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Topological loops with six-dimensional solvable multiplication groups having five-dimensional nilradical*

Ágota Figula, Kornélia Ficzere, Ameer Al-Abayechi

University of Debrecen, Institute of Mathematics, Hungary
figula@science.unideb.hu
ficzerelia@gmail.com
ameer@science.unideb.hu

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Abstract

Using connected transversals we determine the six-dimensional indecomposable solvable Lie groups with five-dimensional nilradical and their subgroups which are the multiplication groups and the inner mapping groups of three-dimensional connected simply connected topological loops. Together with this result we obtain that every six-dimensional indecomposable solvable Lie group which is the multiplication group of a three-dimensional topological loop has one-dimensional centre and two- or three-dimensional commutator subgroup.

Keywords: multiplication group of a topological loop, connected transversals, linear representations of solvable Lie algebras

MSC: 22E25, 17B30, 20N05, 57S20, 53C30

1. Introduction

The multiplication group $Mult(L)$ and the inner mapping group $Inn(L)$ of a loop L are important tools for the investigations in loop theory since there are strong

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relations between the structure of the normal subloops of L and that of the normal subgroups of $Mult(L)$ (cf. [1, 2]). In [9] the authors have obtained necessary and sufficient conditions for a group G to be the multiplication group of L. These conditions say that one can use special transversals A and B with respect to a subgroup K of G. The subgroup K plays the role of the inner mapping group of L whereas the transversals A and B belong to the sets of left and right translations of L.

P. T. Nagy and K. Strambach in [8] investigate thoroughly topological and differentiable loops as continuous and differentiable sections in Lie groups. In this paper we follow their approach and study topological loops L of dimension 3 having a solvable Lie group as their multiplication group. Applying the criteria of [9] we obtained in [3] all solvable Lie groups of dimension ≤ 5 which are the multiplication group of a 3-dimensional connected simply connected topological proper loop. This classification has resulted only decomposable Lie groups as the group $Mult(L)$ of L. Hence we paid our attention to 6-dimensional solvable indecomposable Lie groups. If their Lie algebras have a 4-dimensional nilradical, then among the 40 isomorphism classes of Lie algebras there is only one class depending on a real parameter which consists of the Lie algebras of the group $Mult(L)$ of L (cf. [4]). This result has confirmed the observation that the condition for the multiplication group of a topological loop to be a (finite-dimensional) Lie group is strong. Since the 6-dimensional solvable indecomposable Lie algebras have 4 or 5-dimensional nilradical it remains to deal with the 99 classes of solvable Lie algebras having 5-dimensional nilradical (cf. [7, 10]). In [5] we proved that among them there are 20 classes of Lie algebras which satisfy the necessary conditions to be the Lie algebra of the group $Mult(L)$ of a 3-dimensional loop L. We determined there also the possible subalgebras of the corresponding inner mapping groups.

The purpose of this paper is to determine the indecomposable solvable Lie groups of dimension 6 which have 5-dimensional nilradical and which are the multiplication group of a 3-dimensional connected simply connected topological loop. To find a suitable linear representation of the simply connected Lie groups for the 20 classes of solvable Lie algebras given in [5] is the first step to achieve this classification (cf. Theorem 3.1). Applying the method of connected transversals we show that only those Lie groups G in Theorem 3.1 which have 2- or 3-dimensional commutator subgroup allow continuous left transversals A and B in the group G with respect to the subgroup K given in Theorem 3.1 such that A and B are K-connected and $A \cup B$ generates G (cf. Proposition 3.2 and Theorem 3.3). An arbitrary left transversal A to the 3-dimensional abelian subgroup K of G depends on three continuous real functions with three variables. The condition that the left transversals A and B are K-connected is formulated by functional equations. Summarizing the results of Theorem in [6], of Theorem 16 in [4] and of Theorem 3.3 we obtain that each 6-dimensional solvable indecomposable Lie group which is the multiplication group of a 3-dimensional topological loop has 1-dimensional centre and two- or three-dimensional commutator subgroup.

2. Preliminaries

A loop is a binary system $(L,\cdot)$ if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $x \cdot a = b$ and $a \cdot y = b$ have precisely one solution $x = b/a$ and $y = a\backslash b$. A loop is proper if it is not a group.

The left and right translations $\lambda_a = y \mapsto a{\cdot}y : L \to L$ and $\rho_a : y \mapsto y{\cdot}a : L \to L$, $a \in L$, are bijections of L. The permutation group $Mult(L) = \langle \lambda_a, \rho_a; a \in L\rangle$ is called the multiplication group of L. The stabilizer of the identity element $e \in L$ in $Mult(L)$ is called the inner mapping group $Inn(L)$ of L.

Let G be a group, let $K \le G$, and let A and B be two left transversals to K in G. We say that A and B are K-connected if $a^{-1}b^{-1}ab \in K$ for every $a \in A$ and $b \in B$. The core $Co_G(K)$ of K in G is the largest normal subgroup of G contained in K. If L is a loop, then $\Lambda(L) = \{\lambda_a; a \in L\}$ and $R(L) = \{\rho_a; a \in L\}$ are $Inn(L)$-connected transversals in the group $Mult(L)$ and the core of $Inn(L)$ in $Mult(L)$ is trivial. In [9], Theorem 4.1, the following necessary and sufficient conditions are established for a group G to be the multiplication group of a loop L:

Proposition 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exists a subgroup K with $Co_G(K) = 1$ and K-connected transversals A and B satisfying $G = \langle A, B\rangle$.*

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y,\ (x, y) \mapsto x\backslash y, (x, y) \mapsto y/x : L \times L \to L$ are continuous. In general the multiplication group of a topological loop L is a topological transformation group that does not have a natural (finite dimensional) differentiable structure. In this paper we deal with 3-dimensional connected simply connected topological loops L. We assume that the multiplication group of L is a 6-dimensional solvable indecomposable Lie group G such that its Lie algebra has 5-dimensional nilradical. Then L is homeomorphic to $\mathbb{R}^3$ (cf. [3, Lemma 5]). Since it has nilpotency class 2 (cf. [5, Theorem 3.1]) by Theorem 8 A in [2] the subgroup K in Proposition 2.1 is a 3-dimensional abelian Lie subgroup of G which does not contain any non-trivial normal subgroup of G, A and B are continuous K-connected left transversals to K in G such that $A \cup B$ generates G.

