Multiplicative functions satisfying the equation $f(m^2+n^2) = (f(m))^2 + (f(n))^2$

PHAM VAN CHUNG

Abstract. Purpose of the present paper is to characterize multiplicative functions f which satisfy the equation $f(m^2+n^2)=(f(m))^2+(f(n))^2$ for all positive integers m and n.

1. Introduction

A "multiplicative function" is a function f defined on the set of the positive integers such that $f(mn) = f(m)f(n)$ whenever the greatest common divisor of m and n is 1. The function f is called "completely" multiplicative if the condition $f(mn) = f(m)f(n)$ holds for all m and n.

Claudia A. Spiro [2] proved that if a multiplicative function f satisfies the condition $f(p+q) = f(p) + f(q)$ for all primes p, q and $f(p_0) \neq 0$ for at least one prime p_0, then $f(n) = n$ for each positive integer n.

Replacing the set of primes by the set of squares, in [1], we have inverstigated the multiplicative functions f satisfying the condition

$$f(m^2+n^2) = f(m^2) + f(n^2) \tag{A}$$

for all positive integers m, n. We have shown that if $f \neq 0$ is multiplicative, then f fulfills the condition (A) if and only if
either

(A–1) $f(2^k) = 2^k$ for all integers $k \geq 0$,

(A–2) $f(p^k) = p^k$ for all primes $p \equiv 1 \pmod 4$ and all integers $k \geq 1$, and

(A–3) $f(q^{2k}) = q^{2k}$ for all primes $q \equiv 3 \pmod 4$ and all integers $k \geq 1$

or

(a–1) $f(2) = 2$ and $f(2^k) = 0$ for all integers $k \geq 2$,

(a–2) $f(p^k) = 1$ for all primes $p \equiv 1 \pmod 4$ and all integers $k \geq 1$, and (a–3) $f(q^{2k}) = 1$ for all primes $q \equiv 3 \pmod 4$ and all integers $k \geq 1$.

In the present paper we consider a similar question, and we give solutions of multiplicative functions f satisfying the equation

$$(B) \qquad f(m^2 + n^2) = (f(n))^2 + (f(n))^2$$

for all positive integers m and n. Investigating this question we have same result by using the method of [1]. Our main result is to prove the following.

Theorem. Let $f \neq 0$ be a multiplicative function. The f fulfills the condition

$$(C) \qquad f(m^2 + n^2) = (f(m))^2 + (f(n))^2$$

for all positive integers m and n if and only if

(C–1) $f(2^k) = 2^k$ for all integers $k \geq 0$,

(C–2) $f(p^k) = p^k$ for all primes $p \equiv 1 \pmod 4$ and all integers $k \geq 1$, and

(C–3) $f(q^{2k}) = q^{2k}$ for all primes $q \equiv 3 \pmod 4$ and all integers $k \geq 1$.

We shall prove our theorem in the next two sections. First in Section 2, we verify some auxiliary lemmas. Finally, in Section 3, we give the proof of the theorem.

2. Lemmas

Lemma 1. If f satisfies the hypotheses of the theorem, then we have $f(2) = 2$ and $f(4) = 4$.

Proof. Since $f \neq 0$ multiplicative, we have $f(1) = 1$. Thus, (C) yields $f(2) = f(1^2 + 1^2) = (f(1))^2 + (f(1))^2 = 1 + 1 = 2$. Therefore, by using the equation $5 = 2^2 + 1^2$, (C) implies that $f(5) = (f(2))^2 + (f(1))^2 = 5$. So, we conclude from (C) and the multiplicativity of f that $(f(3))^2 = f(10) - (f(1))^2 = 9$. The fact that $20 = 4^2 + 2^2$, coupled with $f(2) = 2$, $f(5) = 5$ and (C), forces

$$(1) \qquad (f(4))^2 = f(20) - (f(2))^2 = 5f(4) - 4.$$

On the other hand, from (C), the multiplicativity of f, and the equation $52 = 4^2 + 6^2$, the equation

$$(f(4))^2 = f(52) - (f(6))^2 = f(13)f(4) - (f(2))^2(f(3))^2 = f(13)f(4) - 2^2 3^2 \tag{2}$$

follows.

But, by using (C) and the fact $13 = 3^2 + 2^2$, we have $f(13) = (f(3))^2 + (f(2))^2 = 9 + 4 = 13$. So, from (2), it follows that

$$(f(4))^2 = 13f(4) - 36. \tag{3}$$

Thus, by (1) and (3), we have $5f(4) - 4 = 13f(4) - 36$. Consequently, $f(4) = 4$. So, the lemma is proved.

