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SOME REMARKS ON CERTAIN DIOPHANTINE EQUATIONS

ABSTRACT: *The paper gives results on the solutions of the equations*

$a_0x^n+a_1x^{n-1}y+\dots+a_ny^n=m, \quad x^p+y^p=z^2$,

$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ *and* $\frac{5}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$,

where m,n *are integers and* $p\ (>3)$ *is a prime.*

1. Introduction

In this paper we give some remarks concerning the following problemens:

1^o Let $\varphi_n(x,y)$ denotes the form of degree $n\geq 3$ and let $m\in Z$ then the equation

(1.1) $$\varphi_n(x,y)=m$$

has $N_1\leq n$ so called regular solutions in $\langle x,y\rangle\in Z^2$.

2^o If $(x,y)=1$ and $p>3$ is a prime and the equation

(1.2) $$x^p+y^p=z^2$$

has a solution in integers x,y,z then

$$p|z \quad \text{or} \quad p|\varphi(z),$$

where φ denotes the Euler function.

3^o Let

(1.3) $$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

and

(1.4) $$\frac{5}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

Erdös and Straus conjectured, that the equation (1.3) has a positive solution in integers x,y,z for every natural number $n\geq 2$ (see [4], Open problems, p.58 ; No 18). Similar conjecture was posed by Sierpinski (see [4], Open problems, p.58, No 19) for the equation (1.4).

We give explicit form of solutions of (1.3) for $n=4k$, $4k+2$, $4k+3$, $8k+5$; $k=1,2,\dots$ and for $n=4k$, $4k+2$, $4k+3$; $k=1,2,\dots$ of the equation (1.4).

2. Regular solutions of the equation $\varphi_n(x,y)=m$.

J.H.Evertse [2] proved that if

$$(2.1)\qquad N = \text{card}\ \left\{ \langle x,y\rangle \in Z^2 \ ;\ \varphi_n(x,y)=m \right\},$$

where $n\ge 3$, $\varphi_n(x,1)$ is a polynomial, which has three district roots and $t=\omega(|m|)$, where $\omega(|m|)$ denotes the number of distinct prime divisors of $|m|$, then

$$(2.2)\qquad N \le n \left(7^{15\left[\binom{n}{3}+1\right]^2} + 6\cdot 7^{2\binom{n}{3}(t+1)} \right)$$

An improvement of (2.2) was given by E.Bombieri and W.M.Schmidt [1]. Let $F(x,y)\in Z[x,y]$ be an irreducible form of degree $n\ge 3$ and let $N_{n,m}$ be the number of solutions of the equation

$$(2.3)\qquad \varphi_n(x,y) = m$$

in integers x,y such that $(x,y)=1$, then

$$(2.4)\qquad N_{n,m} < C_1 n^{1+t}$$

where C_1 is an absolute constant. If $n > C_2$, then

$$(2.5)\qquad N_{n,m} < 215\ n^{1+t}$$

where $\langle x,y\rangle \equiv \langle -x,-y\rangle$.

In this part we consider so called regular solutions of (2.3).

Let $\varphi_n(x,y)\in Z[x,y]$ and $n\ge 3$, and let

$$(2.6)\qquad \varphi_n(x,y) = a_0x^n+a_1x^{n-1}y+\ \dots\ +a_ny^n = m.$$

Let $d=(a_0, a_1, \ldots, a_n)$, $|m|>1$, $m \nmid d$ and suppose that (2.6) has at least two solutions. Let

(2.7) $$\langle x_k, y_k\rangle \in Z^2 \ , \ k=1,2,\ldots$$

denote the sequence of solutions of (2.6). The sequence (2.7) will be called regular if

(a) for every $i<k$, $\det \begin{bmatrix} x_i , & y_i \\ x_k , & y_k \end{bmatrix} \neq 0$

(b) for every $i<k$, $\left(m \ , \ \det \begin{bmatrix} x_i , & y_i \\ x_k , & y_k \end{bmatrix} \right) = 1$.