3. Six-dimensional solvable Lie multiplication groups with five-dimensional nilradical

Using necessary conditions we found in [5], Theorems 3.6, 3.7, those 6-dimensional solvable indecomposable Lie algebras with 5-dimensional nilradical which can occur as the Lie algebra $\mathbf{g}$ of the multiplication group of a 3-dimensional topological loop L. We obtained also the Lie subalgebras $\mathbf{k}$ of the inner mapping group of L. With the notation in [10] they are the following:

$\mathbf{g}_1 := \mathbf{g}_{6,14}^{a=b=0}$, $\mathbf{k}_{1,1} = \langle e_2, e_4 + e_1, e_5\rangle$, $\mathbf{k}_{1,2} = \langle e_3, e_4 + e_1, e_5\rangle$;

$\mathbf{g}_2 := \mathbf{g}_{6,22}^{a=0}$, $\mathbf{k}_2 = \langle e_3, e_4 + e_1, e_5\rangle$,
$\mathbf{g}_3 := \mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$, $\mathbf{k}_{3,1} = \langle e_3, e_4, e_5 + e_1\rangle$, $\mathbf{k}_{3,2} = \langle e_2, e_4, e_5 + e_1\rangle$;
$\mathbf{g}_4 := \mathbf{g}_{6,51}^{\varepsilon=\pm 1}$, $\mathbf{k}_4 = \langle e_1 + a_1 e_2, e_3 + e_2, e_4\rangle$, $a_1 \in \mathbb{R}$;
$\mathbf{g}_5 := \mathbf{g}_{6,54}^{a=b=0}$, $\mathbf{k}_5 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_6 := \mathbf{g}_{6,63}^{a=0}$, $\mathbf{k}_6 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_7 := \mathbf{g}_{6,25}^{a=b=0}$, $\mathbf{k}_7 = \langle e_1 + e_5, e_2 + \varepsilon e_5, e_4\rangle$, $\varepsilon = 0, 1$;
$\mathbf{g}_8 := \mathbf{g}_{6,15}^{a=0}$, $\mathbf{k}_8 = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5\rangle$, $a_3 \in \mathbb{R} \setminus \{0\}$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_9 := \mathbf{g}_{6,21}^{a=0,0<|b|\leq 1}$, $\mathbf{k}_9 = \langle e_3, e_4 + e_1, e_5 + e_1\rangle$;
$\mathbf{g}_{10} := \mathbf{g}_{6,24}$, $\mathbf{k}_{10} = \langle e_3, e_4, e_5 + e_1\rangle$;
$\mathbf{g}_{11} := \mathbf{g}_{6,30}$, $\mathbf{k}_{11} = \langle e_3, e_4 + a_2 e_1, e_5 + e_1\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_{12} := \mathbf{g}_{6,36}^{a=0,b\geq 0}$, $\mathbf{k}_{12,1} = \langle e_3, e_4, e_5 + e_1\rangle$, $\mathbf{k}_{12,2} = \langle e_3, e_4 + e_1, e_5 + a_3 e_1\rangle$, $a_3 \in \mathbb{R}$;
$\mathbf{g}_{13} := \mathbf{g}_{6,16}$, $\mathbf{k}_{13} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5\rangle$, $a_2, a_3 \in \mathbb{R}$;
$\mathbf{g}_{14} := \mathbf{g}_{6,27}^{a=1,b=\delta=0}$, $\mathbf{k}_{14} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_{15} := \mathbf{g}_{6,49}^{\varepsilon=0,\pm 1}$, $\mathbf{k}_{15} = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3\rangle$, $a_1, a_3 \in \mathbb{R}$;
$\mathbf{g}_{16} := \mathbf{g}_{6,52}^{\varepsilon=0,\pm 1}$, $\mathbf{k}_{16} = \langle e_1 + a_1 e_2, e_3 + e_2, e_4\rangle$, $a_1 \in \mathbb{R}$;
$\mathbf{g}_{17} := \mathbf{g}_{6,57}^{a=0}$, $\mathbf{k}_{17} = \langle e_1 + e_2, e_3 + a_2 e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_{18} := \mathbf{g}_{6,59}^{\delta=1}$, $\mathbf{k}_{18} = \langle e_1 + e_2, e_3 + a_2 e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_{19} := \mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$, $\mathbf{k}_{19} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_{20} := \mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$, $\mathbf{k}_{20} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + a_3 e_4\rangle$, $a_2, a_3 \in \mathbb{R}$.

In [11] a single matrix M is established depending on six variables such that the span of the matrices engenders the given Lie algebra in the list $\mathbf{g}_i$, $i = 1, \dots, 20$. To obtain the matrix Lie group G_i of the Lie algebra $\mathbf{g}_i$ we exponentiate the space of matrices spanned by the matrix M. Simplifying the obtained exponential image we get a suitable simple form of a matrix Lie group such that by differentiating and evaluating at the identity its Lie algebra is isomorphic to the Lie algebra $\mathbf{g}_i$. In case of the Lie algebras $\mathbf{g}_j$, $j = 1, 2, 8, 9, 16$, we take in order the exponential image of the matrices:

$$M_1 = \begin{pmatrix} 0 & -s_3 & s_2 & 0 & -s_6 & 2s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & 0 & 0 & 0 & 2s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; s_i \in \mathbb{R}, i = 1, \dots, 6,$$

$$M_2 = \begin{pmatrix} 0 & -s_3 & s_2 & 0 & -s_6 & 2s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & -s_6 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & 0 & 0 & 0 & 2s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \dots, 6,$$

$$M_8 = \begin{pmatrix} -s_6 & -s_3 & -s_2 & 0 & 0 & 2s_1 \\ 0 & -s_6 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 0 & -s_3 \\ 0 & -s_6 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & -s_6 & 0 & 0 & -s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \dots, 6,$$

$$M_9 = \begin{pmatrix} 0 & -s_3 & s_2 & 0 & 0 & 2s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & -s_6 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & 0 & 0 & -bs_6 & s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \dots, 6,$$

$$M_{16} = \begin{pmatrix} -s_6 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 2s_5 & -\varepsilon s_6 & \varepsilon s_4 & 2s_2 \\ 0 & 0 & 0 & s_5 & 0 & -s_1 \\ 0 & 0 & 0 & 0 & s_5 & s_4 \\ 0 & 0 & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, \varepsilon = 0, \pm 1, i = 1, \dots, 6.$$