Lemma 2. If f fulfills the hypotheses of the theorem, then we have

$$f(2^k) = 2^k \quad \text{for all integers} \quad k \geq 0.$$

Proof. By Lemma 1, we have $f(2^k) = 2^k$ for $k = 1, 2$, and it is clear for cases $k = 0$ and $k = 3$. Assume that n is an integer with $n \geq 3$, and that we have $f(2^k) = 2^k$ for all integers $1 \leq k \leq n$. We will show that $f(2^{n+1}) = 2^{n+1}$. If $n + 1$ is even then $n + 1 = 2k$, where $k + 1 \leq n$. Thus, (C), the multiplicativity of f, $f(5) = 5$, and the hypotheses of the induction yield

$$5f(2^{n+1}) = f(5 \cdot 2^{n+1}) = f(2^{2k+2} + 2^{2k}) = \left(f(2^{k+1})\right)^2 + \left(f(2^k)\right)^2 = 2^{2k+2} + 2^{2k} = 5 \cdot 2^{2k}$$

which gives

$$f(2^{n+1}) = 2^{n+1}.$$

So, it remains to show that $f(2^{n+1}) = 2^{n+1}$, when $n + 1$ is odd. If $n + 1$ is odd, then $n + 1 = 2k + 1$ where $k \leq n$. By using (C) and the hypotheses of induction, we obtain

$$f(2^{n+1}) = f(2^{2k} + 2^{2k}) = 2\left(f(2^k)\right)^2 = 2 \cdot 2^{2k} = 2^{n+1}.$$

Thus, the lemma is proved.

Lemma 3. If f satisfies the hypotheses of the theorem, then we have

$$f(m^2) = (f(m))^2$$

for all positive integers m.

Proof. From the multiplicativity of f and Lemma 2. we easy reduce that $f(2m) = 2f(m)$ for all positive integers m. Thus, by (C) we have $2f(m^2) = f(2m^2) = f(m^2 + m^2) = 2(f(m))^2$, from which $f(m^2) = (f(m))^2$ follows.

Lemma 4. If f fulfills the hypotheses of the theorem, then we have

$$(f(m))^2 = m^2$$

for all positive integers m.

We note that from (C) and Lemma 3 condition (A) follows. So we have, using Lemma 2, that $(A_1) - (A_3)$ are satisfied. From this we have $f(m^2) = m^2$, which proves the lemma. But here we give another proof.

Proof. We argue by induction on m. In the proof of Lemma 1. we have shown that $(f(m))^2 = m^2$ for $m = 1, 2$ and 3.

Suppose that n is an integer with $n \geq 3$, and $(f(m))^2 = m^2$ for all positive integers $m \leq n$. We will show that $(f(n+1))^2 = (n+1)^2$. If $n + 1$ is even, then $n + 1 = 2^k h$ where $h \leq n$ and h is odd. Thus, by the multiplicativity of f, Lemma 2. and the hypotheses of the induction, we obtain

$$(f(n+1))^2 = (f(2^k))^2 (f(h))^2 = 2^{2k} h^2 = (n+1)^2.$$

To show that $(f(n+1))^2 = (n+1)^2$, when $n+1$ is odd, we use the equation $(n-1)^2 + (n+1)^2 = 2(n^2+1)$, where $(2, n^2+1) = 1$. From this and (C) we have $(f(n+1))^2 = f\left[2(n^2+1)\right] - (f(n-1))^2 = f(2)\left[(f(n))^2 + (f(1))^2\right] - (f(n-1))^2$. Since $f(2) = 2$, and the hypotheses of the induction, we reduce that

$$(f(n+1))^2 = 2(n^2+1) - (n-1)^2 = (n+1)^2,$$

which proves the lemma.

Now we return to prove the theorem.

3. The proof of the theorem

First we verify the necessity of the condition.

Assume that f fulfills the condition of the theorem. Then, by Lemma 2., we have shown that (C—1) is satisfied. Moreover, by using Lemmas 3. and 4., we obtain

$$f(q^{2k}) = (f(q^k))^2 = (q^k)^2 = q^{2k}$$

for all positive integers q and k.

So, the condition (C–3) is fulfilled. If p is prim and $p \equiv 1 \pmod 4$, then it is well known that there exist positive integers x and y such that

$$p^k = x^2 + y^2.$$

By using (C), Lemma 4., we have

$$f(p^k) = f(x^2 + y^2) = (f(x))^2 + (f(y))^2 = x^2 + y^2 = p^k,$$

which verified the condition (C–2). So, we have completed the proof the necessity of the condition.

Conversely, suppose that the conditions (C–1), (C–2) and (C–3) are satisfied for a multiplicative function f.

It is well known that we can write

$$m^2 + n^2 = 2^k p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_s^{2\beta_s},$$

where p_i and q_j are primes, $p_i \equiv 1 \pmod 4$ and $q_j \equiv 3 \pmod 4$ for $i = 1, 2 \dots, h$ and $j = 1, 2, \dots, s$, and $k \geq 0$ and α_i, β_i are positive integers. Then by the multiplicativity of f and the conditions (C–1), (C–2) and (C–3),

$$\begin{aligned} f(m^2 + n^2) &= f(2^k) f(p_1^{\alpha_1}) \cdots f(p_h^{\alpha_h}) f(q_1^{2\beta_1}) \cdots f(q_s^{2\beta_s}) \\ &= 2^k p_1^{\alpha_1} \cdots p_h^{\alpha_h} q_1^{2\beta_1} \cdots q_s^{2\beta_s} = m^2 + n^2 = (f(m))^2 + (f(n))^2. \end{aligned}$$

So, this completes the proof of the theorem.

References

[1] P. V. Chung, Multiplicative Functions f Satisfying the Equation $f(m^2 + n^2) = f(m^2) + f(n^2)$. *Mathematica Slovaca,* Vol. **44** (1994), (to appear)

[2] Spiro, Claudia A., Additive Uniwueness Sets for Arithmetic Functions, *Journal of Number Theory,* **42** (1992), 232—246.