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SPECTRAL PROPERTIES OF SOME MATRICES

ABSTRACT: *In this paper we show some spectral properties of matrices. Among others we prove some inequalities for the characteristic roots of matrices satisfying some conditions. We also give a new proof for a theorem of Szulc.*

In 1984 N.W. Kuharenko [2] proved the following spectral property of $n\times n$ real matrix A.

THEOREM A (N.W. KUHARENKO). Let $A = (a_{ij})$ be an $n\times n$ real matrix and let

$$\mathrm{Tr}\ A = 0 \quad \text{and} \quad A_2 = \sum_{1\le i<j\le n} \left(a_{ii}a_{jj}-a_{ij}a_{ji}\right) > 0.$$

Then there exists at least one pair of complex-conjugate eigen-values

$$\lambda_k = c_k \pm id_k$$

of A such that

$$d_k^2 \ge c_k^2 + \frac{2}{n} A_2 .$$

This theorem has application in the theory of dynamical systems.

In 1988 T.Szulc [3] gave the following generalization of Theorem A.

THEOREM B. Let $A = (a_{ij})$ be $n\times n$ real matrix such that

$$\mathrm{Tr}^2\ A < \frac{2n}{n-1}\ A_2$$

Then there exists at least one pair of complex-conjugate eigen-values

$$\lambda_k = c_k \pm id_k$$

of A such that

$$(1.1) \qquad d_k^2 \geq \begin{cases} \dfrac{2nA_2 - (n-1)Tr^2A}{n(n-1)} & \text{if } n \text{ is odd} \\[2ex] \dfrac{2nA_2 - (n-1)Tr^2A}{n^2} & \text{if } n \text{ is even} \end{cases}$$

Moreover if

$$(1.2) \qquad Tr^2 A \leq 2 A_2$$

then

$$(1.3) \qquad d_k^2 \geq \begin{cases} c_k^2 + \frac{1}{n-1}\left(2A_2 - Tr^2A\right) & \text{if } n \text{ is odd} \\[2ex] c_k^2 + \frac{1}{n}\left(2A_2 - Tr^2A\right) & \text{if } n \text{ is even} \end{cases}$$

In the present paper we give another proof of Theorem B. Moreover we prove the following theorems:

THEOREM 1. Let $A \in M_n(\mathbb{Z})$, where $M_n(\mathbb{Z})$ denote the set of all $n\times n$ matrices over $\mathbb{Z}$ and let A be non-singular matrix. The necessary and sufficient condition for λ_j, $j=1,2,\ldots,n$ to be a roots of unity is $|\lambda_j|=1$ for $j=1,2,\ldots,n$.

THEOREM 2. Let $A=(a_{ij})$ be an $n\times n$ complex matrix with $|\det A|>1$ and let $|\overline{\lambda}| = \max\limits_{1\leq j\leq n} |\lambda_j|$ where λ_j are the characteristic roots of A for $j=1,2,\ldots,n$, then

$$|\overline{\lambda}| > 1 + \frac{\log|\det A|}{n}.$$

THEOREM 3. Let $|\overline{\lambda}| = \max\limits_{1\leq j\leq n} |\lambda_j|$ where λ_j for $j=1,2,\ldots,n$ are characteristic roots of $A \in M_n(\mathbb{Z})$, $n\geq 2$ and let the characteristic polynomial of A be irreducible over $\mathbb{Z}$. If for some $j=1,2,\ldots,n$ the root λ_j is not a root of unity then

$$|\overline{\lambda}| > 1 + \frac{\log c}{1+N_0}$$

where $c\geq 2$ is some real number and N_0 depend only on n and c.

We remark that from Theorem 2 it follows immediately:

COROLLARY 1. Let $A = (a_{ij}) \in M_n(\mathbb{Z})$, $n \geq 2$ and $\det A \neq \pm 1$. Then

$$|\overline{\lambda}| > 1 + \frac{\log 2}{n}.$$

From Corollary 1 it follows that a conjecture of Schinzel and Zassenhaus [2] is true for all matrices $A \in M_n(\mathbb{Z})$ with $\det A \neq \pm 1$.

PROOF OF THEOREM 1. It is easy to see that if $\lambda_j^m = 1$ for $j=1,2,\ldots,n$ then $|\lambda_j|=1$. Suppose that $|\lambda_j|=1$ for $j=1,2,\ldots,n$ and let

(1) $$f(\lambda) = \det(\lambda I - A)$$

be the characteristic polynomial of A. Then we have

(2) $$f(\lambda) = \lambda^n + A_1\lambda^{n-1} + \ldots + A_n$$

where $A_i \in \mathbb{Z}$ and $A_n = \det A \neq 0$. Since $|\lambda_j|=1$ for $j=1,2,\ldots,n$ thus by formulas of Viete it follows

(3) $$|A_k| \leq \binom{n}{k}$$

for $k=1,2,\ldots,n$.