This procedure yields the following

Theorem 3.1. *The simply connected Lie group G_i and its subgroup K_i of the Lie algebra $\mathbf{g}_i$ and its subalgebra $\mathbf{k}_i$, $i = 1, \dots, 20$, is isomorphic to the linear group of matrices the multiplication of which is given by:*

For $i = 1$:

$$\begin{aligned} & g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) \\ & = g(x_1 + y_1 + x_2 y_3 - x_3 y_2 - x_6 y_5, x_2 + y_2, x_3 + y_3, x_4 + y_4 e^{-x_6}, x_5 + y_5, x_6 + y_6), \end{aligned}$$

$$K_{1,1} = \{g(u_1, u_3, 0, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$K_{1,2} = \{g(u_1, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 2$:

$$\begin{aligned} & g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) \\ & = g(x_1 + y_1 + x_2 y_3 - x_3 y_2 - x_6 (y_5 + x_2 y_2), \\ & \quad x_2 + y_2, x_3 + y_3 - x_6 y_2, x_4 + y_4 e^{-x_6}, x_5 + y_5, x_6 + y_6), \\ & K_2 = \{g(u_1, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, \end{aligned}$$

for $i = 3$*:*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+y_1-x_6y_4+(\tfrac{1}{2}x_6^2+x_3)y_2,\\&\quad x_2+y_2,x_3+y_3,x_4+y_4-x_6y_2,x_5+y_5e^{-x_6},x_6+y_6),\\&\quad K_{3,1}=\{g(u_2,u_3,0,u_1,u_2,0);u_i\in\mathbb{R},i=1,2,3\},\\&\quad K_{3,2}=\{g(u_2,0,u_3,u_1,u_2,0);u_i\in\mathbb{R},i=1,2,3\},\end{aligned}$$

for $i = 4$*:*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+y_1+x_5y_4,x_2+y_2+x_5y_1+\varepsilon x_4y_6+\tfrac{1}{2}x_5^2y_4,\\&\quad x_3+y_3e^{-x_6},x_4+y_4,x_5+y_5,x_6+y_6),\varepsilon=\pm 1,\\&K_4=\{g(u_1,a_1u_1+u_2,u_2,u_3,0,0);u_i\in\mathbb{R},i=1,2,3\},a_1\in\mathbb{R},\end{aligned}$$

for $i = 5$*:*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+(y_1+x_5y_3)e^{-x_6},x_2+y_2+x_5y_4,x_3+y_3e^{-x_6},x_4+y_4,x_5+y_5,x_6+y_6),\\&K_5=\{g(u_1,u_1+a_2u_2,u_2,u_3,0,0);u_i\in\mathbb{R},i=1,2,3\},a_2\in\mathbb{R},\end{aligned}$$

for $i = 6$*:*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+(y_1+y_3x_5)e^{-x_6},\\&\quad x_2+y_2-(x_5+x_6)y_4,x_3+y_3e^{-x_6},x_4+y_4,x_5+y_5,x_6+y_6),\\&K_6=\{g(u_1,u_1+a_2u_2,u_2,u_3,0,0);u_i\in\mathbb{R},i=1,2,3\},a_2\in\mathbb{R},\end{aligned}$$

for $i = 7$*:*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+(y_1+y_2x_3)e^{-x_6},x_2+y_2e^{-x_6},x_3+y_3,x_4+y_4,x_5+y_5-x_4y_6,x_6+y_6),\\&K_7=\{g(u_1,u_2,0,u_3,u_1+\varepsilon u_2,0);u_i\in\mathbb{R},i=1,2,3\},\ \varepsilon=0,1,\end{aligned}$$

for $i = 8$*:*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+(y_1+y_2x_3)e^{-x_6}-y_3x_2,\\&\quad x_2+y_2e^{-x_6},x_3+y_3,x_4+(y_4-y_2x_6)e^{-x_6},x_5+y_5-x_6y_3,x_6+y_6),\\&K_8=\{g(u_1,u_2,0,u_3,u_1+a_2u_2+a_3u_3,0);u_i\in\mathbb{R},i{=}1,2,3\},a_3\in\mathbb{R}\setminus\{0\},a_2\in\mathbb{R},\end{aligned}$$

for $i = 9$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1 + x_2y_3 - (x_3 + x_2x_6)y_2, x_2 + y_2, \\ &\quad x_3 + y_3 - x_6y_2, x_4 + y_4e^{-x_6}, x_5 + y_5e^{-bx_6}, x_6 + y_6),\ \ 0 < |b| \leq 1, \\ &K_9 = \{g(u_1 + u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, \end{aligned}$$

for $i = 10$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1 - 2x_6y_4 + (x_6^2 - x_2)y_3 - (\tfrac{1}{3}x_6^3 - x_2x_6 - x_3)y_2, x_2 + y_2, \\ &\quad x_3 + y_3 - x_6y_2, x_4 + y_4 - x_6y_3 + \tfrac{1}{2}x_6^2y_2, x_5 + y_5e^{-x_6}, x_6 + y_6), \\ &K_{10} = \{g(u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, \end{aligned}$$

for $i = 11$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1 + x_2y_3 - \tfrac{1}{2}x_2^2y_6, x_2 + y_2, x_3 + y_3 - x_2y_6, \\ &\quad x_4 + y_4e^{-x_6}, x_5 + y_5e^{-x_6} - x_4y_6, x_6 + y_6), \\ &K_{11} = \{g(a_2u_1 + u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \end{aligned}$$

for $i = 12$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1 - x_2y_3 + y_2(x_3 + x_2x_6), x_2 + y_2, x_3 + y_3 - x_6y_2, \\ &\quad x_4 + y_4e^{-bx_6}\cos x_6 + y_5e^{-bx_6}\sin x_6, \\ &\quad x_5 - y_4e^{-bx_6}\sin x_6 + y_5e^{-bx_6}\cos x_6, x_6 + y_6), b \geq 0, \\ &K_{12,1} = \{g(u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, \\ &K_{12,2} = \{g(u_1 + a_3u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_3 \in \mathbb{R}, \end{aligned}$$

