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Markov triples with k-generalized Fibonacci components

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Abstract

We find all triples (x, y, z) of k-Fibonacci numbers which satisfy the Markov equation $x^2 + y^2 + z^2 = 3xyz$. This paper continues and extends previous work by Luca and Srinivasan [6].

Keywords: Markov equation, Markov triples, k-generalized Fibonacci numbers, k-Fibonacci numbers.

MSC: 11B39, 11D61, 11J86

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1. Introduction

A positive integer x is known as a Markov number if there are positive integers y, z, such that the triple (x, y, z) satisfies the equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1.1}$$

Some Markov numbers (sequence A002559 in the OEIS [7]) are

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, \ldots.$$

Note that, if (x, y, z) satisfies (1.1), then y and z are also Markov numbers, hence (x, y, z) is called a Markov triple. Clearly, one can permute the order of the three components and assume that $0 < x \le y \le z$.

It is known that $(1, F_{2n-1}, F_{2n+1})$ is a Markov triple for all $n \ge 0$, where F_r denotes the rth Fibonacci number. Luca and Srinivasan [6] showed these are the only Markov triples whose components are all Fibonacci numbers.

For $k \ge 2$, let $\{F_r^{(k)}\}_{r \ge -(2-k)}$ denote the k-generalized Fibonacci sequence given by the recurrence

$$F_r^{(k)} = F_{r-1}^{(k)} + \cdots + F_{r-k}^{(k)}, \quad \text{for all} \quad r \ge 2,$$

with $F_j^{(k)} = 0$ for $j = 2-k, \ldots, 0$ and $F_1^{(k)} = 1$.

We determine all Markov triples of the form $(F_s^{(k)}, F_m^{(k)}, F_n^{(k)})$, where s, m, n are positive integers. That is, we find all the solutions of the Diophantine equation

$$\left(F_s^{(k)}\right)^2 + \left(F_m^{(k)}\right)^2 + \left(F_n^{(k)}\right)^2 = 3F_s^{(k)}F_m^{(k)}F_n^{(k)}. \tag{1.2}$$

By symmetry and since $F_1^{(k)} = F_2^{(k)} = 1$, we assume that $2 \le s \le m \le n$. Many arithmetic properties have recently been studied for the k-generalized Fibonacci sequences. Some Diophantine equations similar to the one discussed in this paper can be found in [1] and [4].

Here is our main result.

Main Theorem. *The only solutinos (k, s, m, n) of equation (1.2) with $k \ge 2$ and $2 \le s \le m \le n$ are the trivial solutions $(k, 2, 2, 2)$ and $(k, 2, 2, 3)$ and the parametric one $(2, 2, 2l-1, 2l+1)$ for some integer $l \ge 2$.*

In particular, there are no non-trivial Markov triples of k-generalized Fibonacci numbers for any $k \ge 3$.

2. Preliminaries

To start, let us assume that (x, y, z) is a Markov triple with $x \le y \le z$. Suppose that $x = y$. Then

$$2x^2 + z^2 = 3x^2 z,$$

which implies $(z/x)^2 = 3z - 2 \in \mathbb{Z}$. Therefore, $z = rx$ where r is some positive integer. We thus get

$$2 + r^2 = 3xr. \tag{2.1}$$

Hence, $r|2$, so $r = 1, 2$ and we obtain the triples $(x, y, z) = (1, 1, 1), (1, 1, 2)$.

Suppose next that $y = z$. Then,

$$x^2 + 2z^2 = 3z^2x,$$

which implies $(x/z)^2 = 3x - 2 \in \mathbb{Z}$. Hence, $z \mid x$, but since $x \le z$, we get $x = y = z$, and again the only possibility is $(x, y, z) = (1, 1, 1)$. The previous observation shows that aside from the triples $(1, 1, 1)$ and $(1, 1, 2)$, each Markov triple consists of different integers. Thus, we obtained for the Diophantine equation (1.2) the trivial solutions (k, s, m, n) given by $(k, 2, 2, 2)$ and $(k, 2, 2, 3)$. From now on, we assume that $1 \le x < y < z$, so $2 \le s < m < n$.

