JAMES P. JONES and PÉTER KISS

PROPERTIES OF THE LEAST COMMON MULTIPLE FUNCTION*

ABSTRACT: In this paper we show some properties of the function $L(x)$, the least common multiple of the natural numbers not greater than an integer x, and the function $Q(x) = x!/L(x)$.

The subject of this paper is to show the number-theoretic properties of the function $L(x) = LCM[1,2,\ldots,x]$, the least common multiple of the numbers $\leq x$. This function has a connection with the function $\Pi(x)$, the number of primes $\leq x$ and it is related to the two Chebyshev functions $\psi(x) = \ln(L(x))$ and $\theta(x)$ for which

$$\theta(x) = \sum_{p \leq x} \ln(p) \text{ and } \psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \ldots$$

* Research was supported by the Hungarian National Foundation (Grant No. 1641) for Scientific Research and the National Scienfitic and Engineering Reseurch Council of Canada.

I. The properties of the function L(x)

For any positive integer $x, L(x)$ can be written in the form

(1) $$L(x) = \prod_{p \leq x} p^{k_p}$$

Where k_p is defined for primes p by

$$p^{k_p} \leq x < p^{k_p+1}$$

and so for any prime $p \leq x$ we have

(2) $$k_p = \left[\frac{\ln(x)}{\ln(p)}\right].$$

where [ ] is the integer part function. From (1) and (2)

(3) $$\ln(L(x)) = \sum_{p \leq x}\left[\frac{\ln(x)}{\ln(p)}\right] \cdot \ln(p)$$

follows, which was shown also in [6].

In the followings we prove some other properties of $L(x)$.

LEMMA 1.1.

$$\lim_{x \to \infty} \frac{\ln(L(x))}{\theta(x)} = 1.$$

PROOF. By (3), using that $k_p = 1$ for p's with $\sqrt{x} < p \leq x$, we get

$$\ln(L(x)) = \sum_{p \leq \sqrt{x}} k_p \cdot \ln(p) + \sum_{\sqrt{x} < p \leq x} \ln(p) =$$
$$= \sum_{p \leq \sqrt{x}} \ln\left(p^{k_p - 1}\right) + \theta(x).$$

But

$$\sum_{p \leq \sqrt{x}} \ln\left(p^{k_p - 1}\right) < \sum_{p \leq \sqrt{x}} \ln(x) = \ln(x) \cdot \Pi\left(\sqrt{x}\right)$$

and

$$\frac{\ln(x).\Pi(\sqrt{x})}{\theta(x)} \to 0 \quad \text{as } x \to \infty$$

from which the lemma follows.

LEMMA 1.2. $0 \le \Pi(x)\ln(x) - \ln(L(x)) \le \theta(x)$

PROOF. From the inequality $r - l < [r] \leqq r$, using (3), we obtain

$$\Pi(x)\ln(x) - \theta(x) \leqq \ln(L(x)) \leqq \Pi(x)\ln(x)$$

and so the lemma is proved.

LEMMA 1.3. $\theta(x) \leqq \ln(L(x)) \leqq \Pi(x)\ln(x)$.

PROOF. $p \le x$ implies $1 \le \ln(x)/\ln(p)$. Hence from (3) we obtain

$$\theta(x) = \sum_{p \le x} \ln(p) \leqq \sum_{p \le x} \left[\frac{\ln(x)}{\ln(p)}\right] \ln(p) = \ln(L(x)).$$

The other inequality is part of Lemma 1.2.

We will need also the following result

LEMMA 1.4. $$\lim_{x \to \infty} \frac{\theta(x)}{\ln(x)\Pi(x)} = 1.$$

PROOF. We can deduce this from $\theta(x) \approx x$ plus the Prime Number Theorem (cf. e. g. in [2]). Or one can prove it from inequalities

(5) $$x\left(1 - \frac{1}{\ln(x)}\right) < \theta(x), \qquad \Pi(x) < \frac{x}{\ln(x)}\left(1 + \frac{3}{2\ln(x)}\right)$$

for $41 \le x$, which can be found in [5], since by Lemma 1.3 we have

$$1-\frac{1}{\sqrt{\ln(x)}}<\frac{\left(1-\frac{1}{\ln(x)}\right)}{\left(1+\frac{3}{2\ln(x)}\right)}=\frac{x\cdot\left(1-\frac{1}{\ln(x)}\right)}{x\cdot\left(1+\frac{3}{2\ln(x)}\right)}<\frac{\theta(x)}{\ln(x)\cdot\Pi(x)}\leq 1.$$

The function $\Pi(x)$ can be approximated through the function $L(x)$. We will show that $\Pi(x)$ is asymptotic to $\ln(L(x))/\ln(x)$

COROLLARY 1.5.

$$\lim_{x\to\infty}\frac{\ln(L(x))}{\ln(x)\cdot\Pi(x)}=1.$$

PROOF. From Lemmas 1.3 and 1.4, we have for $x\geq 41$,

$$1-\frac{1}{\sqrt{\ln(x)}}<\frac{\theta(x)}{\ln(x)\Pi(x)}\leq\frac{\ln(L(x))}{\ln(x)\cdot\Pi(x)}\leq 1.$$

Now we can show that $\psi(x)=\ln(L(x))$ is asymptotic to x.

COROLLARY 1.6.

$$\lim_{x\to\infty}\frac{\ln(L(x))}{x}=1.$$

PROOF. Using Lemma 1.3 and the inequalities (5) we get

$$1-\frac{1}{\ln(x)}<\frac{\theta(x)}{x}\leq\frac{\ln(L(x))}{x}\leq\frac{\Pi(x)\ln(x)}{x}<1+\frac{3}{2\ln(x)}.$$

This actually shows that $\ln(L(x))=x+o\left(\frac{x}{\ln(x)}\right)$ which implies the following two corollaries.

