:

```
\AddToHook{huaz}{\def\myfont{}}
\newcommand{\myfont}[1]{{\usefont{T1}{yv1d}{m}{n}#1}}
\newcommand{\mytext}{X.A~osztály}
\az{\myfont{\mytext}}
```

The previous case can be solved without the `huaz` hook in the following way:

```
\newcommand{\myfont}[1]{{\usefont{T1}{yv1d}{m}{n}#1}}
\newcommand{\mytext}{X.A~osztály}
\az*{\mytext}~\myfont{\mytext}
```

If the `\myfont` is included in the `\mytext` definition, the usage of the `huaz` hook is definitely necessary.

```
\AddToHook{huaz}{\def\myfont{}}
\newcommand{\myfont}[1]{{\usefont{T1}{yv1d}{m}{n}#1}}
\newcommand{\mytext}{\myfont{X.A~osztály}}
\az{\mytext}
```

In the following case, it is also necessary to use the `huaz` hook.

```
\AddToHook{huaz}{\def\myfont{}}
\DeclareRobustCommand{\myfont}[1]{{\usefont{T1}{yv1d}{m}{n}#1}}
\renewcommand{\thesection}{\myfont{\arabic{section}}}
\section{Cím}\label{sec}
\aref{sec}
```

Here the `\myfont` is defined as a robust command, because it goes into a moving argument. The case of the previous example rarely occurs, because it is not customary to use a text formatting command when specifying a counter type (Arabic, Roman, etc.).

4. Implementation

We do not publish the complete code here, which is approx. 500 input lines, we just show two essential details.

4.1. Detecting accented characters

All the accented vowels (Hungarian and non-Hungarian) are collected into the `\huaz@list@A` list command. On the other hand, `\huaz@list@X` contains the list of non-Hungarian accented consonants whose pronunciation begin with a vowel (e.g. ń, ñ, ň, ŕ, ř, ś, š, etc.).

The detection of accented characters must be solved in different ways in the following three cases:

Case 1. If the compiler is `pdflatex` and the source file is UTF-8 encoded, then for example, the letter "ö" can be detected with the ASCII characters `^^c3^^b6`, where `c3b6` is the UTF-8 hexadecimal code of "ö". So, with these conditions the definition of the `\huaz@list@A` and `\huaz@list@X` are

```
\def\huaz@list@A{^^c3^^b6,^^c3^^bc,^^c3^^b3,^^c5^^91,^^c3^^ba,^^c3^^a9,...}
\def\huaz@list@X{^^c4^^b9,^^c5^^81,^^c4^^bc,^^c4^^ba,^^c5^^82,^^c5^^85,...}
```

Case 2. If the compiler is `pdflatex` and the source file is ISO-8859-2 encoded, then for example, the letter "ö" can be detected with the ASCII characters `^^f6`, where `f6` is the ISO-8859-2 hexadecimal code of "ö". (Although in this case the accent commands are also suitable for detection.) So then the definition of the `\huaz@list@A` and `\huaz@list@X` are

```
\def\huaz@list@A{^^a1,^^b1,^^c1,^^c2,^^c3,^^c4,^^c9,^^ca,^^cb,^^cc,^^cd,^^ce,...}
\def\huaz@list@X{^^a3,^^a5,^^a6,^^a9,^^aa,^^b3,^^b6,^^b9,^^ba,^^c0,^^c5,^^d1,...}
```

Case 3. If the compiler is `xelatex` or `lualatex` then the source file can only be UTF-8 encoded. In this case for example, the letter "ö" can be detected with the ASCII characters `^^^^00f6`, where `U+00f6` is the Unicode code point of "ö". So, on these terms the definition of the `\huaz@list@A` and `\huaz@list@X` are

```
\def\huaz@list@A{^^^^00f6,^^^^00fc,^^^^00f3,^^^^0151,^^^^00fa,^^^^00e9,...}
\def\huaz@list@X{^^^^0139,^^^^0141,^^^^013c,^^^^013a,^^^^0142,^^^^0145,...}
```

The `\ifpdftex` and the `\inputencodingname` commands are used to distinguish these cases.

4.2. The central inside command of the huaz package

The most important inside command of the `huaz` package is the `\huaz@z{`⟨*input*⟩`}`. The effect of this command: The value of `\ifhuaz@must@z@` will be true, if and only if the ⟨*input*⟩ must be preceded by the definite article "az", otherwise its value will be false. If the `\ifhuaz@must@z@` is true, then the output of the `\huaz@z{`⟨*input*⟩`}` is "z", otherwise nothing. The definition of this macro consists of the following major blocks:

Block 1. The first syntax unit of ⟨*input*⟩ is included in the following list: `\AA,\aa, \AE,\ae,e,u,i,o,a,E,U,I,O,A,5`.

```
\StrChar{#1}{1}[\huaz@temp]%
\@for\huaz@list:={\AA,\aa,\AE,\ae,e,u,i,o,a,E,U,I,O,A,5}%
\do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
```