for $i = 13$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + [y_1 - y_4x_6 + y_2(\tfrac{1}{2}x_6^2 + x_3)]e^{-x_6} - x_2y_3, x_2 + y_2e^{-x_6}, \\ &\quad x_3 + y_3, x_4 + (y_4 - y_2x_6)e^{-x_6}, x_5 + y_5 - x_6y_3, x_6 + y_6), \\ &K_{13} = \{g(u_1, u_2, 0, u_3, u_1 + a_2u_2 + a_3u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2, a_3 \in \mathbb{R}, \end{aligned}$$

for $i = 14$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1e^{-x_6} + x_2y_3, x_2 + y_2e^{-x_6}, x_3 + y_3, \\ &\quad x_4 + y_4 - x_6y_3, x_5 + y_5 - x_6y_4 + \tfrac{1}{2}x_6^2y_3, x_6 + y_6), \end{aligned}$$

$$K_{14} = \{g(u_1, u_2, 0, u_3, u_1 + a_2u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},$$

for $i = 15$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1e^{-x_6} + x_4y_5, x_2 + (y_2 - 2\varepsilon y_4x_6 - y_1x_5)e^{-x_6} + (x_1 - x_4x_5)y_5, \\ &\quad x_3 + y_3 - x_6y_5, x_4 + y_4e^{-x_6}, x_5 + y_5, x_6 + y_6), \varepsilon = 0, \pm 1, \\ &K_{15} = \{g(u_1, u_2, a_1u_1 + u_2 + a_3u_3, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_1, a_3 \in \mathbb{R}, \end{aligned}$$

for $i = 16$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1 + x_5y_4 + \tfrac{1}{2}x_5^2y_6, \\ &\quad x_2 + y_2 + 2x_5y_1 + (x_5^2 - \varepsilon x_6)y_4 + (\tfrac{1}{3}x_5^3 + \varepsilon(x_4 - x_5x_6))y_6, \\ &\quad x_3 + y_3e^{-x_6}, x_4 + y_4 + x_5y_6, x_5 + y_5, x_6 + y_6), \varepsilon = 0, \pm 1, \\ &K_{16} = \{g(u_1, a_1u_1 + u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_1 \in \mathbb{R}, \end{aligned}$$

for $i = 17$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + (y_1 + x_5y_3)e^{-x_6}, x_2 + y_2 + x_5y_4 - \tfrac{1}{2}x_5^2y_6, \\ &\quad x_3 + y_3e^{-x_6}, x_4 + y_4 - x_5y_6, x_5 + y_5, x_6 + y_6), \\ &K_{17} = \{g(u_1, u_1 + a_2u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \end{aligned}$$

for $i = 18$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + (y_1 + y_3x_5)e^{-x_6}, x_2 + y_2 - (x_5 + x_6)y_4 - \tfrac{1}{2}(x_5 + x_6)^2y_5, \\ &\quad x_3 + y_3e^{-x_6}, x_4 + y_4 + (x_5 + x_6)y_5, x_5 + y_5, x_6 + y_6), \\ &K_{18} = \{g(u_1, u_1 + a_2u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \end{aligned}$$

for $i = 19$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + y_1e^{-ax_6} + x_3y_2, x_2 + y_2, x_3 + y_3e^{-ax_6}, \\ &\quad x_4 + y_4 - x_6y_2, x_5 + y_5e^{-x_6}, x_6 + y_6), a \in \mathbb{R} \setminus \{0\}, \\ &K_{19} = \{g(u_1, 0, u_2, u_1 + a_2u_2 + u_3, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R}, \end{aligned}$$

for $i = 20$*:*

$$\begin{aligned} &g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ &= g(x_1 + (y_1 - x_6y_5 + y_2x_3)e^{-x_6}, x_2 + y_2e^{-x_6}, \\ &\quad x_3 + y_3, x_4 + y_4 - x_3y_6, x_5 + y_5e^{-x_6}, x_6 + y_6), \\ &K_{20} = \{g(u_1, u_2, 0, u_1 + a_2u_2 + a_3u_3, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2, a_3 \in \mathbb{R}. \end{aligned}$$

Among the Lie groups in Theorem 3.1 only the group G_1 has 2-dimensional commutator subgroup and the groups G_i, $i = 2, \dots, 7$, have 3-dimensional commutator subgroup. We show that among the 6-dimensional solvable indecomposable Lie groups with 5-dimensional nilradical precisely these Lie groups are the multiplication groups of three-dimensional connected simply connected topological loops.

Proposition 3.2. *There does not exist 3-dimensional connected topological proper loop L such that the Lie algebra $\mathbf{g}$ of the multiplication group of L is one of the Lie algebras $\mathbf{g}_i$, $i = 8, \dots, 20$.*

Proof. If L exists, then there exists its universal covering loop $\tilde{L}$ which is homeomorphic to $\mathbb{R}^3$. The pairs (G_i, K_i) in Theorem 3.1 can occur as the group $Mult(\tilde{L})$ and the subgroup $Inn(\tilde{L})$. We show that none of the groups G_i, $i = 8, \dots, 20$, satisfies the condition that there exist continuous left transversals A and B to K_i in G_i such that for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in K_i$. By Proposition 2.1 the groups G_i, $i = 8, \dots, 20$, are not the multiplication group of a loop $\tilde{L}$. Hence no proper loop $\tilde{L}$ exists which yields that also no proper loop L exists. This proves the assertion.

