What positive integers $n$ can be presented in the form
\[ n = (x + y + z)(1/x + 1/y + 1/z)? \]

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Submitted: November 9, 2020
Accepted: April 19, 2021
Published online: April 27, 2021

Abstract
This paper shows that the equation in the title does not have positive integer solutions when $n$ is divisible by 4. This gives a partial answer to a question by Melvyn Knight. The proof is a mixture of elementary $p$-adic analysis and elliptic curve theory.

Keywords: Elliptic curves, $p$-adic numbers
AMS Subject Classification: 11G05, 11D88

1. Introduction

According to Bremner, Guy, and Nowakowski [1], Melvyn Knight asked what integers $n$ can be represented in the form
\[ n = (x + y + z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right), \] (1.1)
where $x, y, z$ are integers. In the same paper [1], the authors made an extension study of (1.1) in integers when $n$ is in the range $|n| \leq 1000$. Integer solutions are found except for 99 values of $n$. The question becomes more interesting if we ask for positive integer solutions, which was also briefly discussed in [1, Section 2]. In this paper, we will prove the following theorem:
Theorem 1.1. Let $n$ be a positive integer. Then equation
\[
(x + y + z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = n
\]
does not have positive integer solutions if $4 \mid n$.

This theorem gives the first parametric family when (1.1) does not have positive integer solutions. The proof technique is a nice combination of $p$-adic analysis and elliptic curve theory, which was successfully applied to prove the insolubility of the equation
\[
(x + y + z + w) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) = n
\]
for the families $n = 4m^2, 4m^2 + 4, m \in \mathbb{Z}$ and $m \not\equiv 2 \pmod{4}$, see [2].

2. The Hilbert symbol

Let $p$ be a prime number, and let $a, b \in \mathbb{Q}_p$. The Hilbert symbol $(a, b)_p$ is defined as
\[
(a, b)_p = \begin{cases} 
1, & \text{if the equation } ax^2 + by^2 = z^2 \text{ has a solution } \\
(X, Y, Z) \neq (0, 0, 0) \text{ in } \mathbb{Q}_p^3, \\
-1, & \text{otherwise.}
\end{cases}
\]
The symbol $(a, b)_\infty$ is defined similarly but $\mathbb{Q}_p$ is replaced by $\mathbb{R}$. The following properties of the Hilbert symbol are true, see Serre [3, Chapter III]:

(i) For all $a, b,$ and $c$ in $\mathbb{Q}_p^*$, then
\[
(a, bc)_p = (a, b)_p(a, c)_p,
\]
\[
(a, b^2)_p = 1.
\]

(ii) For all $a$ and $b$ in $\mathbb{Q}_p^*$, then
\[
(a, b)_\infty \prod_{p \text{ prime}} (a, b)_p = 1.
\]

(iii) Let $p$ be a prime number, and let $a$ and $b$ in $\mathbb{Q}_p^*$. Write $a = p^\alpha u$ and $b = p^\beta v$, where $\alpha = v_p(a)$ and $\beta = v_p(b)$. Then
\[
(a, b)_p = (-1)^{\alpha\beta(p-1)/2} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha \quad \text{if } p \neq 2,
\]
\[
(a, b)_p = (-1)^{(u-1)(v-1)/4 + \alpha(u^2-1)/8 + \beta(v^2-1)/8} \quad \text{if } p = 2,
\]
where $\left(\frac{u}{p}\right)$ denotes the Legendre symbol.
3. A main theorem

**Theorem 3.1.** Let \( n \) be a positive integer divisible by 4. Let \( u \) and \( v \) be nonzero rational numbers such that

\[
v^2 = u(u^2 + (n^2 - 6n - 3)u + 16n).
\]

Then

\[ u > 0. \]

Let \( A = n^2 - 6n - 3 \), \( B = 16n \) and \( D = n - 1 \). Then

\[
A^2 - 4B = (n - 9)(n - 1)^2.
\]

Now

\[
v^2 = u(u^2 + Au + B). \tag{3.1}
\]

The proof of Theorem 3.1 is achieved by means of the following three lemmas.

**Lemma 3.2.** If \( p \) is an odd prime number, then

\[
(u, -D)_p = 1.
\]

**Proof.** Let \( r = v_p(u) \). Then \( u = p^r s \), where \( s \in \mathbb{Z}_p \), and \( p \nmid s \).

Case 1: \( r < 0 \). Then from (3.1), we have

\[
v^2 = p^{3r} s (s^2 + p^{-r} A + Bp^{-2r}).
\]

Therefore \( 2v_p(v) = 3r \), hence \( 2 \mid r \). Now

\[
(p^{-3r/2}v)^2 = s(s^2 + p^{-r} A + Bp^{-2r}). \tag{3.2}
\]

Note that \( p \nmid s \). Taking (3.2) modulo \( p \) gives \( s \) is a square modulo \( p \). Hence \( s \in \mathbb{Z}^2_p \).

