

Visual argumentations in teaching trigonometry

Francesco Laudano

Università degli studi del Molise

francesco.laudano@unimol.it

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Abstract

In this paper, we study the possibility of building a learning path that allows students to develop trigonometric knowledge and skills by the end of Grade 10 of secondary science-based schools. In particular, we describe an action research experiment, in part done through distance learning, aimed at incorporating all trigonometry topics within the framework of the study of Euclidean geometry. The inquiry-based learning methodology and the support of dynamic geometry software with a laboratory teaching approach were used. The learning path is based on several “visual/dynamic proof” and is explained by an example lesson on the Cosines Law.

The experiment could be extended by teachers into physical/virtual classrooms and could offer practical strategies and tools for teaching trigonometry.

Keywords: Trigonometric path, Euclidean geometry, trigonometric relationships, visual proofs

AMS Subject Classification: 97G10, 97G60

1. Introduction

The multiple applications of trigonometry have made it a fundamental milestone in the training of students, which has led many governments to introduce the study of this topic since the first year of secondary science-based schools. This practice can be seen, for example, in Canada ([13, 14]), France ([8–10]), Hungary ([6]),

Italy ([11]), the UK ([5]), and the USA ([1]). However, the study of trigonometry is performed mainly in the third and fourth years, probably because some trigonometric reasoning seems too abstract for 14- to 15-year-old students. Moreover, as highlighted in [2], the learning of trigonometry is a highly difficult area of mathematics for both students and teachers. Furthermore, learning outcomes are strongly influenced by the teaching approach, which in some cases points towards the application of trigonometric relations, almost entirely omitting proofs, and in other cases, in contrast, emphasizes algebraic-formal aspects at the risk of losing students' attention. In the following, without neglecting these two important aspects of trigonometry training, we present a geometric path that allows to develop the applications of trigonometry without overlooking the educational potential of the proofs. The core idea, which is developed in the following, consists of incorporating trigonometry into the elementary geometry path, replacing the classic formal arguments with visual and dynamic proofs, which are simpler to build, especially in distance learning. This is in order to improve students' demonstrative skills, since, as pointed out in [18] and [7], even students who are particularly talented in mathematics, they still have difficulty in solving problems that require proof.

The experiment, in part done through distance learning, was built on the inquiry-based learning (IBL) methodology (see [4] for a literature review). The use of educational technologies in the teaching of trigonometry has been analyzed by Ross et al. [16] and can provide decisive help to teachers. Therefore, in the development of the path, dynamic geometry software (GDS) with a laboratory teaching approach was used. The construction of mathematical ideas is based upon real problems and follows the different phases of inquiry until students' knowledge is deeply rooted through the relevant proofs.

2. The didactic project

Since the IBL is founded on concrete problems and stimulates questions and actions to solve them, it seemed as the most suitable methodology to use in order to implement an innovative learning activity for students in G9–G10. The fundamental ideas of IBL can be found in Dewey's thought [3]. According to him, students build their knowledge and skills through a sequence of research stages, in which they formulate hypothesis, verify them and discuss the results of their investigations [17]. The IBL activities are based on a workshop format: the teacher acts as a facilitator by guiding students' development and exchange of ideas by asking them appropriate questions; students work in small groups and take active part in their learning process [15]. In the teaching of geometry, we start from a concrete problem and, after having identified its essence, we search for its solution. In the activity that we describe, the basic idea consists of making students observe and manipulate through GDS some geometric constructions which had been appropriately prepared, and then gradually direct students towards the discovery of the formulas and trigonometric theorems. In the next phase, the teacher raises

students' awareness to the need to prove the relations that have been discovered, and guides them to the search of the proofs. Later, in order to establish the effectiveness of the activity, the initial problem from which the activity started is solved by using the discovered properties. In the course of the didactic activities, we have tried to chain the evolution of geometric knowledge and students' gratification in the awareness of their mutual influence. In particular, efforts have been made to concatenate the two aspects in order to trigger a virtuous circle in the learning process. In fact, we believe that the possibility of discovering "something new" provides pupils with a "motivational lever" that can be decisive for the development of skills. What is presented in Table 1 is the path that incorporates the study of trigonometry in the development of geometry, thus valuing also this part of Mathematics that is considered by many authors fundamental in the training of students' logical abilities. The trigonometry Units are developed in greater detail in Table 2.

Table 1. Units, General outcomes and Chapters.