Two arbitrary left transversals to the group K_i in G_i are:

For $i = 9, 10, 11, 12$,

$$A = \{g(u, v, h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(k, l, f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

for $i = 8, 13, 14, 15$,

$$A = \{g(h_1(u, v, w), h_2(u, v, w), u, h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), f_2(k, l, m), k, f_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

for $i = 16, 17, 18$,

$$A = \{g(h_1(u, v, w), u, h_2(u, v, w), h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), k, f_2(k, l, m), f_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

for $i = 19$

$$A = \{g(h_1(u, v, w), u, h_2(u, v, w), v, h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), k, f_2(k, l, m), l, f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

for $i = 20$

$$A = \{g(h_1(u, v, w), h_2(u, v, w), u, v, h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$
$$B = \{g(f_1(k, l, m), f_2(k, l, m), k, l, f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

where $h_i(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$ and $f_i(k, l, m) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0, 0, 0) = h_i(0, 0, 0) = 0$. Taking in G_i, $i = 9, 11, 12$, the elements

$$a = g(0, v, h_1(0, v, 0), h_2(0, v, 0), h_3(0, v, 0), 0) \in A,$$

$$b = g(0, 0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), m) \in B$$

and in G_{17} the elements

$$a = g(h_1(0, v, 0), 0, h_2(0, v, 0), h_3(0, v, 0), v, 0) \in A,$$
$$b = g(f_1(0, 0, m), 0, f_2(0, 0, m), f_3(0, 0, m), 0, m) \in B$$

one has $a^{-1}b^{-1}ab \in K_i$ if and only if
for $i = 9$

$$mv^2 - 2vf_1(0, 0, m) = h_2(0, v, 0)(1 - e^m) + h_3(0, v, 0)(1 - e^{bm}), \tag{3.1}$$

for $i = 11$

$$\tfrac{1}{2}mv^2 + vf_1(0, 0, m) = (e^m - 1)(h_3(0, v, 0) + a_2h_2(0, v, 0)) - e^m mh_2(0, v, 0), \tag{3.2}$$

for $i = 12$ and for $K_{12,1}$

$$2vf_1(0, 0, m) - mv^2 = (1 - e^{bm} \cos m)h_3(0, v, 0) - e^{bm} \sin mh_2(0, v, 0), \tag{3.3}$$

for $i = 12$ and for $K_{12,2}$

$$\begin{aligned} 2vf_1(0, 0, m) - mv^2 = (1 - e^{bm} \cos m)(h_2(0, v, 0) + a_3h_3(0, v, 0)) \\ + e^{bm} \sin m(h_3(0, v, 0) - a_3h_2(0, v, 0)), \end{aligned} \tag{3.4}$$

for $i = 17$

$$\begin{aligned} -\tfrac{1}{2}mv^2 - vf_3(0, 0, m) = (1 - e^m)[h_1(0, v, 0) + (a_2 - v)h_2(0, v, 0)] \\ - e^m vf_2(0, 0, m) \end{aligned} \tag{3.5}$$

is satisfied for all $m, v \in \mathbb{R}$. On the left hand side of equations (3.1), (3.2), (3.3), (3.4), (3.5) is the term mv^2 hence there does not exist any function $f_i(0, 0, m)$ and $h_i(0, v, 0)$, $i = 1, 2, 3$, satisfying these equations. Taking in G_{10} the elements

$$a = g(0, v, h_1(0, v, w), h_2(0, v, w), h_3(0, v, w), w) \in A$$
$$b = g(0, 0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), m) \in B,$$

respectively in G_{18} the elements

$$a = g(h_1(0, v, w), 0, h_2(0, v, w), h_3(0, v, w), v, w) \in A,$$
$$b = g(f_1(0, 0, m), 0, f_2(0, 0, m), f_3(0, 0, m), 0, m) \in B,$$

respectively in G_{16} the elements

$$a = g(h_1(0, v, 0), 0, h_2(0, v, 0), h_3(0, v, 0), v, 0) \in A,$$
$$b = g(f_1(0, l, m), 0, f_2(0, l, m), f_3(0, l, m), l, m) \in B$$

we obtain that $a^{-1}b^{-1}ab \in K_i$ if and only if in case $i = 10$ the equation

$$\begin{aligned} &e^w(1-e^m)h_3(0,v,w) + e^m(e^w-1)f_3(0,0,m) \\ &\qquad = (w^2+2v+2mw)f_1(0,0,m) + 2wf_2(0,0,m) \\ &\qquad\quad - (m^2+2wm)h_1(0,v,w) - 2mh_2(0,v,w) \\ &\qquad\quad - m^2wv - w^2mv - mv^2 - \tfrac{1}{3}vm^3, \end{aligned} \tag{3.6}$$

respectively in case $i = 18$ the equation

$$\begin{aligned} &e^m(e^w-1)(f_1(0,0,m) + a_2f_2(0,0,m)) \\ &\quad + e^w(1-e^m)[h_1(0,v,w) + (a_2-v)h_2(0,v,w)] \\ &= e^{m+w}vf_2(0,0,m) + (w+v)f_3(0,0,m) \\ &\quad - mh_3(0,v,w) + v^2m + \tfrac{1}{2}m^2v + wvm, \end{aligned} \tag{3.7}$$

respectively in case $i = 16$ the equation

$$\begin{aligned} &-\tfrac{1}{3}v^3m - v^2lm - l^2vm - \tfrac{1}{2}a_1v^2m - \varepsilon m^2v - a_1vlm \\ &= (1-e^m)h_2(0,v,0) - 2lh_1(0,v,0) + (l^2+2vl+a_1l+2\varepsilon m)h_3(0,v,0) \\ &\quad + 2vf_1(0,l,m) - (v^2+2vl+a_1v)f_3(0,l,m) \end{aligned} \tag{3.8}$$

holds for all $m, l, v, w \in \mathbb{R}$. Substituting into (3.6)

$$f_2(0,0,m) = f_2'(0,0,m) - mf_1(0,0,m),\ h_2(0,v,w) = h_2'(0,v,w) - wh_1(0,v,w),$$

respectively into (3.7)

$$f_1(0,0,m) = f_1'(0,0,m) - a_2f_2(0,0,m),\ h_1(0,v,w) = h_1'(0,v,w) + (v-a_2)h_2(0,v,w),$$

respectively into (3.8)

$$\begin{aligned} h_1(0,v,0) &= h_1'(0,v,0) + \left(v + \tfrac{1}{2}a_1\right)h_3(0,v,0), \\ f_1(0,l,m) &= f_1'(0,l,m) + \left(l + \tfrac{1}{2}a_1\right)f_3(0,l,m), \end{aligned}$$

we get in case $i = 10$

$$\begin{aligned} &e^w(1-e^m)h_3(0,v,w) + e^m(e^w-1)f_3(0,0,m) \\ &\quad = (w^2+2v)f_1(0,0,m) - m^2h_1(0,v,w) + 2wf_2'(0,0,m) \\ &\qquad - 2mh_2'(0,v,w) - m^2wv - w^2mv - mv^2 - \tfrac{1}{3}vm^3, \end{aligned} \tag{3.9}$$

respectively in case $i = 18$

$$\begin{aligned} &e^m(e^w-1)f_1'(0,0,m) - e^{m+w}vf_2(0,0,m) + e^w(1-e^m)h_1'(0,v,w) \\ &= (w+v)f_3(0,0,m) - mh_3(0,v,w) + v^2m + \tfrac{1}{2}m^2v + wvm, \end{aligned} \tag{3.10}$$

respectively in case $i = 16$

$$\begin{aligned}(1-e^m)h_2(0,v,0) + (l^2+2\varepsilon m)h_3(0,v,0)\\ - v^2 f_3(0,l,m) - 2lh_1'(0,v,0) + 2vf_1'(0,l,m)\\ = -\tfrac{1}{3}v^3m - v^2lm - l^2vm - \tfrac{1}{2}a_1v^2m - \varepsilon m^2 v - a_1vlm. \end{aligned} \tag{3.11}$$