We need some facts about k-generalized Fibonacci numbers. For $k \ge 2$ fixed, by [3] we have the following Binet-like formula for the rth k-generalized Fibonacci number

$$F_r^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{r-1}, \tag{2.2}$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are the roots of the characteristic polynomial

$$\Phi_k(x) = x^k - x^{k-1} - \dots - 1,$$

and

$$f_k(x) := \frac{x - 1}{2 + (k + 1)(x - 2)}.$$

It is known that this polynomial has only one real root larger than 1, let's denote it by $\alpha(= \alpha_1)$. It is in the interval $(2(1 - 2^{-k}), 2)$, see [5, Lemma 2.3] or [8, Lemma 3.6]. The remaining roots $\alpha_2, \dots, \alpha_k$ are all smaller than 1 in absolute value. Furthermore, powers of α can be used to bound $F_r^{(k)}$ (see [2]) from above and below as in the inequality

$$\alpha^{r-2} < F_r^{(k)} < \alpha^{r-1}, \quad \text{which holds for all} \quad r \ge 1. \tag{2.3}$$

It is known from [4] that the coefficient $f_k(\alpha)$ in the Binet formula (2.2) satisfies the inequalities

$$\frac{1}{2} \le f_k(\alpha) \le \frac{3}{4}, \quad \text{for all} \quad k \ge 2. \tag{2.4}$$

It is also known (see [3]) that

$$F_r^{(k)} = f_k(\alpha)\alpha^{r-1} + e_k(r), \quad \text{for all} \quad r \ge 1, \quad \text{with} \quad |e_k(r)| < 1/2, \tag{2.5}$$

and it follows from the recurrence formula that

$$F_r^{(k)} = 2^{r-2} \quad \text{for all} \quad 2 \le r \le k + 1. \tag{2.6}$$

Sometimes we write $\alpha(k) := \alpha$ in order to emphasize the dependence of α on k. It is easy to check that $\alpha(k)$ is increasing as a function of k. In particular, the inequality

$$\phi := \frac{1+\sqrt{5}}{2} = \alpha(2) \leq \alpha(k) < \alpha(k+1) < 2 \tag{2.7}$$

holds for all $k \geq 2$

By (1.2) and (2.3), we have the following relations between our variables:

$$\alpha^{2(n-2)} < (F_n^{(k)})^2 < 3F_s^{(k)}F_m^{(k)}F_n^{(k)} < \alpha^{s+m+n}$$

and

$$3\alpha^{s+m+n-6} < 3F_s^{(k)}F_m^{(k)}F_n^{(k)} < (3F_n^{(k)})^2 < 3\alpha^{2(n-1)},$$

which imply $n \leq s+m+3$ and $s+m \leq n+3$, respectively. We record this intermediate result.

Lemma 2.1. *Assume that* (k, s, m, n) *is a solution of equation* (1.2) *with* $k \geq 2$ *and* $2 \leq s < m < n$. *Then*

$$|n-(s+m)| \leq 3. \tag{2.8}$$

3. The proof of the Main Theorem

To avoid notational clutter, we omit the superscript (k), so we write F_r instead of $F_r^{(k)}$ but understand that we are working with the k-generalized Fibonacci numbers. We use (2.5) to rewrite (1.2), as

$$\begin{aligned} F_s^2 + F_m^2 + f_k^2\alpha^{2(n-1)} + 2e_kf_k\alpha^{n-1} + e_k^2 \\ = 3(f_k\alpha^{s-1} + e_k'')(f_k\alpha^{m-1} + e_k')(f_k\alpha^{n-1} + e_k). \end{aligned} \tag{3.1}$$