Unit	General Outcome	Chapter
Isometry G9	Solve problems involving congruence between polygons, using both measurements and geometric proofs	Congruences Triangles Quadrilaterals Circles
Similarity G9	Solve problems involving similitude between polygons, using both measurements and geometric proofs	Similarities Triangles Circle
Trigonometry 1 G9	Solve triangles using trigonometric theorems	Trigonometric Ratios Right Triangles Any Triangles
Equivalence G10	Solve problems using trigonometric theorems	Equivalences Triangles and Trapezoids Special n -gons Euclidean Theorems
Trigonometry 2 G10	Solve problems involving trigonometric functions using trigonometric identities and trigonometric equations	Trigonometric Functions Trigonometric Identities Equations and Problems

Table 2. Detail of the Trigonometry Chapters.

Unit	Chapter	Lesson
Trigonometry 1	Trigonometric Ratios	Sine, Cosine, Tangent ratio Basic trigonometric identities The ratio of 30° , 60° and 45°
	Right Triangle	Pythagoren Theorem Right Triangle Theorems Solving righth triangles
	Any Triangle	Triangle Area Formula Length of a cord, Sines Law Cosines Law Solving any triangles
Trigonometry 2	Trigonometric Functions	Graph of Trigonometric Functions Periodicity of trigonometric functions Associated angles relations
	Trigonometric Identities	Ptolemy's identities Double angle formulas Half angle formulas Product-sum identities Parametric formulas
	Equations and Problems	Basic equations Some non linear equation Application problems

3. A lesson plan on the Cosines Law

In this section, we describe an example lesson on Cosines Law directed to G9 students. In particular, we get an overview of how students discovered the cosine law while using dynamic geometry software in the classroom.

The activity, which lasts approximately 90 minutes, is conceived as a DGS lab and entailed the use of worksheets to guide the students to start the inquiry.

Objectives:

- Discover the Cosines Law.
- Proof Cosines Law.
- Solve problems using Cosines Law.

For this activity it seemed appropriate to give students many suggestions in order to guide their search in the right direction. Instead, for the activities that followed, suggestions were gradually reduced in order to make students become more and more autonomous in the management of the inquiry.

If we get an overview of how students discovered the cosine law while using dynamic geometry software in the classroom, there is no need to mention the test.

Step 1. Problem posing

Everyone knows the Pythagorean Theorem and how it allows to solve problems related to right triangles. But what happens when the problem concerns non-right triangles? I.e. consider the following text:

“An aircraft tracking station determines the distance from a common point O to each aircraft and the angle between the aircraft. If angle O between two aircraft is equal to 49° and the distances from point O to the two aircraft are 50 km and 72 km, find distance between the two aircraft (round answers to 1 decimal place).”

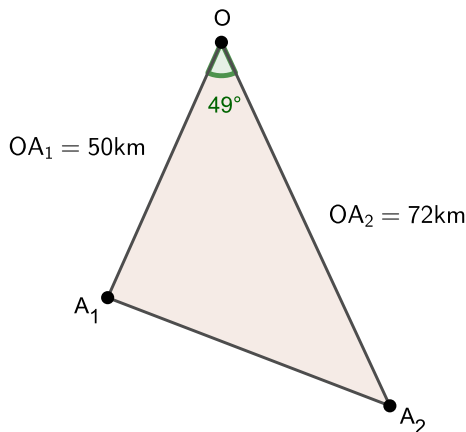


Figure 1. Sketch of the problem.

After a few minutes of reflection followed by a teacher-led discussion, the students propose two possible solution strategies. The first starts by drawing a perpendicular line from point A_1 to OA_2 and applies the Pythagorean formula twice. The second one starts from the GDS construction of a triangle B_1PB_2 (similar to A_1OA_2), with $B_1P = 5$ cm, $PB_2 = 7.2$ cm and $B_1\hat{P}B_2 = 49^\circ$ and find the distance between the two aircraft multiplying the length of B_1B_2 for the ratio 10^6 . The teacher points out that both strategies are valid and general. However, a formula that expresses the length of one side of a triangle as a function of the other sides and the angle between them would make it possible to solve the problem more effectively. Therefore the teacher invites the students to reflect and discuss to answer the following questions.

Question 1. *In principle, could there be a general formula that expresses the length of one side of a triangle as a function of the other sides and the angle between them?*

Some students point out that two sides and the angle between them uniquely determine all the triangle's elements, so they concur that such a formula might exist. Then, the teacher propose an investigation aimed at an extension of Pythagoras' theorem that would be useful to solve the aircraft problem.

Step 2. Working with DGS

The search for the formula involves several stages. The first phase is aimed at discovering the role played by the angle and, in particular, at understanding which of the functions $\sin \alpha$ and $\cos \alpha$ could intervene in the formula. The teacher invites students to work on a file suitably prepared with DGS (see Figure 2) so that they can change the width of the angle \widehat{BCA} by acting on the slider, while BC and DC are segments with a given length equal to 10 cm.