Since on the right hand side of (3.9), respectively (3.10), respectively (3.11) there is the term $-\frac{1}{3}vm^3$, respectively $\frac{1}{2}m^2v$, respectively $-\frac{1}{3}v^3m$ there does not exist any function $f_i(0,0,m)$ and $h_i(0,v,w)$, $i=1,2,3$, respectively $f_i(0,l,m)$, $i=1,3$, and $h_j(0,v,0)$, $j=1,2,3$, satisfying equation (3.9), respectively (3.10), respectively (3.11).

Taking in G_i, $i=8,13,14$, the elements

$$\begin{aligned}a &= g(h_1(0,0,w),h_2(0,0,w),0,h_3(0,0,w),0,w)\in A,\\ b &= g(f_1(k,0,m),f_2(k,0,m),k,f_3(k,0,m),0,m)\in B,\end{aligned}$$

respectively in G_{19} the elements

$$\begin{aligned}a &= g(h_1(0,0,w),0,h_2(0,0,w),0,h_3(0,0,w),w)\in A,\\ b &= g(f_1(k,0,m),k,f_2(k,0,m),0,f_3(k,0,m),m)\in B,\end{aligned}$$

respectively in G_{20} the elements

$$\begin{aligned}a &= g(h_1(0,0,w),h_2(0,0,w),0,0,h_3(0,0,w),w)\in A,\\ b &= g(f_1(k,0,m),f_2(k,0,m),k,0,f_3(k,0,m),m)\in B\end{aligned}$$

we have $a^{-1}b^{-1}ab\in K_i$ precisely if for $i=8$ the equation

$$\begin{aligned}wk = {}& e^w(1-e^m)[(a_2+a_3w)h_2(0,0,w) + a_3h_3(0,0,w) + h_1(0,0,w)]\\ &+ e^m(e^w-1)[(a_3m+a_2-k)f_2(k,0,m) + a_3f_3(k,0,m) + f_1(k,0,m)]\\ &+ e^{m+w}[a_3wf_2(k,0,m) + (2k-a_3m)h_2(0,0,w)],\end{aligned} \tag{3.12}$$

for $i=13$ the equation

$$\begin{aligned}wk = {}& e^w(1-e^m)[(\tfrac{1}{2}w^2+a_2+a_3w)h_2(0,0,w) + (a_3+w)h_3(0,0,w) + h_1(0,0,w)]\\ &+ e^m(e^w-1)[(\tfrac{1}{2}m^2-k+a_3m+a_2)f_2(k,0,m)\\ &\qquad + (m+a_3)f_3(k,0,m) + f_1(k,0,m)]\\ &+ e^{m+w}[((m+a_3)w+\tfrac{1}{2}w^2)f_2(k,0,m) + (2k-\tfrac{1}{2}m^2-(w+a_3)m)h_2(0,0,w)]\\ &+ e^{m+w}(wf_3(k,0,m) - mh_3(0,0,w)),\end{aligned} \tag{3.13}$$

for $i=14$ the equation

$$\begin{aligned}&\tfrac{1}{2}w^2k + mwk + wf_3(k,0,m) - mh_3(0,0,w)\\ &= e^w(1-e^m)(h_1(0,0,w) + a_2h_2(0,0,w))\\ &\quad + e^m(e^w-1)(f_1(k,0,m) + a_2f_2(k,0,m)) - e^{m+w}kh_2(0,0,w),\end{aligned} \tag{3.14}$$

for $i = 19$ the equation

$$\begin{aligned} wk = e^w(1-e^m)h_3(0,0,w) - e^m(1-e^w)f_3(k,0,m) - e^{a(m+w)}kh_2(0,0,w) \\ + e^{aw}(1-e^{am})(h_1(0,0,w) + a_2h_2(0,0,w)) \\ - e^{am}(1-e^{aw})(f_1(k,0,m) + a_2f_2(k,0,m)), \end{aligned} \quad (3.15)$$

for $i = 20$ the equation

$$\begin{aligned} -wk = e^w(1-e^m)(h_1(0,0,w) + a_2h_2(0,0,w) + (w+a_3)h_3(0,0,w)) \\ + e^m(1-e^w)((k-a_2)f_2(k,0,m) - f_1(k,0,m) - (m+a_3)f_3(k,0,m)) \\ + e^{m+w}(kh_2(0,0,w) - mh_3(0,0,w) + wf_3(k,0,m)) \end{aligned} \quad (3.16)$$

is satisfied for all $k, m, w \in \mathbb{R}$, $a_2, a_3 \in \mathbb{R}$. Putting into (3.12)

$$h_1(0,0,w) = h_1'(0,0,w) - (a_3w + a_2)h_2(0,0,w) - a_3h_3(0,0,w),$$
$$f_1(k,0,m) = f_1'(k,0,m) + (k - a_3m - a_2)f_2(k,0,m) - a_3f_3(k,0,m),$$

respectively into (3.13)

$$h_1(0,0,w) = h_1'(0,0,w) - (\tfrac{1}{2}w^2 + a_3w + a_2)h_2(0,0,w) - (a_3+w)h_3(0,0,w),$$
$$f_1(k,0,m) = f_1'(k,0,m) + (k - \tfrac{1}{2}m^2 - a_3m - a_2)f_2(k,0,m) - (m+a_3)f_3(k,0,m),$$
$$f_3(k,0,m) = f_3'(k,0,m) - (m+a_3)f_2(k,0,m),$$
$$h_3(0,0,w) = h_3'(0,0,w) - (w+a_3)h_2(0,0,w),$$

