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RECIPROCAL SUM OF PRIME DIVISORS OF LUCAS NUMBERS

ABSTARCT: *In this paper, among others, we show that for any non-degenerate Lucas sequence the reciprocal sum of the prime divisors of the n^{th} term is less than $\log\log\log n + c$ for any $n>n_0$. The constant c is an absolute one, only n_0 depends on the parameters of the sequence. It is an extension of a result of P. Erdős who proved it for Mersenne numbers.*

Let $R = \{R_n\}_{n=0}^{\infty}$ be a Lucas sequence of integers defined by

$$R_n = A\,R_{n-1} + B\,R_{n-2} \quad (n>1),$$

where A,B are fixed non-zero integers and the initial terms are $R_0=0$, $R_1=1$. Throughout the paper we assume that $(A,B)=1$ and the sequence is non-degenerate, that is if α and β denote the roots of the characteristic polynomial x^2-Ax-B, then α/β is not a root of unity. It is known that in this case

$$(1) \qquad R_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$$

for any $n\geq 0$. Furthermore if p is a prime and $p\nmid B$, then there are terms in R such that $p|R_n$. We shall denote the

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least index with such property by $r(p)$, i.e. $r(p)=n$ if $p|R_n$ but $p\nmid R_m$ for $0<m<n$. In this case we say p is a primitive prime divisor of R_n. For primes p with $p|B$ there is no term $R_n (n\geq 1)$ divisible by p if $(A,B)=1$; in this case we assume $r(p)=\infty$. If p is a prime, $p\nmid B$, $D=A^2+4B$ and (D/p) denotes the Legendre's symbol with $(D/p)=0$ if $p|D$ then, as it is well known,

(2) $$r(p) \mid (p - (D/p))$$

and

(3) $$p \mid R_n \quad \text{if and only if} \quad r(p)|n$$

(see e.g. [2]).

In the special case $(A;B)=(3;-2)$ the terms of the sequence R are $R_n=2^n-1$. For this sequence P.Erdős [1] proved that there are positive constants c' and c'' such that

$$\sum_{p|(2^n-1)} \frac{1}{p} < \log\log\log n + c'$$

for the distinct prime divisors and

$$\sum_{d|(2^n-1)} \frac{1}{d} < c''.\log\log n$$

for the distinct positive divisors of the terms. Erdős noted that similar results hold for the divisors of the numbers a^n-1 ($a>1$ is an integer) but he asked wether the constants c' and c'' in this case depend on a or not.

In this paper, using a little modification of Erdős' argument, we extend these results for Lucas numbers furthermore we give their improvements by showing that the constants in the inequalities do not depend on the sequence.

We note that $R_n\neq 0$ if $n\neq 0$ (since α/β is not a root of unity) and we shall write 1 for the reciprocal sum of the divisors of R_n if $R_n=1$.

THEOREM. There are positive absolute constants c and c_0, which do not depend on the sequence R, such that

$$(4) \qquad \sum_{p|R_n} \frac{1}{p} < \log\log\log n + c$$

and

$$(5) \qquad \sum_{d|R_n} \frac{1}{d} < c_0 \log\log n$$

for any $n>n_0$, where n_0 depends only on the sequence R.

We note that similar results can be obtained if $(A,B)>1$ but in this case the constants c and c_0 are not absolute ones, they depend on A and B

We also note that in the case $R_n=2^n-1$ C. Pomerance [3] obtained results for special divisors. Let $E(n) = \sum 1/d$, where the summation is extended for positive integers d for which $d|(2^n-1)$ and $d\nmid(2^m-1)$ if $0<m<n$, further let $F(n)= \sum 1/d$, where d runs over the integers for which $d|(2^n-1)$ and $d>n$. Among others Pomerance proved that

$$E(n) \geq \frac{1}{n} \exp\left\{(1+o(1)) \sqrt{\log\log n}\right\}$$

for infinitely many n and the set of n with

$$E(n) < \frac{1}{n} (\log n)^{-f(n)}$$

has logarithmic density 1 for any function f for which $f(n) \to 0$ as $n \to \infty$, furthermore.

$$F(n) < \exp\left\{ - \log n \log\log\log n / 2\cdot\log\log n\right\}$$

for all large n.