Block 2. The first syntax unit of ⟨*input*⟩ is an accent command and the second one is one of the following `e,u,i,o,a,E,U,I,O,A`.

```
\huaz@temp@if@false%
\StrChar{#1}{1}[\huaz@temp]%
\@for\huaz@list:={\",\',\H,\`,\~,\^,\v,\u,\=,\k}%
\do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@temp@if@true}{}}%
\ifhuaz@temp@if@%
  \StrChar{#1}{2}[\huaz@temp]%
  \@for\huaz@list:={e,u,i,a,E,U,I,O,A}%
  \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
\fi%
```

Block 3. The first syntax unit of ⟨input⟩ is included in the `\huaz@list@A`.

```
\ifhuaz@must@oneunit@%
  \StrLeft{#1}{1}[\huaz@tempa]%
\else%
  \StrLeft{#1}{2}[\huaz@tempa]%
\fi%
\@for\huaz@list:=\huaz@list@A%
\do{\StrLeft{\huaz@list}{2}[\huaz@tempb]%
    \IfStrEq{\huaz@tempa}{\huaz@tempb}{\huaz@must@z@true}{}}%
```

Block 4. The second syntax unit of ⟨input⟩ (if any) is not a letter, but a number, punctuation mark or space, while the first syntax unit is included in the following list: `f,l,m,n,r,x,y,F,L,M,N,S,R,X,Y` or (in case `latin2`, `xelatex`, `lualatex`) in `\huaz@list@X`.

```
\huaz@temp@if@false%
\StrChar{#1}{2}[\huaz@temp]%
\@for\huaz@list:={;,`,',",+,!,/,=,(,),<,>,@,.,?,:,-,*,0,1,2,3,4,5,6,7,8,9,{,},
                  { },{},\&,\#,\_,\unskip,\kern}%
\do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@temp@if@true}{}}%
\ifhuaz@temp@if@%
  \StrChar{#1}{1}[\huaz@temp]%
  \@for\huaz@list:={f,l,m,n,r,x,y,F,L,M,N,S,R,X,Y}%
  \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
  \ifhuaz@must@oneunit@%
    \ifhuaz@must@z@\else%
      \@for\huaz@list:=\huaz@list@X%
      \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
    \fi%
  \fi%
\fi%
```

Block 5. The third syntax unit of ⟨input⟩ (if any) is not a letter, but a number, punctuation mark or space, the second syntax unit is included in the following list: `f,l,m,n,r,x,y,F,L,M,N,S,R,X,Y` and the first one is an accent command.

```
\huaz@temp@if@false%
\StrChar{#1}{3}[\huaz@temp]%
\@for\huaz@list:={;,`,',",+,!,/,=,(,),<,>,@,.,?,:,-,*,0,1,2,3,4,5,6,7,8,9,{,},
                  { },{},\&,\#,\_,\unskip,\kern}%
\do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@temp@if@true}{}}%
\ifhuaz@temp@if@%
  \huaz@temp@if@false%
  \StrChar{#1}{2}[\huaz@temp]%
  \@for\huaz@list:={f,l,m,n,r,x,y,F,L,M,N,S,R,X,Y}%
  \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@temp@if@true}{}}%
  \ifhuaz@temp@if@%
    \StrChar{#1}{1}[\huaz@temp]%
    \@for\huaz@list:={\",\',\H,\`,\~,\^,\v,\u,\=,\k}%
    \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
```

```
  \fi%
\fi%
```

Block 6. The third syntax unit of ⟨*input*⟩ (if any) is not a letter, but a number, punctuation mark or space, and the first two syntax unit is included in the following list: `ly,Ly,LY,ny,Ny,NY,sz,Sz,SZ` or (in case `pdflatex + utf8`) in `\huaz@list@X`.

```
\huaz@temp@if@false%
\StrChar{#1}{3}[\huaz@temp]%
\@for\huaz@list:={;,`,',",+,!,/,=,(,),<,>,@,.,?,:,-,*,0,1,2,3,4,5,6,7,8,9,{,},
                  { },{},\&,\#,\_,\unskip,\kern}%
\do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@temp@if@true}{}}%
\ifhuaz@temp@if@%
  \StrLeft{#1}{2}[\huaz@temp]%
  \@for\huaz@list:={ly,Ly,LY,ny,Ny,NY,sz,Sz,SZ}%
  \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
  \ifhuaz@must@oneunit@\else%
    \ifhuaz@must@z@\else%
      \@for\huaz@list:=\huaz@list@X%
      \do{\IfStrEq{\huaz@temp}{\huaz@list}{\huaz@must@z@true}{}}%
    \fi%
  \fi%
\fi%
```

Block 7. The first syntax unit of ⟨*input*⟩ is 1 (“egy”) and the second one is not number.

```
\StrChar{#1}{1}[\huaz@temp]%
\IfStrEq{\huaz@temp}{1}{%
  \StrChar{#1}{2}[\huaz@temp]%
  \IfInteger{\huaz@temp}{}{\huaz@must@z@true}%
}{}%
```