Here, for simplicity, we wrote $f_k := f_k(\alpha)$, $e_k := e_k(n)$, $e_k' := e_k(m)$, $e_k'' := e_k(s)$. Therefore, after some calculations, we get

$$|f_k^2\alpha^{2(n-1)} - 3f_k^3\alpha^{s+m+n-3}| \leq |G_1(k,s,m,n,\alpha)| + F_s^2 + F_m^2, \tag{3.2}$$

where $G_1(k,s,m,n,\alpha)$ is the contributions of those terms in the right-hand side expansion of (3.1). Therefore,

$$\begin{aligned} |G_1(k,s,m,n,\alpha)| \leq \frac{27}{32}\alpha^{s+m-2} + \frac{27}{32}\alpha^{s+n-2} + \frac{9}{16}\alpha^{s-1} \\ + \frac{27}{32}\alpha^{m+n-2} + \frac{9}{16}\alpha^{m-1} + \frac{21}{16}\alpha^{n-1} + \frac{5}{8}. \end{aligned}$$

Now, we divide both sides of (3.2) by $3f_k^3\alpha^{s+m+n-3}$. By (2.3) and (2.4), we get

$$|1-(3f_k)^{-1}\alpha^{n-(m+s)+1}| \leq \frac{8}{3}\left(\frac{27\alpha}{32\alpha^n} + \frac{27\alpha}{32\alpha^m} + \frac{9\alpha^2}{16\alpha^{m+n}}\right.$$

$$+ \frac{27\alpha}{32\alpha^s} + \frac{9\alpha^2}{16\alpha^{s+n}} + \frac{21\alpha^2}{16\alpha^{s+m}} + \frac{5\alpha^3}{8\alpha^{s+m+n}} + \frac{\alpha}{\alpha^{m+n-s}} + \frac{\alpha}{\alpha^{s+n-m}} \Bigg).$$

Since $2 \leq s < m < n$, we have $m \geq 3$, $n \geq 4$, $m \geq s+1$ and $n \geq s+2$. Therefore, after some calculations, we arrive at

$$|1 - (3f_k)^{-1}\alpha^{n-(m+s)+1}| < \frac{15.2}{\alpha^s}. \tag{3.3}$$

We put $t := n - (m+s)$. By (2.8), we have that $t \in \{\pm 3, \pm 2, \pm 1, 0\}$. We proceed by cases. If $t + 1 \leq 0$, then

$$\frac{1}{3} \leq 1 - (3f_k)^{-1}\alpha^{t+1} \leq 1 - \frac{2^{t+3}}{9},$$

which implies

$$1/3 < |1 - (3f_k)^{-1}\alpha^{t+1}|. \tag{3.4}$$

Now, if $t + 1 \geq 2$, then $\phi^2 \leq \alpha^{t+1} \leq 2^{t+1}$. Thus, we obtain

$$1 - \frac{2}{3}2^{t+1} \leq 1 - (3f_k)^{-1}\alpha^{t+1} \leq 1 - \frac{4}{9}\phi^2.$$

Since $1 - 4\phi^2/9 < -0.16$ and $1 - 2^{t+2}/3 < -1.6$, we get

$$0.16 < |1 - (3f_k)^{-1}\alpha^{t+1}|. \tag{3.5}$$

Finally, we treat the case $t = 0$. Let us consider, for $k \geq 2$, the function

$$g(x, k) = \frac{2x + (k+1)(x^2 - 2x)}{3(x-1)}.$$

Clearly, for $x > \sqrt{2}$ fixed, the function $g(x, k)$ is increasing as a function of k. On the other hand,

$$\frac{\partial}{\partial x} g(k, x)\Big|_{x = x_k} = 0, \quad \text{where} \quad x_k := \frac{1 + k \pm \sqrt{1 - k^2}}{k+1}.$$