PROOF OF THE THEOREM. In the proof we shall use positive real numbers c_1, c_2, ... , wich are absolute constants, and

$k_1, k_2, \ldots,$ which depend on the sequence R.

First we consider inequality (4). Let

$$A(n) = \sum_{p \mid R_n} \frac{1}{p} .$$

By (3) we can write

$$A(n) = \sum_{d \mid n} \sum_{r(p)=d} \frac{1}{p} .$$

and A(n) can be divided into three parts:

$$A_1(n) = \sum_{\substack{d \mid n \\ d \le \log n}} \sum_{r(p)=d} \frac{1}{p} ,$$

$$A_2(n) = \sum_{\substack{d \mid n \\ d > \log n}} \sum_{\substack{r(p)=d \\ p \le n}} \frac{1}{p}$$

and

$$A_3(n) = \sum_{\substack{d \mid n \\ d > \log n}} \sum_{\substack{r(p)=d \\ p > n}} \frac{1}{p}$$

such that

(6) $$A(n) = A_1(n) + A_2(n) + A_3(n).$$

By (1) and (2) it is easy to see that there are at most $k_1 d/\log d$ primes such that $r(p)=d$, so the number of primes in sum $A_1(n)$ is at most

$$(\log n)\cdot k_1 \cdot \frac{\log n}{\log\log n} = k_1 \cdot \frac{\log^2 n}{\log\log n} .$$

It implies, using the estimation $p_i < c_1 i \cdot \log i$ for the i^{th} prime, that

$$A_1(n) < \sum_{p \le y} \frac{1}{p} ,$$

where

(7) $$y = k_1 c_1 \frac{\log^2 n}{\log \log n} \cdot \log \frac{k_1 \cdot \log^2 n}{\log \log n} < \log^3 n$$

for any sufficiently large n. We know that

(8) $$\sum_{p \le x} \frac{1}{p} < \log \log x + c_2$$

thus by (7) and (8)

(9) $$A_1(n) < \log \log \log n + c_3$$

follows.

Now we give an estimation for $A_2(n)$ supposing that n is sufficiently large. It is enough to deal with the sum

(10) $$A_2'(n) = \sum_{\substack{d \mid n \\ d > \log n}} \sum_{\substack{r(p)=d \\ p \le n \\ p < d^3}} \frac{1}{p}$$

since

$$\sum_{\substack{d \mid n \\ d > \log n}} \sum_{\substack{r(p)=d \\ p \ge d^3}} \frac{1}{p} \le \sum_{d > \log n} \frac{1}{d^3} \cdot \frac{k_1 d}{\log d} < c_4$$

and so

(11) $$A_2(n) < A_2'(n) + c_4 .$$

We can write $A_2'(n)$ in the form

(12) $$A_2'(n) = \sum_{i=0}^{N} \sum_i \frac{1}{p} + O\left(\frac{1}{\log n}\right) ,$$

where N is an integer defined by

$$(\log n)^{2^N} < n \le (\log n)^{2^{N+1}}$$

and the summation in $\sum_i$ is extended for primes p for which

$$(\log n)^{2^i} < p \le (\log n)^{2^{i+1}}$$

and for which the conditons in (10) are satisfied. We note that $O(1/\log n)$ in (12) can be different from zero only if there is a prime p such that $\log n - 1 < p \le \log n$, $r(p)=p+1$ and $(p+1)|n$. Since $r(p)=d$, $d|n$ and $p<d^3$ for the primes in $\sum_i$, by (2),

$$(p-(D/p),n) \ge d > \sqrt[3]{p} > \left[\log n\right]^{2^i/3}$$

follows and $(D/p)\ne 0$ if n is large.