Block 8. The first four characters of ⟨*input*⟩ is a number from 1000 to 1999 (“ezer. . .”) and the fifth one is not number.

```
\StrLen{#1}[\huaz@temp]%
\setcounter{huaz@temp@count}{\huaz@temp}%
\ifnum\value{huaz@temp@count}>3%
  \StrChar{#1}{1}[\huaz@temp]%
  \IfStrEq{\huaz@temp}{1}{%
    \StrMid{#1}{2}{4}[\huaz@temp]%
    \IfInteger{\huaz@temp}{%
      \StrChar{#1}{5}[\huaz@temp]%
      \IfInteger{\huaz@temp}{}{\huaz@must@z@true}%
    }{}%
  }{}%
\fi%
```

Block 9. The first seven characters of ⟨*input*⟩ is a number from 1000000 to 1999999 ("egymillió. . . ") and the eighth one is not number.

```
\StrLen{#1}[\huaz@temp]%
\setcounter{huaz@temp@count}{\huaz@temp}%
\ifnum\value{huaz@temp@count}>6%
  \StrChar{#1}{1}[\huaz@temp]%
  \IfStrEq{\huaz@temp}{1}{%
    \StrMid{#1}{2}{7}[\huaz@temp]%
    \IfInteger{\huaz@temp}{%
      \StrChar{#1}{8}[\huaz@temp]%
      \IfInteger{\huaz@temp}{}{\huaz@must@z@true}%
    }{}%
  }{}%
\fi%
```

Block 10. The first ten characters of ⟨*input*⟩ is a number from 1000000000 to 1999999999 ("egymilliárd. . . ") and the eleventh one is not number.

```
\StrLen{#1}[\huaz@temp]%
\setcounter{huaz@temp@count}{\huaz@temp}%
\ifnum\value{huaz@temp@count}>9%
  \StrChar{#1}{1}[\huaz@temp]%
  \IfStrEq{\huaz@temp}{1}{%
    \StrMid{#1}{2}{10}[\huaz@temp]%
    \IfInteger{\huaz@temp}{%
      \StrChar{#1}{11}[\huaz@temp]%
      \IfInteger{\huaz@temp}{}{\huaz@must@z@true}%
    }{}%
  }{}%
\fi%
```

There are still a lot of interesting and important parts of the code. For example, the conversion of the Roman numerals to Arabic, the expandable version of the `\cite` command, replacing the cross-reference commands with their expandable versions, ignoring the formatting commands.

5. Future work

The package has some limitations. It only recognizes the standard cross-reference commands, furthermore it handles the `\cite` command in default case, with `natbib` and `bibtex`, but only in certain cases with `biblatex`. Narrowing these limitations is an important development direction. Of course, at the end of testing, the goal is to publish the package on the The Comprehensive TeX Archive Network, which is the central place for all kinds of material around TEX.

6. Summary

Currently, there are two tools (`magyar.ldf`, `nevelok.sty`) in the LaTeX system, with which the user can automatically insert Hungarian definite articles in front of cross-references and macros storing text. Both have numerous errors and omissions. The most serious problem is that the UTF-8 encoded source files are not handled. The `huaz` package provides an alternative to these two packages, that is of course UTF-8 compatible, less buggy, much more flexible and modern.

References

[1] American Mathematical Society: *User's Guide for the amsmath Package (Version 2.1)*, The Comprehensive TeX Archive Network (2020), url: https://www.ctan.org/pkg/amsmath.

[2] Z. Bodó: *nevelok – LaTeX package for automatic definite articles for Hungarian*, The Comprehensive TeX Archive Network (2015), url: https://ctan.org/pkg/nevelok.

[3] L. Lamport: *LaTeX: A document preparation system, User's guide and reference manual*, 2nd edition, Addison Wesley, 1994, isbn: ISBN-10: 0201529831, ISBN-13: 978-0201529838.

[4] H. Oberdiek: *The refcount package*, The Comprehensive TeX Archive Network (2019), url: https://ctan.org/pkg/refcount.

[5] S. Rahtz, H. Oberdiek: *Hypertext marks in LaTeX: a manual for hyperref*, The Comprehensive TeX Archive Network (2022), url: https://ctan.org/pkg/hyperref.

[6] P. Szabó: *Typing Hungarian text with MagyarLaTeX*, Hungarian (2009), url: https://math.bme.hu/latex/magyarldf-doc.pdf.

[7] C. Tellechea: *xstring User's manual*, The Comprehensive TeX Archive Network (2021), url: https://ctan.org/pkg/xstring.

[8] The LaTeX3 Project Team: *LaTeX News, Issues 1–35*, The Comprehensive TeX Archive Network (2022), url: https://www.latex-project.org/news/latex2e-news/ltnews.pdf.

[9] The LaTeX Project Team: *The iftex package*, The Comprehensive TeX Archive Network (2022), url: https://ctan.org/pkg/iftex.

[10] F. Wettl, Gy. Mayer, P. Szabó: *LaTeX handbook*, Hungarian, Budapest, Hungary: Panem, 2004, isbn: ISBN 963 545 398 1.