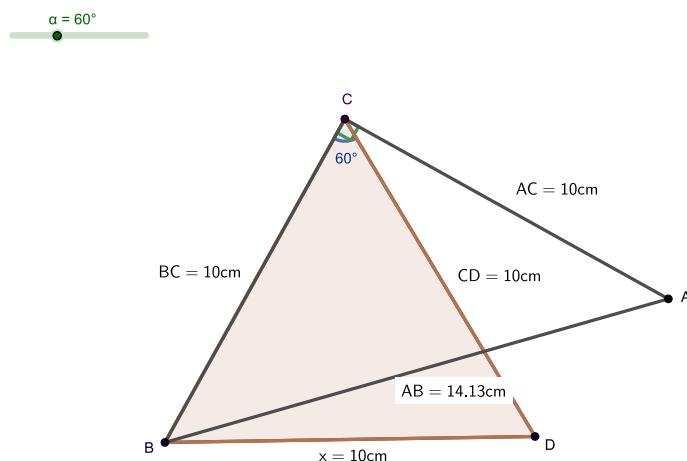


Figure 2. Working with isosceles triangles.

Using the slider, students immediately realize that increasing the width of the angle \widehat{BCA} also increases the length of AB . Then the teacher guides them through the following questions and the related discussion.

Question 2. *How does the value of AB^2 change as the sine and cosine of the angle α ?*

To answer this question, the teacher invites the students to note down on a worksheet the sine and cosine values of the 30° , 45° , 60° , 90° , 120° , 150° angles and the corresponding values of AB^2 .

Table 3. Relations between $\sin \alpha$, $\cos \alpha$ and AB^2 .

α	$\sin \alpha$	$\cos \alpha$	AB^2
30°	0.5	0.866	...
45°
60°	0.866	0.5	14.13 cm
90°
120°
150°

Comparing the relationships shown in the table, the students note that the value of the cosine constantly decreases as the angle increases. Instead that of the sine increases for $0^\circ < \alpha \leq 90^\circ$ while it decreases for $90^\circ \leq \alpha < 180^\circ$. Therefore the length of AB does not seem to be related to $\sin \alpha$.

Moreover, they observe that as α increases, the quantity $-\cos \alpha$ increases, and so does the length of AB .

Therefore the teacher advises students to look for the more simple formula verifying the conditions that emerged with the use of the DSG. Then, since the relation sought must extend the Pythagorean theorem, it guides them through the following questions and related discussion.

Question 3. *Can the formula be obtained from the Pythagorean relation by subtracting an appropriate quantity that depends on $\cos \alpha$?*

The teacher leads the discussion to obtain AB^2 by subtracting a term proportional to $\cos \alpha$ from $BC^2 + AC^2$. Then he invites the students to calculate the proportionality factor in various cases using DGS from Figure 2, to answer the following question.

Question 4. *What is the relationship between AB^2 and α in the previous cases?*

The teacher checks that students, by calculating the quantities $(AB^2 - AC^2 - BC^2)/\cos \alpha$, always get the value 200, as α changes. Then it check to see if students really discover that $AB^2 = 100 + 100 - 2 \cdot 100 \cos \alpha$.

Therefore, the teacher invites students to modify the length of AC and BC to check if the above relation can be extended to other isosceles triangles. By carrying out these tests, students discover the formula $AB^2 = AC^2 + BC^2 - 2 \cdot AC \cdot BC \cdot \cos \alpha$ for isosceles triangles.

In the next phase, the teacher makes the students check if the formula they have just found can be extended to the scalene triangle BCD in Figure 3. The related file has been prepared so that the students can change both the width of the angle \widehat{BCD} and the length of BC and DC , using the dynamism of the software. As for isosceles triangles, the teacher leads the discussion to obtain BD^2 by subtracting from $BC^2 + CD^2$ a term proportional to $\cos \alpha$. Then, he invites the students to

calculate the proportionality factor in various cases through DGS to answer the following question.

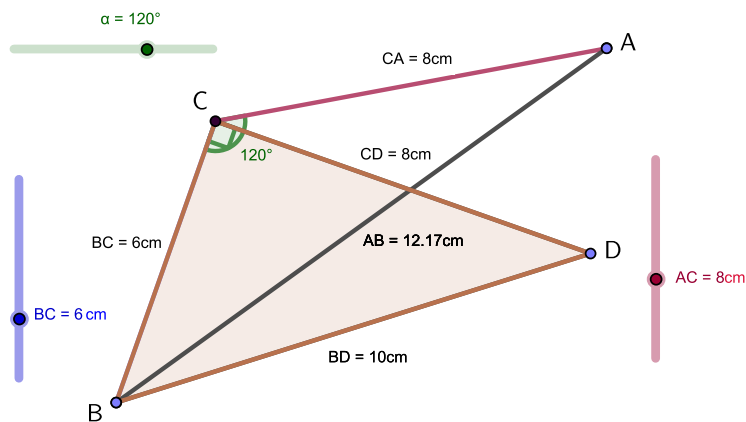


Figure 3. Working with scalene triangles.