Let x be a real number with conditons

$$y_1 = (\log n)^{2^i} < x \le (\log n)^{2^{i+1}} = y_2$$

and let $Q(i,x)$ be a set of primes p defined by

$$Q(i,x) = \left\{p : p<x,\ (p-(D/p),\ n) > \left[\log n\right]^{2^i/3}\right\}.$$

If $q(i,x)$ denotes the cardinality of the set $Q(i,x)$, then evidently

$$(13) \qquad \prod_{p<x} (p-(D/p),\ n) > \left[\log n\right]^{2^i q(i,x)/3}.$$

On the other hand Erdős proved that

$$\prod_{p<x} (p-1,\ n) < \exp\left\{c_6 x\cdot \log\log n/\log x\right\}$$

for any $x>(\log n)^{16}$ and $x\le n$ (see (15) and (21) in [1]) and we can similarly obtain that

$$\prod_{p<x} (p+1,\ n) < \exp\left\{c_6 x\cdot \log\log n/\log x\right\},$$

thus

$$(14) \qquad \prod_{p<x} (p-(D/p),\ n) < \prod_{p<x} (p-1,\ n)\cdot(p+1,\ n) <$$

$$< \exp\left\{c_7 x\cdot \log\log n/\log x\right\}.$$

Comparing (13) with (14)

$$q(i,x) < \frac{c_8 x}{2^i \cdot \log x} \tag{15}$$

follows for $i \geq 4$.

Let $a(m)$ be an arithmetical function such that $a(m)=1$ if m is a prime satisfying the conditions for the primes in $\sum_i$ and $a(m)=0$ otherwise. Then

$$\sum_{m \leq x} a(m) \leq q(i,x)$$

by the definiton of $q(i,x)$ and, using (15) and the fact $y_2 = y_1^2$, by Abel's identity we have

$$\sum_i \frac{1}{p} = \sum_{y_1 < m \leq y_2} a(m) \cdot \frac{1}{m} \leq \tag{16}$$

$$\leq \frac{c_8}{2^{2i+1} \cdot \log \log n} + \frac{c_8}{2^i} \cdot \int_{y_1}^{y_2} \frac{dt}{t \cdot \log t} < \frac{c_9}{2^i}$$

for any $i \geq 4$. But by (8)

$$\sum_{i=0}^{3} \sum_i \frac{1}{p} \leq \sum_{p \leq (\log n)^{16}} \frac{1}{p} - \sum_{p \leq \log n} \frac{1}{p} + O\left(\frac{1}{\log n}\right) = O(1)$$

since $p \geq d-1$ if $r(p)=d$, and so by (8) and (16) we obtain

$$A_2'(n) = c_9 \cdot \sum_{i=4}^{N} \frac{1}{2^i} + O(1) < c_{10} . \tag{17}$$

For the third summand of $A(n)$ we also get $A_3(n)=O(1)$. Namely R_n has at most $k_1 n/\log n$ distinct prime divisors greater than n, and so really

$$A_3(n) \leq \sum_{\substack{p \mid R_n \\ p > n}} \frac{1}{p} < \frac{1}{n} \cdot \frac{k_1 n}{\log n} < c_{11} \tag{18}$$

if n is sufficiently large.

Thus by (6), (9), (11), (17) and (18)

$$(19) \qquad \sum_{p|R_n} \frac{1}{p} = A(n) < \log\log\log n + c_{12}$$

follows which proves (4) with $c=c_{12}$.

For the reciprocal sum of the positive divisors of R_n we have

$$\sum_{p|R_n} \frac{1}{d} < \prod_{p|R_n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) = \prod_{p|R_n} \left(1 + \frac{1}{p-1}\right) <$$

$$< \prod_{p|R_n} e^{1/(p-1)} = \exp\left\{ \sum_{p|R_n} \frac{1}{p-1} \right\} =$$

$$= \exp\left\{ c_{13} + \sum_{p|R_n} \frac{1}{p} \right\}$$

and so by (19)

$$\sum_{d|R_n} \frac{1}{d} < \exp\left\{ c_{14} + \log\log\log n \right\}$$

follows which implies inequality (5) with $c_0=e^{c_{14}}$.

REFERENCES

[1] P. Erdős, On the sum $\sum_{d|2^n-1} d^{-1}$, Israel J. Math., 9 (1971), 43-48.

[2] D.H. Lehmer, An extended theory of Lucas' function, Ann. of Math., 31 (1930), 419-448.

[3] C. Pomerance, On primitive divisors of Mersenne numbers, Acta Arithm., 46 (1986), 355-367.