We also have \( 2 \mid r \), so \( u = 2^r s \in \mathbb{Q}^2_p \). Therefore \( (u, -D)_p = 1 \).

Case 2: \( r = 0 \).

If \( p \nmid D \), then both \( u \) and \( -D \) are units in \( \mathbb{Z}_p \). Therefore \( (u, -D)_p = 1 \).

If \( p \mid D \), then \( n \equiv 1 \pmod{p} \). Hence \( A = n^2 - 6n - 3 \equiv -8 \pmod{p} \) and \( B = 16n \equiv 16 \pmod{p} \). Thus

\[
v^2 \equiv u(u^2 - 8u + 16) \pmod{p}
\]

\[
\equiv u(u - 4)^2 \pmod{p}. \tag{3.3}
\]

If \( u \equiv 4 \pmod{p} \), then \( u \in \mathbb{Z}^2_p \). Hence \( (u, -D)_p = 1 \). If \( u \not\equiv 4 \pmod{p} \), then from (3.3), we have

\[
u \equiv \left(\frac{v}{u - 4}\right)^2 \pmod{p}.
\]

Therefore \( u \in \mathbb{Z}^2_p \). Hence \( (u, -D)_p = 1 \).
Case 3: $r > 0$. Then (3.1) becomes
\[
v^2 = p^r s (p^{2r}s^2 + Ap^r s + B). \tag{3.4}
\]
If $p \mid B$, then $p \mid n$. Therefore $-D = 1 - n \equiv 1 \pmod{p}$. Hence $-D \in \mathbb{Z}_p^2$. Thus $(u, -D)_p = 1$.

If $p \nmid B$, then from (3.4) we have $r = 2v_p(v)$. Thus $2 \mid r$.

If $p \nmid D$, then both $s$ and $-D$ are units in $\mathbb{Z}_p$. Therefore $(s, -D)_p = 1$. Hence
\[
(u, -D)_p = (p^r s, -D)_p = (s, -D)_p = 1.
\]

If $p \mid D$, then $n \equiv 1 \pmod{p}$. Therefore $A = n^2 - 6n - 3 \equiv -8 \pmod{p}$ and $B = 16n \equiv 16 \pmod{p}$. Let $\omega = p^{-\frac{r}{2}} v$. Because $r = 2v_p(v)$, we have $p \nmid \omega$. From (3.4) we have
\[
\omega^2 \equiv s(p^{2r}s^2 - 8p^r s + 16) \equiv 16s \pmod{p},
\]
so that
\[
s \equiv (\omega/4)^2 \pmod{p}.
\]
Thus $s \in \mathbb{Z}_p^2$. Hence $(s, -D)_p = 1$. Note that $2 \mid r$, therefore
\[
(u, -D)_p = (p^r s, -D)_p = (s, -D)_p = 1. \tag{\Box}
\]

**Lemma 3.3.** We have
\[
(u, -D)_2 = 1.
\]

**Proof.** Let $n = 4k$, where $k \in \mathbb{Z}^+$. 

If $2 \mid k$, then $-D = 1 - 4k \equiv 1 \pmod{8}$. Therefore $-D \in \mathbb{Z}_2^2$. Hence $(u, -D)_2 = 1$. So we only need to consider the case $2 \nmid k$. Let $r = v_2(u)$. Then $u = 2^r s$, where $2 \nmid s$.

Case 1: $2 \mid r$. Then
\[
(u, -D)_2 = (2^r s, 1 - 4k)_2
= (s, 1 - 4k)_2
= (-1)^{(s-1)(1-4k-1)/4}
= 1.
\]

Case 2: $2 \nmid r$. We show that this case is not possible.

If $r < 0$, then from (3.1), we have
\[
v^2 = 2^{3r} s(s^2 + 2^{-r} As + 2^{-2r} B).
\]
Therefore $3r = 2v_2(v)$. Hence $2 \mid r$, a contradiction.