We prove the following theorem:

<u>THEOREM 1</u>. Let N_1 denote the number of regular solutions of (2.6) and let $|m|>1$, $m\nmid d$, where $d=(a_0, a_1, \ldots, a_n)$. Then

(2.8) $$N_1 \leq n.$$

<u>PROOF</u>. Suppose that the equation (2.6) has $N_1 > n$ regular solutions

$$\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots , \langle x_{n+1}, y_{n+1}\rangle,$$

then from (2.6) we get

(2.9) $$\begin{cases} a_0x_1^n+a_1x_1^{n-1}y_1+ \ldots \qquad +a_ny_1^n = m \\ a_0x_2^n+a_1x_2^{n-1}y_2+ \ldots \qquad +a_ny_2^n = m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_0x_{n+1}^n+a_1x_{n+1}^{n-1}y_{n+1}+ \ldots +a_ny_{n+1}^n = m \end{cases}$$

The fundamental determinant of the system (2.9) has the form

(2.10) $$\Delta = \begin{vmatrix} x_1^n, & x_1^{n-1}y_1, & \dots, & y_1^n \\ x_2^n, & x_2^{n-1}y_2, & \dots, & y_2^n \\ \dots & \dots & \dots & \dots \\ x_{n+1}^n, & x_{n+1}^{n-1}y_{n+1}, & \dots, & y_{n+1}^n \end{vmatrix}$$

It is easy to see that from (2.10)

(2.11) $$\Delta = \prod_{1\le i<k\le n+1} \left(y_k x_i - x_k y_i\right) =$$

$$= \prod_{1\le i<k\le n+1} \det \begin{bmatrix} x_i , & y_i \\ x_k , & y_k \end{bmatrix}$$

follows. Thus by (2.11) and (a) it follows that $\Delta \neq 0$, and therefore by Crammer's ruele we obtain

(2.12) $$a_0 = \frac{m \cdot \Delta'_1}{\Delta} , \quad a_1 = \frac{m \cdot \Delta'_2}{\Delta} , \dots , a_n = \frac{m \cdot \Delta'_{n+1}}{\Delta}$$

Since $a_i \in Z$, from (2.12) we have

(2.13) $$\Delta \mid m \cdot \Delta'_i \quad \text{for every } i=1,2,\dots,n+1.$$

But $(m,\Delta)=1$, since from (b) $\left(m , \det \begin{bmatrix} x_i , y_i \\ x_k , y_k \end{bmatrix} \right) = 1$ and from (2.13) we get

(2.14) $$\Delta \mid \Delta'_i \quad \text{for every } i=1,2,\dots,n+1.$$

Thus by (2.14) and (2.12) it follows

$$a_0 = m \frac{\Delta'_1}{\Delta} , \quad a_1 = m \frac{\Delta'_2}{\Delta} , \quad \dots , \quad a_n = m \frac{\Delta'_{n+1}}{\Delta}$$

where

$$\frac{\Delta'_i}{\Delta} \in Z \quad \text{for every } i=1,2,\dots,n+1,$$

and therefore $m|a_j$ for $j=0,1,\ldots,n$. Since $d = (a_0, a_1, \ldots, a_n)$, thus $d|a_j$ and therefore $m|d$. It contradicts to our conditions $|m|>1$ and $m\nmid d$ so the proof is complete.

We remark that by similar method we can prove the well-known Lagrange's theorem concerning the number of solutions of the congruence

$$f(x)\equiv 0 \pmod{p},$$

where $f \in Z[x]$ and p is a prime (comp.[3]).

3. On the equation $x^p + y^p = z^2$

In 1977 G.Terjanian [8] proved that if the equation

$$x^{2p} + y^{2p} = z^{2p},$$

where p is odd, has a solution in integers x,y,z, then

$$2p|x \quad \text{or} \quad 2p|y.$$

In 1981 A.Rotkiewicz [5] improved this result showing that if $x^{2p} + y^{2p} = z^{2p}$ has a solution in integers x,y,z, where p is and odd prime, then $8p^3|x$ or $8p^3|y$.

In 1982 A.Rotkiewicz [6] obtained that if $(x,y)=1$ and $p>3$ is a prime and $p|z$ and $2\nmid z$ or $p\nmid z$ and $2|z$, then the equation

$$x^p + y^p = z^2 \tag{3.1}$$

has no solution in integers x,y,z.

For other results see also [7].

In this part we prove the following theorem:

THEOREM 2. Suppose that $(x,y)=1$ and $p>3$ is a prime. If the equation (3.1) has a solution in integers x,y,z, then

$$p|z \quad \text{or} \quad p|\varphi(z), \tag{3.2}$$

where φ denotes the Euler function.

PROOF. First, we prove that if $(x,y)=1$, $n<m$ and

(3.3) $$x^n + y^n \mid x^m - y^m,$$

then

(3.4) $$n \mid m .$$

Since $m > n$ thus $m = nq + r$, where $0 \le r < n$. We consider two cases:

(i) q is an odd number,

(ii) q is an even number.