respectively into (3.14)

$$h_1(0,0,w) = h_1'(0,0,w) - a_2h_2(0,0,w),$$
$$f_3(k,0,m) = f_3'(k,0,m) - mk,$$
$$f_1(k,0,m) = f_1'(k,0,m) - a_2f_2(k,0,m),$$

respectively into (3.15)

$$h_1(0,0,w) = h_1'(0,0,w) - a_2h_2(0,0,w),$$
$$f_1(k,0,m) = f_1'(k,0,m) - a_2f_2(k,0,m),$$

respectively into (3.16)

$$h_1(0,0,w) = h_1'(0,0,w) - a_2h_2(0,0,w) - (w+a_3)h_3(0,0,w),$$
$$f_1(k,0,m) = f_1'(k,0,m) + (k-a_2)f_2(k,0,m) - (m+a_3)f_3(k,0,m)$$

in order equations (3.12), (3.13), (3.14), (3.15), (3.16) reduce in case $i = 8$ to

$$\begin{aligned} wk = e^w(1-e^m)h_1'(0,0,w) + e^m(e^w-1)f_1'(k,0,m) \\ + e^{m+w}[a_3wf_2(k,0,m) + (2k - a_3m)h_2(0,0,w)], \end{aligned} \quad (3.17)$$

in case $i = 13$ to

$$\begin{aligned} wk = e^{w}(1-e^{m})h_1'(0,0,w) + e^{m}(e^{w}-1)f_1'(k,0,m) \\ + e^{m+w}[\tfrac{1}{2}w^2 f_2(k,0,m) + (2k - \tfrac{1}{2}m^2)h_2(0,0,w) \\ + wf_3'(k,0,m) - mh_3'(0,0,w)], \end{aligned} \tag{3.18}$$

in case $i = 14$ to

$$\begin{aligned} &\tfrac{1}{2}w^2 k + wf_3'(k,0,m) - mh_3(0,0,w) \\ &= e^{w}(1-e^{m})h_1'(0,0,w) + e^{m}(e^{w}-1)f_1'(k,0,m) - e^{m+w}kh_2(0,0,w), \end{aligned} \tag{3.19}$$

in case $i = 19$ to

$$\begin{aligned} wk = e^{w}(1-e^{m})h_3(0,0,w) - e^{m}(1-e^{w})f_3(k,0,m) - e^{a(m+w)}kh_2(0,0,w) \\ + e^{aw}(1-e^{am})h_1'(0,0,w) - e^{am}(1-e^{aw})f_1'(k,0,m), \end{aligned} \tag{3.20}$$

and in case $i = 20$ to

$$\begin{aligned} -wk = e^{w}(1-e^{m})h_1'(0,0,w) + e^{m}(e^{w}-1)f_1'(k,0,m) \\ + e^{m+w}(kh_2(0,0,w) - mh_3(0,0,w) + wf_3(k,0,m)). \end{aligned} \tag{3.21}$$

Since on the left hand side of (3.17), (3.18), (3.20), (3.21), respectively of (3.19) is the term wk, respectively $\frac{1}{2}w^2k$ there does not exist any function $f_i(k,0,m)$, $h_i(0,0,w)$, $i = 1,2,3$, satisfying equation (3.17), (3.18), (3.20), (3.21), respectively (3.19).

Taking in G_{15} the elements

$$\begin{aligned} a &= g(h_1(0,0,w), h_2(0,0,w), 0, h_3(0,0,w), 0, w) \in A, \\ b &= g(f_1(0,l,m), f_2(0,l,m), 0, f_3(0,l,m), l, m) \in B \end{aligned}$$

the product $a^{-1}b^{-1}ab$ lies in K_{15} if and only if the equation

$$\begin{aligned} wl = e^{w}(1-e^{m})[h_2(0,0,w) + (a_3 + 2w\varepsilon)h_3(0,0,w) + a_1h_1(0,0,w)] \\ + e^{m}(e^{w}-1)[f_2(0,l,m) + (l + a_1)f_1(0,l,m) + (a_3 + 2m\varepsilon)f_3(0,l,m)] \\ + e^{m+w}[2w\varepsilon f_3(0,l,m) - 2lh_1(0,0,w) - (l^2 + 2m\varepsilon + a_1l)h_3(0,0,w)] \end{aligned} \tag{3.22}$$

is satisfied for all $m, l, w \in \mathbb{R}$. Substituting into (3.22)

$$\begin{aligned} h_1(0,0,w) &= h_1'(0,0,w) - \tfrac{1}{2}a_1h_3(0,0,w), \\ h_2(0,0,w) &= h_2'(0,0,w) - a_1h_1(0,0,w) - (a_3 + 2w\varepsilon)h_3(0,0,w), \\ f_2(0,l,m) &= f_2'(0,l,m) - (l + a_1)f_1(0,l,m) - (a_3 + 2m\varepsilon)f_3(0,l,m), \end{aligned}$$

we obtain

$$wl = e^{w}(1-e^{m})h_2'(0,0,w) + e^{m}(e^{w}-1)f_2'(0,l,m)$$

$$+ e^{m+w}[2w\varepsilon f_3(0,l,m) - 2lh_1'(0,0,w) - (l^2 + 2m\varepsilon)h_3(0,0,w)]. \qquad (3.23)$$

On the left hand side of equation (3.23) is the term wl hence there does not exist any function $f_i(0,l,m)$, $i = 2,3$, and $h_j(0,0,w)$, $j = 1,2,3$ such that equation (3.23) holds. □

Theorem 3.3. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group is a 6-dimensional solvable indecomposable Lie group having 5-dimensional nilradical. Then the pairs of Lie groups (G_i, K_i), $i = 1,\dots,7$, are the multiplication groups $Mult(L)$ and the inner mapping groups $Inn(L)$ of L.*