Assume first that $k \geq 4$ fixed. Then $g(x, k)$ is increasing for $x \in (1, 2)$ and $1.93 < \alpha(4) \leq \alpha(k)$. Therefore,

$$1.14 < g(1.93, 4) \leq g(\alpha, k) = (3f_k)^{-1}\alpha.$$

Thus, we conclude that

$$0.14 < |1 - (3f_k)^{-1}\alpha|. \tag{3.6}$$

Now, for $k = 2$ and $k = 3$, we get

$$(3f_2)^{-1}\phi < 0.75 \quad \text{and} \quad (3f_3)^{-1}\alpha(3) < 0.992, \tag{3.7}$$

respectively. By (3.4), (3.5), (3.6) and (3.7), we conclude that the inequality

$$0.008 < |1 - (3f_k)^{-1}\alpha^{n-(m+s)+1}|, \tag{3.8}$$

holds in all the cases when $k \geq 2$ and $|n-(m+s)| \leq 3$. Thus, by the previous estimate (3.8) together with (3.3) and the inequality (2.8), we get

$$2 \leq s \leq 15 \quad \text{and} \quad 1 \leq n-m \leq 18.$$

Now, we rewrite equation (1.2) as

$$\begin{aligned} F_s^2 + f_k^2\alpha^{2(m-1)} + 2e_k' f_k\alpha^{m-1} + (e_k')^2 + f_k^2\alpha^{2(n-1)} + 2e_k f_k \alpha^{n-1} + e_k^2 \\ = 3F_s(f_k\alpha^{m-1} + e_k')(f_k\alpha^{n-1} + e_k). \end{aligned} \tag{3.9}$$

After some calculations, we obtain

$$|f_k^2\alpha^{2(n-1)} + f_k^2\alpha^{2(m-1)} - 3F_s f_k^2 \alpha^{n+m-2}| \leq |G_2(k,s,m,n,\alpha)| + F_s^2, \tag{3.10}$$

where $G_2(k,s,m,n,\alpha)$ correspond to those terms in the right-hand side expansion of (3.9). Therefore,

$$|G_2(k,s,m,n,\alpha)| \leq \left(\frac{9\alpha^{13}}{8} + \frac{3}{4\alpha}\right)(\alpha^m + \alpha^n) + \frac{3\alpha^{14}}{4} + \frac{1}{2}. \tag{3.11}$$

Now, we divide both sides of (3.10) by $3F_s f_k^2\alpha^{n+m-2}$ and use the previous estimate (3.11) together with the fact that the inequality $F_s^2 < \alpha^{28}$ holds for all $2 \leq s \leq 15$, to get

$$|(3F_s)^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) - 1| < \frac{1.37 \times 10^8}{\alpha^m} \tag{3.12}$$

Let us assume that $k \geq 14$. By (2.6), we have that $F_s = 2^{s-2}$ for $2 \leq s \leq 15$. We now put $t := n-m$ and we study the function

$$h(s,t,x) = \frac{1}{3 \cdot 2^{s-2}}\left(\frac{x^{2t}+1}{x^t}\right),$$

where $(s,t) \in [2,15] \times [1,18]$ and $x \in (\alpha(14), 2)$. Clearly this function is increasing in terms of x, therefore

$$h(s,t,\alpha(14)) \leq (3F_s)^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) \leq h(s,t,2).$$

We check computationally that

$$h(s,t,2) < 0.9 \quad \text{and} \quad 1.1 < h(s,t,\alpha(14)),$$

hold in the entire range of our variables $(s,t) \in [2,15] \times [1,18] \cap (\mathbb{Z} \times \mathbb{Z})$. Therefore, for $k \geq 14$, $2 \leq s \leq 15$ and $1 \leq n-m \leq 18$, we get

$$0.1 < |(3F_s)^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) - 1|.$$

On the other hand, for $3 \leq k \leq 13$, $2 \leq s \leq 15$ and $1 \leq n - m \leq 18$, we find computationally that

$$0.004 < \min |(3F_s)^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) - 1|. \tag{3.13}$$

Therefore, comparing the above lower bound (3.13) with (3.12), we get that for $k \geq 3$,

$$2 \leq s \leq 15, \quad 3 \leq m \leq 50 \quad \text{and} \quad 4 \leq n \leq 68. \tag{3.14}$$

The remaining case $k = 2$ has already been treated but we can include it in our analysis nevertheless. We start noting that for $3 \leq s \leq 15$ and $1 \leq n - m \leq 18$, we have

$$0.16 < \min |(3F_s)^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) - 1|. \tag{3.15}$$