From (3) it follows that there exist only finite number of polynomials with integer coefficients such that

$$|\lambda_j|=1 \text{ for } j=1,2,\ldots,n.$$

Consider the following sequence

(4) $$f_m(\lambda)=\left(\lambda-\lambda_1^m\right)\left(\lambda-\lambda_2^m\right)\ldots\left(\lambda-\lambda_n^m\right)=\lambda^n + A_1^{(m)}\lambda^{n-1}+\ldots+A_n^{(m)}$$

where $A_j^{(m)} \in \mathbb{Z}$. The sequence (4) has only finite number of distinct elements, because

$$|\lambda_j^m| = |\lambda_j|^m = 1 \text{ and } |A_j^{(m)}| \leq \binom{n}{k} \;;\; j,k=1,2,\ldots,n.$$

Therefore we can take a subsequence $\left\{f_{m_i}(\lambda)\right\}$ such that

(5) $\qquad f_{m_0}(\lambda) = f_{m_1}(\lambda) = f_{m_2}(\lambda) = \ldots$

where $m_0 < m_1 < m_2 < \ldots$. From (4) and (5) we get

(6) $\qquad \lambda_1^{m_i} = \lambda_{\sigma(1)}^{m_0},\ \lambda_2^{m_i} = \lambda_{\sigma(2)}^{m_0},\ \ldots\ ,\ \lambda_n^{m_i} = \lambda_{\sigma(n)}^{m_0}$

where $\left\{\ \sigma(1),\ \ldots,\ \sigma(n)\ \right\}$ denotes some permutation of $\left\{1,2,\ldots,n\right\}$. There are infinitely many exponents m_i such that $m_1 < m_2 < \ldots$. On the other hand there exists only $n!$ permutations of the set $\left\{1,2,\ldots,n\right\}$. Therefore there are exponents m_{i_1} and m_{i_2} for which

(7) $\qquad \lambda_1^{m_{i_1}} = \lambda_1^{m_{i_2}}\ ,\ \lambda_2^{m_{i_1}} = \lambda_2^{m_{i_2}}\ ,\ \ldots\ ,\ \lambda_n^{m_{i_1}} = \lambda_n^{m_{i_2}}$

From (7) we get

$$\lambda_j^m = 1 \quad \text{for} \quad j=1,2,\ldots,n$$

where $m=m_{i_1}-m_{i_2}$ and $m_{i_1}-m_{i_2} > 0$ and the proof is complete.

PROOF OF THEOREM 2. Let $f(x)$be the characteristic polynomial of the matrix A. It is well-known that

(8) $\qquad |\lambda_1\cdots\lambda_n| = |\det A|.$

Since $|\lambda_j| \le |\overline{\lambda}|$ for $j=1,2,\ldots,n$, thus by (8) it follows

(9) $\qquad |\overline{\lambda}| \ge |\det A|^{1/n}$.

Since $|\det A|>1$ thus we have

(10) $\qquad |\det A|^{\frac{1}{n}} = e^{\frac{1}{n}\log|\det A|} = 1 + \frac{1}{n}\log|\det A| + \ldots$

From (10) we get

(11) $\qquad |\det A|^{\frac{1}{n}} > 1 + \frac{1}{n}\log|\det A|$.

From (11) and (9) we obtain

$$|\overline{\lambda}| > 1 + \frac{\log|\det A|}{n}$$

and the proof is complete.

PROOF OF THEOREM 3. Let N_o denotes the set of all roots of the characteristic polynomials of fixed degree $n \geq 2$ with integer coefficients and irreducible over $\mathbb{Z}$ such that $|\lambda| \leq c$, where $c \geq 2$ is some real number. Suppose that for every $j=1,2,\dots,n$ we have

$$(12) \qquad |\lambda_j| \leq c^{\frac{1}{1+N_o}}$$

From (12) we get

$$(13) \qquad |\lambda_j^k| \leq |\lambda_j|^k \leq c^{\frac{k}{1+N_o}} \leq c$$

for $k=1,2,\dots,1+N_o$. From (13) and definition of N_o we get that there exists $k_1^{(j)} \neq k_2^{(j)}$ such that $k_1^{(j)}, k_2^{(j)} \in \{1,2,\dots,1+N_o\}$ and

$$(14) \qquad \lambda_j^{k_1^{(j)}} = \lambda_j^{k_2^{(j)}} \qquad \text{for } j=1,2,\dots,n.$$

From (14) we get

$$\lambda_j^m = 1 \quad \text{for } j=1,2,\dots,n \quad \text{where } m = k_1^{(j)}-k_2^{(j)}$$

and λ_j is a root of unity for $j=1,2,\dots,n$.