Proof. The sets

$$A = \{g(k, 1 - e^m, l, me^{-m}, 2l, m); k, l, m \in \mathbb{R}\},$$
$$B = \{g(u, w, v, 2ve^{-w}, 1 - e^w, w); u, v, w \in \mathbb{R}\},$$

respectively

$$C = \{g(k, l, 1 - e^m, me^{-m}, -2l, m); k, l, m \in \mathbb{R}\},$$
$$D = \{g(u, v, w, -2ve^{-w}, 1 - e^w, w); u, v, w \in \mathbb{R}\}$$

are $K_{1,1}$-, respectively $K_{1,2}$-connected left transversals in G_1. The sets

$$A = \{g(k, l, l, me^{-m}, l^2 - 1 + e^m, m); k, l, m \in \mathbb{R}\},$$
$$B = \{g(u, v, v, -we^{-w}, v^2 + 1 - e^w, w); u, v, w \in \mathbb{R}\}$$

are K_2-connected left transversals in G_2. The sets

$$A = \{g(k, \tfrac{1}{2}m^2 - l, l, e^m - 1 - m(\tfrac{1}{2}m^2 - l), me^{-m}, m); k, l, m \in \mathbb{R}\},$$
$$B = \{g(u, \tfrac{1}{2}w^2 - v, v, 1 - e^w - w(\tfrac{1}{2}w^2 - v), -we^{-w}, w); u, v, w \in \mathbb{R}\},$$

respectively

$$C = \{g(k, l, \tfrac{1}{2}m^2 + e^m - 1, -lm + m, le^{-m}, m); k, l, m \in \mathbb{R}\},$$
$$D = \{g(u, v, \tfrac{1}{2}w^2 - e^w + 1, -vw + w, -ve^{-w}, w); u, v, w \in \mathbb{R}\}$$

are $K_{3,1}$-, respectively $K_{3,2}$-connected left transversals in G_3. The sets

$$A = \{g((l + a_1)(1 - e^m) + l, k, -e^{-m}(\tfrac{1}{2}l^2 + \varepsilon m), 1 - e^m, l, m); k, l, m \in \mathbb{R}\},$$
$$B = \{g((v + a_1)(e^w - 1) + v, u, e^{-w}(\tfrac{1}{2}v^2 + \varepsilon w), e^w - 1, v, w); u, v, w \in \mathbb{R}\}$$

are K_4-connected left transversals in G_4. The sets

$$A = \{g(le^{-k}(a_2 - l + 1), m, -le^{-k}, 1 - le^k - e^k, l, k); k, l, m \in \mathbb{R}\},$$
$$B = \{g(ve^{-u}(v - 1 - a_2), w, ve^{-u}, ve^u + e^u - 1, v, u); u, v, w \in \mathbb{R}\}$$

are K_5-connected left transversals in G_5. The sets

$$A = \{g((l - a_2)l + (l + m)e^{-m}, k, l, e^m - 1, l, m); k, l, m \in \mathbb{R}\},$$
$$B = \{g((v - a_2)v - (v + w)e^{-w}, u, v, 1 - e^w, v, w); u, v, w \in \mathbb{R}\}$$

are K_6-connected left transversals in G_6. The sets

$$A = \{g((\varepsilon - k)me^{-m}, -me^{-m}, k, -ke^m, l, m), k, l, m \in \mathbb{R}\},$$
$$B = \{g((u - \varepsilon)we^{-w}, we^{-w}, u, ue^w, v, w), u, v, w \in \mathbb{R}\}$$

are K_7-connected left transversals in G_7. For all $i = 1, \dots, 7$, the sets A, B, respectively C, D generate the group G_i. According to Proposition 2.1 the pairs (G_i, K_i), $i = 1, \dots, 7$, are multiplication groups and inner mapping groups of L which proves the assertion. □

Corollary 3.4. *Each* 3*-dimensional connected topological proper loop* L *having a solvable indecomposable Lie group of dimension* 6 *as the group* $Mult(L)$ *of* L *has* 1*-dimensional centre and* 2*- or* 3*-dimensional commutator subgroup.*

Proof. If L has a 6-dimensional indecomposable nilpotent Lie group as its multiplication group, then the assertion follows from case b) of Theorem in [6]. If it has a 6-dimensional indecomposable solvable Lie group with 4-dimensional nilradical, then the assertion is proved in Theorem 16 in [4]. If it has a 6-dimensional indecomposable solvable Lie group with 5-dimensional nilradical, then Theorems 3.6 and 3.7 in [5] and Theorem 3.3 give the assertion. □

References

[1] A. A. Albert: *Quasigroups I*, Trans. Amer. Math. Soc. 54 (1943), pp. 507–519.

[2] R. H. Bruck: *Contributions to the Theory of Loops*, Trans. Amer. Math. Soc. 60 (1946), pp. 245–354.

[3] Á. Figula: *Three-dimensional topological loops with solvable multiplication groups*, Comm. Algebra 42 (2014), pp. 444–468.

[4] Á. Figula, A. Al-Abayechi: *Topological loops having solvable indecomposable Lie groups as their multiplication groups*, submitted to Transform. Groups (2018).

[5] Á. Figula, A. Al-Abayechi: *Topological loops with solvable multiplication groups of dimension at most six are centrally nilpotent*, Int. J. Group Theory (2019), pp. 14, doi: 10.22108/ijgt.2019.114770.1522.

[6] Á. Figula, M. Lattuca: *Three-dimensional topological loops with nilpotent multiplication groups*, J. Lie Theory 25 (2015), pp. 787–805.

[7] G. M. Mubarakzyanov: *Classification of Solvable Lie Algebras in dimension six with one non-nilpotent basis element*, Izv. Vyssh. Uchebn. Zaved. Mat. 4 (1963), pp. 104–116.

[8] P. T. Nagy, K. Strambach: *Loops in Group Theory and Lie Theory (De Gruyter Expositions in Mathematics, 35)*, Berlin: Walter de Gruyter GmbH & Co. KG, 2002.

[9] M. Niemenmaa, T. Kepka: *On Multiplication Groups of Loops*, J. Algebra 135 (1990), pp. 112–122.

[10] A. Shabanskaya, G. Thompson: *Six-dimensional Lie algebras with a five-dimensional nil-radical*, J. Lie Theory 23 (2013), pp. 313–355.

[11] G. Thompson, C. Hettiarachchi, N. Jones, A. Shabanskaya: *Representations of Six-dimensional Mubarakazyanov Lie algebras*, J. Gen. Lie Theory Appl. 8.1 (2014), Art. ID 1000211, 10 pp.