If $r \geq 0$, then (3.1) becomes
\[
v^2 = 2^r s(2^{2r}s^2 + 2^r (16k^2 - 24k - 3)s + 2^6k). \tag{3.5}
\]
If $r \geq 7$, then
\[ v^2 = 2^{r+6}s(2^{2r-6}s^2 + (16k^2 - 24k - 3)2^{r-6}s + k). \]
Therefore $r + 6 = 2v_2(v)$. Hence $2 \mid r$, a contradiction.
If $r < 7$, then $r \leq 5$. Let $\phi = \frac{r}{2^s}$. Then from (3.5), we have
\[ \phi^2 = s(2^r s^2 + (16k^2 - 24k - 3)s + 2^{6-r}k). \]  
(3.6)

If $r = 5$, then taking (3.6) modulo 8 gives $\phi^2 \equiv s(-3s + 2k) \pmod{8}$. Hence $2sk \equiv \phi^2 + 3s^2 \equiv 4 \pmod{8}$, which is not possible because $2 \nmid sk$.
If $r = 3$, then taking (3.6) modulo 8 gives $\phi^2 \equiv -3s^2 \pmod{8}$. Hence $0 \equiv \phi^2 + 3s^2 \equiv 4 \pmod{8}$, a contradiction.
If $r = 1$, then taking (3.6) modulo 8 gives $\phi^2 \equiv s(2 - 3s) \pmod{8}$. So $2s \equiv 3s^2 + \phi^2 \equiv 4 \pmod{8}$, which is not possible because $2 \nmid s$. \hfill \Box

**Lemma 3.4.**
\[ (u, -D)_\infty = 1. \]

**Proof.** From the product formula for the Hilbert symbol, we have
\[ (u, -D)_\infty \prod_{p \text{ prime, } p < \infty} (u, -D)_p = 1. \]  
(3.7)

By Lemma 3.2, Lemma 3.3, and (3.7), we have $(u, -D)_\infty = 1$. \hfill \Box

To complete the proof of Theorem 3.1, we see that Lemma 3.4 shows that the equation $uX^2 + (1 - n)Y^2 = Z^2$ has a solution $(X, Y, Z) \neq (0, 0, 0)$ in $\mathbb{R}^3$. Because $1 - n < 0$, we have $u > 0$. Hence Theorem 3.1 is proved.

### 4. A proof of Theorem 1.1

We follow [1, Section 2]. Write (1.1) as
\[ x^2(y + z) + x(y^2 + (3 - n)yz + z^2) + yz = 0. \]  
(4.1)

Hence
\[ x = \frac{-y^2 + (n - 3)yz - z^2 \pm \Delta}{2(y + z)}, \]
where $\Delta$ satisfies
\[ \Delta^2 = y^4 - 2(n - 1)yz(y^2 + z^2) + (n^2 - 6n - 3)y^2z^2 + z^4. \]  
(4.2)

Then (4.2) is birationally equivalent to the elliptic curve
\[ v^2 = u(u^2 + (n^2 - 6n - 3)u + 16n), \]  
(4.3)
and we can write out the maps between (4.1) and (4.3):

\[ u = \frac{-4(xy + yz + zx)}{z^2}, \quad v = \frac{2(u - 4n)y}{z} - (n - 1)u, \]

and

\[ x, y \quad \frac{z}{z} = \pm v - (n - 1)u \quad \frac{2(4n - u)}{2(4n - u)}. \]

Then the following is true.

**Proposition 4.1.** The necessary and sufficient conditions for (4.1) to have positive integer solutions \((x, y, z)\) are \(n > 0\) and \(u < 0\).

**Proof.** See Bremner, Guy, and Nowakowski [1, Section 2].

Now, let \(n = 4k\), where \(k \in \mathbb{Z}^+\). Assume there exists a positive integer solution \((x, y, z)\) to (1.1). Then Proposition 4.1 shows that \(u < 0\). If \(v = 0\), then (4.3) implies \(u^2 + (n^2 - 6n - 3)u + 16n = 0\). Therefore \((n - 9)(n - 1)^3 = (n^2 - 6n - 3)^2 - 4 \times 16n\) is a perfect square. Hence \((n - 9)(n - 1)\) is a perfect square. Let \((n - 9)(n - 1) = m^2\). Then \((n - 5)^2 - 16 = m^2\). The equation \(X^2 - 16 = Y^2\) only has integer solutions \((X, Y) = (\pm 5, \pm 3)\). Thus \(n - 5 = \pm 5\), giving no solutions \(n = 4k\). Therefore \(v \neq 0\). Hence \(u, v \neq 0\). From Theorem 3.1, we have \(u > 0\), contradicting Proposition 4.1. Hence (1.1) does not have solutions in positive integers.

**Acknowledgement.** The author would like to thank the referee for his careful reading and valuable comments.

**References**

