Note on Abel's result about roots of polynomials

DARIUSZ FREJMAN

Abstract. In this Note we give an algebraic proof of the well-know Abel's result about roots of polynomials. Our proof is other than the proof given by M. S. Grosof and G. Taiani in the paper [3].

A theorem of Abel [2] states: if $P(x)$, $Q(x)$ are any polynomials such that $\deg Q = n \geq 3, Q(x)$ has no multiple roots and $\deg P = m \leq n-2$, then

(1) $$\sum_{i=1}^{n} \frac{P(r_i)}{Q'(r_i)} = 0,$$

where r_i's are distinct roots of $Q(x)$ and $Q'(x)$ denotes the derivate of $Q(x)$.

We remark that the original proof given by Abel uses integrals. The modern proof can be based on residue theory. In the paper [3] given by M. S. Grosof and G. Taiani has been presented an algebraic proof of (1) in spirit of classical theory of equations by using some property of the Vandermonde's determinant. The purpose of this note is to present also algebraic proof of (1) by different method. Nemely, we prove the following:

Theorem. Let $F(x) = \sum_{k=0}^{\infty} a_k x^{m_k}$, where $0 \leq m_k \leq n-2$ and $Q(x)$ be the polynomial of the degree $n \geq 3$, such that $Q(x)$ has no multiple roots. Then

(2) $$\sum_{i=1}^{n} \frac{F(r_i)}{Q'(r_i)} = 0,$$

where r_i's are distinct roots of $Q(x)$ and $Q'(x)$ denotes the derivate of $Q(x)$.

Proof. In the proof of (2) we use of the following result (see [1], p. 87 and solution on p. 220). If $Q(x)$ has no multiple roots and $\deg Q(x) = n \geq 3$, then for every natural fixed s, such that $0 \leq s \leq n-2$, we have;

(3) $$\sum_{i=1}^{n} \frac{r_i^s}{Q'(r_i)} = 0,$$

where r_i's are distinct roots of $Q(x)$.

The proof of (3) easily follows by an algebraic method.

Since $F(x) = \sum_{k=0}^{\infty} a_k x^{m_k}$, and $0 \leq m_k \leq n-2$, then

$$\sum_{i=1}^{n} \frac{F(r_i)}{Q'(r_i)} = \sum_{i=1}^{n} \frac{1}{Q'(r_i)} \sum_{k=0}^{\infty} a_k r_i^{m_k}. \tag{4}$$

From (4) we obtain

$$\sum_{i=1}^{n} \frac{F(r_i)}{Q'(r_i)} = a_0 \sum_{i=1}^{n} \frac{r_i^{m_0}}{Q'(r_i)} + a_1 \sum_{i=1}^{n} \frac{r_i^{m_1}}{Q'(r_i)} + \cdots + a_k \sum_{i=1}^{n} \frac{r_i^{m_k}}{Q'(r_i)} + \cdots \tag{5}$$

Since $0 \leq m_k \leq n-2$ then by (3) and (5) we obtain

$$\sum_{i=1}^{n} \frac{F(r_i)}{Q'(r_i)} = 0,$$

and the proof of the Theorem is complete.

Remark. Putting $m_k = m - k, k = 0, 1, \ldots, m$; we have

$$F(x) = \sum_{k=0}^{m} a_k x^{m_k} = \sum_{k=0}^{m} a_k x^{m-k} = P(x),$$

where $\deg P(x) = m \leq n-2$, and we obtain (1).

Rereferences

[1] D. K. Faddieĭev and I. S. Sominskiĭ, Collection of the problems in higheralgebra, Moskov, 1964 (in Russian).

[2] P. Griffiths, "Variations on a theorem of Abel", *Invent. Math.* **35** (1976), 321–390.

[3] M. S. Grosof and G. Taiani, "Vandermonde strikes again", *Amer. Math. Monthly*, 1993, 575–577.