Question 5. *Is the above relation still valid for the triangle BCD in Figure 3?*

Students verify that as α changes, the amount $(BD^2 - BC^2 - CD^2)/\cos \alpha$ is constant. Then the teacher invites them to identify the algebraic relationship between AB , BC and this factor, and check to see if students really discover that $AB^2 = 36 + 64 - 2 \cdot 6 \cdot 8 \cos \alpha$. At this point, the teacher invites the students to modify the length of sides AC and BC using the related sliders, to extend to any scalene triangles the relationship between the proportionality constant and their lengths. In this way the students discover that the above formula could hold for every triangle.

Step 3. Formulating a conjecture

The teacher proposes to students to build more triangles, using the prepared DGS file, to test the formula we seem to have discovered. The questions posed by the teacher guide the students in formulating the conjecture.

Question 6. *Is the above relation still valid for any triangle, in particular for obtuse triangles?*

Question 7. *At this point, can we formulate a conjecture to generalize the Pythagorean Theorem?*

From the previous observations the students obtain the formula

$$AB^2 = AC^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos \alpha.$$

Through the next question and the peer discussion that follows, the teacher lets emerge the need for a proof.

Question 8. *Can we be sure that the previous formula holds for all triangles?*

Step 4. Proving the law

Since the formula to prove is an extension of Pythagorean Theorem, the teacher proposes to prove it by trying to generalize a proof of this Theorem. In particular, he suggests starting from the recently studied proof, which showed in the left of Figure 4, based on Tangent-Secant Theorem.

Question 9. *Can the reasoning showed in the left side of Figure 4 be extended to non-right triangles?*

If necessary, the teacher suggests to use the Two Secant Theorem, since it naturally extends the Tangent-Secant Theorem, and guide students in understanding proof. In the right side of the Figure 4 we show a sketch of the proof, that will only be shown to pupils if they can't find out for themselves.

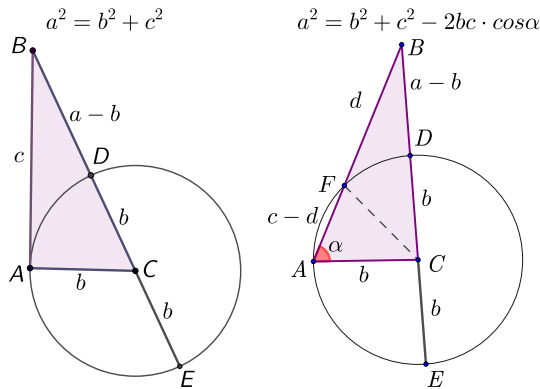


Figure 4. In the left we have $(a + b) : c = c : (a - b)$, i.e. the Pythagorean Theorem. In the right we have $(a + b) : c = d : (a - b)$ and $c - d = 2b \cdot \cos \alpha$, hence the Cosines Law.

Step 5. Solving the problem

At this point the teacher points out to the students that they have the tools to solve the initial problem and asks them to find the solution. He also invites them to reflect and discuss by asking the following questions.

Question 10. *What kind of problems can we solve by using the Cosines Law?*

Question 11. *Can we find another way to solve the aircraft problem?*

Question 12. *Can we find another way to prove the Cosines Law?*

4. Conclusions

The experimentation of the path that we have presented involved approximately 100 pupils from different geographical and socio-cultural backgrounds. At the end

of each unit the efficiency of the activity was assessed by considering the following aspects:

- students' ability to complete the research path by discovering and formulating the theorems that have been studied, and in particular the trigonometric relations;
- students' ability to use the acquired knowledge to solve applied problems.

The data collection was of an exploratory research; therefore, it could be considered only as a prelude to a more in-depth research plan aimed at comparing the results obtained using more traditional teaching approaches. In particular, we consider it appropriate to analyze the levels reached by students, according to van Hiele classification [19].

With this work we want also to highlight the importance of geometric reasoning in the didactic field, also due to the fact that in the last decade the teaching of Euclidean geometry in secondary schools seems to have lost its vigor. Yet, this part of mathematics is considered fundamental in the development of students' logical abilities by many authors [12].

While taking into account the limitations of our research, the initial results obtained by the students, if confirmed by further and more in-depth experiments, seem to indicate the achievement of the objectives of the experimentation. In particular, the visual/dynamic proofs seemed to be useful in the DL, where it is more difficult to maintain students' attention on long formal argumentations, whereas it would be more effective to stimulate them to think about geometric figures.

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