COROLLARY 1.7. For all $\varepsilon>0$ and all sufficiently large x,

$$(e-\varepsilon)^x < L(x) < (e+\varepsilon)^x.$$

COROLLARY 1.8. (see also in [6])

$$\lim_{x\to\infty} L(x)^{\frac{1}{x}} = e.$$

LEMMA 1.9. $\lim_{x\to\infty} \frac{\ln(x!)}{x\ln(x)} = 1.$

PROOF. We use the following inequality, a form of Stirling's Theorem:

$$x\cdot\ln(x) - x + \frac{1}{2}\ln(x) + \frac{\ln(2\pi)}{2} < \ln(x!) \le x\cdot\ln(x) - x + \frac{1}{2}\ln(x) + 1.$$

Hence $1 - \frac{1}{\ln(x)} < \frac{\ln(x!)}{x\ln(x)} < 1$, and so $\frac{\ln(x!)}{x\ln(x)} = 1 + o\left(\frac{1}{\ln(x)}\right)$.

THEOREM 1. $\lim_{x\to\infty} \frac{\ln(x!)}{\ln(L(x))\ln(x)} = 1.$

PROOF. From Corollary 1.6. Lemma 1.9. we have

$$\lim_{x\to\infty} \frac{\ln(x!)}{\ln(L(x))\ln(x)} = \lim_{x\to\infty} \frac{\frac{\ln(x!)}{x\ln(x)}}{\frac{\ln(L(x))}{x}} = \frac{\lim_{x\to\infty} \frac{\ln(x!)}{x\ln(x)}}{\lim_{x\to\infty} \frac{\ln(L(x))}{x}} = \frac{1}{1} = 1.$$

Using Stirling's Formula again and Corollary 1.6 we may obtain π as a limit. Some other similar result for π was obtained in [3] and [4].

THEOREM 2.

(6) $$\lim_{x\to\infty} \frac{x!^2 e^{2x}}{2\ln(L(x))\cdot x^{2x}} = \pi$$

PFOOF. After we multiply the left side of (6) by 2 and take the log we obtain

$$2\ln x! + 2x - 2x\ln x - \ln\ln L(x) =$$
$$= 2\ln x! + 2x - 2x\ln(x) - \ln(x) - (\ln\ln L(x) - \ln x).$$

The term $\ln(\ln(L(x))) - \ln(x) \to 0$, as $x \to \infty$, by Corollary 1.6. One of the formulations of Stirling's Formula (cf. e. g. Artin [1]) says that there exists a δ, such that

$$\ln(x!) + x - x\cdot\ln(x) - \frac{1}{2}\cdot\ln(x) = \frac{1}{2}\cdot\ln(2\pi) + \frac{\delta}{12x} \qquad (0 < \delta < 1).$$

Multiply this by 2 and take the limit as $x \to \infty$, the theorem follows.

II. Quotient after $x!$ is divided by $L(x)$.

We derive some induction results about the quotient $x!/L(x)$, which is here denoted by $Q(x)$.

LEMMA 2.1. $L(x+1) = \dfrac{L(x)\cdot(x+1)}{(L(x), x+1)}$.

PROOF. Since $L(x+1) = [L(x), x+1]$.

LEMMA 2.2. $L(x+1)$ divides $L(x)\cdot(x+1)$.
PROOF. By Lemma 2.1.

LEMMA 2.3. $L(x)$ divides $x!$.

PROOF. Induction on x, using Lemm 2.2. from $L(x+1)|L(x)(x+1)$ and $L(x)|x!$, $L(x+1)|x!(x+1)$. follows.

DEFINITION 2.4. $Q(x)=\dfrac{x!}{L(x)}$.

LEMMA 2.5. $Q(x)$ is an integer and $Q(x)\,|\,x!$.

PFOOF. By Lemma 2.3.

LEMMA 2.6. $Q(x+1)=\left(Q(x)\cdot(x+1),x!\right)$.

FROOF. From Lemma 2.1. using $Q(x+1)=Q(x).\left(L(x),x+1\right)$.

LEMMA 2.7. p is a prime if and only if $L(p)=p\cdot L(p-1)$.

LEMMA 2.8. p is a prime if and only if $Q(p)=Q(p-1)$.

DEFINITION 2.9. $K(x)=\dfrac{Q(x)}{Q(x-1)}$.

LEMMA 2.10. $K(x)$ is an integer, $K(x)=(L(x-1),x)$ and $K(x)\,|\,x$.

FROOF. Use Lemma 2.6.

LEMMA 2.11. p is prime iff $K(p)=1$.

LEMMA 1.12. p is composite iff $1<K(p)$.

REFERENCES

[1] E. Artin, *Einfuhrung in die Theori der Gammafunktion,* Hamburger Mathematische Einzelschriften, Heft / 1931, Verlag B. G. Teubner, Leipzig. English translation: *The Gamma Function,* Holt Rinehart and Winston, N.Y., 1964.

[2] T. M. Apostol, *Introduction to Analytic Number Theory,* Springer-Verlag, 1976.

[3] Péter Kiss and Ferenc Mátyás, An asymptotic formula for π, Journal of Number Theory, 31 (1989), 255—259.

[4] Y. V. Matijasevic and R. K. Guy, *A new formula for* π, Amer. Math. Monthly, 93 (1986), 631—635.

[5] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers,* Illinois Jour. Math. 6 (1962), 64—94.

[6] E. Trost, *Primzahlen,* Verlag Birkhauser, Basel-Stuttgart, 1953. Russian translation by N. J. Feldman and A. O. Gelfand, Moscow, Nauka, 1959.