In the case (i), we have

(3.5) $$x^m - y^m =$$

$$= x^r \left(x^n+y^n\right)\left(x^{n(q-1)} - x^{n(q-2)}y + \ldots - y^{n(q-1)}\right) - y^{nq}\left(x^r+y^r\right).$$

Since $\left(x^n + y^n , y^{nq}\right) = 1$ thus from (3.3) and (3.5) we get

(3.6) $$x^n + y^n \mid x^r + y^r ,$$

which is impossible if $r>0$. Thus $r=0$ and therefore $n|m$.

In case (ii) we have $q = 2^\alpha q'$, where $(2,q')=1$, $\alpha \ge 1$ and therefore for $m = n\,2^\alpha q' + r$, $0 \le r < n$ we get

(3.7) $$x^m - y^m = x^r\left[\left(x^{nq'}\right)^{2^\alpha} - \left(y^{nq'}\right)^{2^\alpha}\right] + \left(y^{nq'}\right)^{2^\alpha}\left(x^r - y^r\right).$$

Since

$$x^n+y^n \;\Big|\; x^r\left[\left(x^{nq'}\right)^{2^\alpha} - \left(y^{nq'}\right)^{2^\alpha}\right]$$

thus by assumption (3.3) and (3.7) we obtain

$$x^n + y^n \mid x^r - y^r ,$$

and similarly as above it is impossible if $r>0$. Thus $r=0$ and $n|m$ follows. Since $(x,y)=1$,

(3.8) $$\left(x, x^n+y^n\right)=1 \quad \text{and} \quad \left(y, x^n+y^n\right)=1$$

and therefore from Euler's theorem we have

(3.9) $$x^{\varphi\left(x^n+y^n\right)} \equiv 1 \quad \left(\mathrm{mod}\left(x^n+y^n\right)\right)$$

and

(3.10) $$y^{\varphi\left(x^n+y^n\right)} \equiv 1 \quad \left(\mathrm{mod}\left(x^n+y^n\right)\right).$$

From (3.9) and (3.10) we get

(3.11) $$x^n+y^n \;\Big|\; x^{\varphi\left(x^n+y^n\right)} - y^{\varphi\left(x^n+y^n\right)}.$$

By (3.3) , (3.14) it follows that

(3.12) $$n \;\Big|\; \varphi\left(x^n+y^n\right).$$

If the equation (3.1) has a solution in integers x,y,z such that $(x,y)=1$, then

(3.13) $$\varphi\left(x^p+y^p\right) = \varphi\left(z^2\right).$$

From (3.13) and (3.12) we get

(3.14) $$p \;\Big|\; \varphi\left(z^2\right).$$

It is easy to see that $\varphi\left(z^2\right) = z\ \varphi(z)$ for arbitrary fixed integer z, therefore we get

$$p|z \quad \text{or} \quad p|\varphi(z)$$

and the proof is finished.

4. On a conjecture of Erdös - Straus and Sierpinski.

In this part we prove two theorems.

THEOREM 3. The equation

(4.1) $$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has a solution in positive integers x,y,z for every n=4k, 4k+2, 4k+3, 8k+5, where k=1,2,...

The proof follows from the following identities:

$$\frac{4}{4k} = \frac{1}{k+2} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$\frac{4}{4k+2} = \frac{1}{k+2} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(2k+1)}$$

$$\frac{4}{4k+3} = \frac{1}{k+2} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(4k+3)}$$

$$\frac{4}{8k+5} = \frac{1}{2(k+1)} + \frac{1}{(k+1)(8k+5)} + \frac{1}{2(k+1)(8k+5)}$$

<u>THEOREM 4</u>. The equation

(4.2) $$\frac{5}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has a solution in positive integers x,y,z for every n=4k, 4k+2, 4k+3, where k=1,2,...

The proof follows from the following identities:

$$\frac{5}{4k} = \frac{1}{k+1} + \frac{1}{4k} + \frac{1}{k(k+1)}$$

$$\frac{5}{4k+2} = \frac{1}{k+1} + \frac{1}{2(2k+1)} + \frac{1}{(k+1)(2k+1)}$$

$$\frac{5}{4k+3} = \frac{1}{k+1} + \frac{1}{4k+3} + \frac{1}{(k+1)(4k+3)} .$$

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