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FUNCTIONAL RECURRENCES AND DIFFERENTIAL EQUATIONS

ABSTRACT: *In this paper we show connections among solutions of differential equations, power matrices and functional recurrences. The results can be used to investigate some functional recurrences connected with number theoretic polynomials.*

In the paper [1], some connections was given among solutions of differential equations of second order, power matrices and functional recurrences. We have proved the following theorem:

THEOREM A. Let s_0, t_0, u, v be functions of x and let λ be a constant satisfying the following conditions:

(i) $s_0, t_0, u, v \in C^2(J)$ where $J=(x_1, x_2) \subset R$

(ii) $u \neq 0$, $v \neq 0$ on J and $\lambda \in R_+$.

Then the functions

$$y_1 = s_0 \cdot u^\lambda \quad , \quad y_2 = t_0 \cdot v^\lambda$$

are the solutions of the differential equation

$$D_0 y'' + D_1 y' + D_2 y = 0$$

where

$$D_0 = \det\begin{bmatrix} s_0 & s_1 \\ t_0 & t_1 \end{bmatrix}, \quad D_1 = \det\begin{bmatrix} s_2 & s_0 \\ t_2 & t_0 \end{bmatrix}, \quad D_2 = \det\begin{bmatrix} s_1 & s_2 \\ t_1 & t_2 \end{bmatrix}$$

and

$$s_1 = s_0' + \lambda\, s_0 \frac{u'}{u}\,, \qquad s_2 = s_1' + \lambda\, s_1 \frac{u'}{u}$$

$$t_1 = t_0' + \lambda\, t_0 \frac{v'}{v}\,, \qquad t_2 = t_1' + \lambda\, t_1 \frac{v'}{v}\,.$$

Theorem A has been used to investigation of some functional recurrences connected with number theoretic polynomials, in particular Pell, Lucas and Fibonacci polynomials.

In the present paper we give some generalization of Theorem A. Namely we prove the following theorem:

THEOREM B. Let $s_{0,k}$, u_k be functions of x for $k=1,2,\dots,n$ and let λ be a constant satisfying the following conditions:

(a) $s_{0,k}, u_k \in C^{(n)}(J)$, where $J=(x_1,x_2)\subset R$ for $k=1,2,\dots n$;

(b) $u_k \neq 0$ on J for $k=1,2,\dots,n$ and $\lambda \in R_+$.

Then the functions

(1) $y_1 = s_{0,1}\cdot u_1^{\lambda}$, $y_2 = s_{0,2}\cdot u_2^{\lambda}$, $\dots y_n = s_{0,n}\cdot u_n^{\lambda}$

are the solutions of the differential equation

(2) $P_0 y^{(n)} + P_1 y^{(n-1)} + \dots + P_{n-1} y^{(1)} + P_n\cdot y = 0$,

where

(3) $P_{j-1} = (-1)^j \cdot D_{1,j-1}$; $j=1,2,\dots,n+1$

and $D_{1,j-1}$ denotes the minor of the matrix S which we get by deleting the first row and j-kolumn of the matrix S, where

(4) $$S = \begin{pmatrix} s_{n,1} & s_{n-1,1} & \dots & s_{0,1} \\ s_{n,1} & s_{n-1,1} & \dots & s_{0,1} \\ s_{n,2} & s_{n-1,2} & \dots & s_{0,2} \\ \dots & \dots & \dots & \dots \\ s_{n,n} & s_{n-1,n} & \dots & s_{0,n} \end{pmatrix}$$

and

(5) $$s_{l,k} = s_{l-1,k}' + \lambda\cdot s_{l-1,k} \cdot \frac{u_k'}{u_k}$$

for $l,k=1,2,\dots,n$.

PROOF OF THEOREM B. Let us denote

$$L(y) = P_0 y^{(n)} + P_1 y^{(n-1)} + \dots + P_n y$$

then for $y_1 = s_{0,1} \cdot u_1^{\lambda}$ we have

$$L(y_1) = L\left(s_{0,1} \cdot u_1^{\lambda}\right) = P_0 \left(s_{0,1} \cdot u_1^{\lambda}\right)^{(n)} + \dots + P_n \left(s_{0,1} \cdot u_1^{\lambda}\right).$$

It is easy to see that

$$\left(s_{0,1} \cdot u_1^{\lambda}\right)' = s'_{0,1} \cdot u_1^{\lambda} + \lambda \cdot s_{0,1} \cdot u_1^{\lambda-1} \cdot u'_1 = u_1^{\lambda} \left(s'_{0,1} + \lambda \cdot \frac{u'_1}{u_1}\right).$$

Putting in the last equality

$$s'_{0,1} + \lambda \cdot \frac{u'_1}{u_1} = s_{1,1}$$

we get

$$\left(s_{0,1} \cdot u_1^{\lambda}\right)' = s_{1,1} \, u_1^{\lambda} .$$

Similarly we get

$$\left(s_{0,1} \cdot u_1^{\lambda}\right)^{(k)} = \left(s_{k-1,1} \cdot u_1^{\lambda}\right)' = s_{k,1} \cdot u_1^{\lambda}$$

for $k=2,3,\dots,n$. Thus we get

$$L\left(s_{1,0} \cdot u_1^{\lambda}\right) = P_0 \cdot s_{n,1} \cdot u_1^{\lambda} + P_1 \cdot s_{n-1,1} \cdot u_1^{\lambda} + \dots + P_n \cdot s_{0,1} \cdot u_1^{\lambda} =$$

$$= \left(P_0 \cdot s_{n,1} + P_1 \cdot s_{n-1,1} + \dots + P_n \cdot s_{0,1}\right) \cdot u_1^{\lambda} .$$

We remark that from (4)

(6) $\qquad \det S = 0.$

follows. On the other hand from (4) we have

(7) $\qquad \operatorname{Det} S = s_{n,1} \cdot D_{1,0} - s_{n-1,1} \cdot D_{1,1} + \dots + s_{0,1} (-1)^n D_{1,n+1}.$

Since

$$P_{j-1} = (-1)^j \, D_{1,j-1} \qquad \text{for } j=0,1,\dots,n+1$$

thus by (6) and (7) it follows

$$L\left(s_{0,k} \cdot u_k^{\lambda}\right) = 0 \qquad \text{for } k=2,\dots,n$$

and the proof is complete.

From Theorem B we get the following.

COROLLARY. Let the assumption of theorem B be satisfied and let

$$s_{0,1} = s_{0,2} = \dots = s_{0,n} = 1.$$

If the functions $u_1, \dots, u_n$ are linearly independent over R then the general solution of the differential equation (2) is of the form

$$y = c_1 u_1^{\lambda} + c_2 u_2^{\lambda} + \dots + c_n u_n^{\lambda}$$

where $c_1, c_2, \dots, c_n$ are arbitrary constans.

REFERENCE

[1] A. Grytczuk and K. Grytczuk, Functional recurrences, Proceedings of the Pisa Conference (1988), to appear.