By the assumption of our theorem if follows that there exist some $j \in \{1,2,\dots,n\}$ such that

$$(15) \qquad |\lambda_j| > c^{\frac{1}{N_o+1}} .$$

From (15)

$$|\overline{\lambda}| = \max_{1\leq j\leq n} |\lambda_j| \geq c^{\frac{1}{1+N_o}} = e^{\frac{1}{1+N_o}\log c} > 1 + \frac{\log c}{1+N_o}$$

follows and the proof is complete.

PROOF OF THEOREM B. We have the well-known identity

(16) $\qquad \operatorname{Tr}(A^2) = \operatorname{Tr}^2 A - 2A_2 .$

On the other hand we have

(17) $$\operatorname{Tr}(A^2) = \lambda_1^2 + \ldots + \lambda_n^2 = \sum_{i=1}^{n} \left(\operatorname{Re} \lambda_i\right)^2 - \sum_{i=1}^{n} \left(\operatorname{Im} \lambda_i\right)^2$$

and

(18) $$\operatorname{Tr}^2 A = \left(\lambda_1 + \ldots + \lambda_n \right)^2 = \left[\sum_{i=1}^{n} \operatorname{Re} \lambda_i \right]^2$$

From (16) - (18) we obtain

$$\operatorname{Tr}(A^2) = \frac{1}{n} \operatorname{Tr}^2 A + \frac{(n-1)\operatorname{Tr}^2 A - 2nA_2}{n} ,$$

thus

$$\lambda_1^2 + \ldots + \lambda_n^2 - \frac{1}{n} \left(\lambda_1 + \ldots + \lambda_n\right)^2 = \frac{(n-1)\operatorname{Tr}^2 A - 2nA_2}{n} < 0.$$

Since the left hand side cannot be negative for all real λ_i then we get that there exists at least one characteristic root which is a complex number.

From (16) - (18) we obtain

$$\sum_{i=1}^{n} \left(\operatorname{Re} \lambda_i\right)^2 - \sum_{i=1}^{n} \left(\operatorname{Im} \lambda_i\right)^2 = \frac{1}{n} \left[\sum_{i=1}^{n} \operatorname{Re} \lambda_i \right]^2 + \frac{(n-1)\operatorname{Tr}^2 A - 2nA_2}{n}$$

and therefore we get

(19) $$\sum_{i=1}^{n} \left(\operatorname{Im} \lambda_i\right)^2 = \sum_{i=1}^{n} \left(\operatorname{Re} \lambda_i\right)^2 - \frac{1}{n} \left[\sum_{i=1}^{n} \operatorname{Re} \lambda_i \right]^2 + \frac{2nA_2 - (n-1)\operatorname{Tr}^2 A}{n}$$

Let

(20) $$d = \max_{1\le i\le n} |\mathrm{Im}\,\lambda_i|$$

Then from (19) and (20) we obtain

(21) $$k\, d^2 \ge \sum_{i=1}^{n} \left(\mathrm{Im}\,\lambda_i\right)^2 \ge \frac{2nA_2-(n-1)\mathrm{Tr}^2A}{n},$$

where $k\le n$ if n is even and $k\le n-1$ if n is odd. From (21) we obtain (1.1).

Now, suppose that (1.2) is true and let

$$x_k = \left(\mathrm{Re}\,\lambda_k\right)^2 - \left(\mathrm{Im}\,\lambda_k\right)^2$$

Then we have

(22) $$\sum_{j=1}^{n} \lambda_j^2 = \sum_{k=1}^{n} x_k = \mathrm{Tr}^2A - 2A_2 \le 0.$$

From (22) we get that there exists at least one complex number such that $\mathrm{Im}\,\lambda \ne 0$. Let

(23) $$\sum_{j=1}^{n} \lambda_j^2 = \sum_{k=1}^{n} x_k = B + \sum_{l=1}^{m} x_{k_l} = \mathrm{Tr}^2A - 2A_2$$

where B denotes the sum of squares of all real eigenvalues of A and let

(24) $$x_{k_d} = \min_{1\le k_j\le m} \left(x_{k_1}, x_{k_2}, \dots, x_{k_m}\right) = \min_{1\le j\le m} \left(x_{k_j}\right).$$

Then from (23) and (24) we obtain

(25) $$m\, x_{k_d} \le \sum_{l=1}^{n} x_{k_l} \le \mathrm{Tr}^2A - 2A_2.$$

From (25) we get

$$x_{k_d} = \left(\mathrm{Re}\ \lambda_{k_d}\right)^2 - \left(\mathrm{Im}\ \lambda_{k_d}\right)^2 \le \frac{1}{m}\left(\mathrm{Tr}^2 A - 2A_2\right) \tag{26}$$

where $m \le n$ if n is even and $m \le n-1$ if n is odd.

It is easy to see that from (25) we get (1.3) and therefore the proof of Theorem B is complete.

REFERENCES

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