If $s = 2$, we have that $3^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) = 1$ when $n - m = 2$. Therefore, for $1 \leq n - m \leq 18$ with $n \neq m + 2$,

$$0.25 < \min |3^{-1}(\alpha^{n-m} + \alpha^{-(n-m)}) - 1|.$$

Thus, comparing the above lower bound (3.15) with (3.12), for $k = 2$ and $n \neq m+2$, we get,

$$2 \leq s \leq 15, \quad 3 \leq m \leq 42 \quad \text{and} \quad 4 \leq n \leq 50. \tag{3.16}$$

By (3.14) and (3.16), we conclude that:

Lemma 3.1. *If (k, s, m, n) is a solution of equation (1.2) with $2 \leq s < m < n$ and $k \geq 2$, then either $k = 2$ and $n = m + 2$ or $k \geq 3$,*

$$2 \leq s \leq 15, \quad 3 \leq m \leq 50 \quad \textit{and} \quad 4 \leq n \leq 68.$$

Now, we need to bound the variable k. Let us assume first that $k > 67$. Then, by Lemma 3.1, we have

$$n \leq 68 < k + 1.$$

Thus, the formula $F_r = 2^{r-2}$ holds for all three $r \in \{s, m, n\}$. Hence, equation (1.2) may be rewritten as

$$2^{2(s-2)} + 2^{2(m-2)} + 2^{2(n-2)} = 3 \cdot 2^{n+m+s-6}.$$

Dividing both sides of this equality by $2^{2(s-2)}$, we get

$$1 + 2^{2(m-s)} + 2^{2(n-s)} = 3 \cdot 2^{n+m-s-2}. \tag{3.17}$$

Since $m - s \geq 1$ and $n - s \geq 2$, the left-hand side of (3.17) is an odd integer greater than or equal to 21. If $n + m > s + 2$, the right-hand side is an even number, if $n + m < s + 2$ the right-hand side is not an integer and if $n + m = s + 2$ the right-hand side is 3 and none of these situations is possible. Thus, $k \leq 67$.

Assume next that $k = 2$ and $n = m + 2$ for some $m \geq 3$. Recall that the case $k = 2$ was treated in [6], so, the following has already been done and we present it here just to end our analysis. By their Lemma 3.2, we have $s = 2$. Thus,

$$1 + F_m^2 + F_{m+2}^2 = 3F_m F_{m+2}. \tag{3.18}$$

If m is an even number, then one of m or $m + 2$ is a multiple of 4, so one of F_m or F_{m+2} is a multiple of 3, which leads to

$$1 + F_j^2 \equiv 0 \bmod 3,$$

for some $j \in \{m, m + 2\}$, which is not possible. Therefore, $m = 2l - 1$ for some $l \geq 2$. Thus, equation (3.18) may rewritten as

$$1 + F_{2l-1}^2 + F_{2l+1}^2 = 3F_{2l-1}F_{2l+1} \quad \text{for} \quad l \geq 2,$$

which holds since it is equivalent to $1 + F_{2l}^2 = F_{2l-1}F_{2l+1}$, which is a particular case of Cassini's formula.

In summary, we have the following result:

Lemma 3.2. *If* (k, s, m, n) *is a solution of* (1.2) *with* $2 \leq s < m < n$, *then either* $k = 2$, $s = 2$, $m = 2l - 1$ *and* $n = 2l + 1$ *for some* $l \geq 2$ *or*

$$2 \leq k \leq 67, \quad 2 \leq s \leq 15, \quad 3 \leq m \leq 50 \quad \text{and} \quad 4 \leq n \leq 68.$$

Finally, a brute force search for solutions (k, s, m, n) of the equation (1.2), using the respective range given by the previous lemma, finishes the proof of our Main Theorem. Here, we used

$$F[r\_, k\_] := \text{SeriesCoefficient}[\text{Series}[x/(1 - \text{Sum}[x^j, \{j, 1, k\}]), \{x, 0, 1400\}], r],$$

to create the rth k-Fibonacci number.

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