

A computer-aided demonstration of the Poincare model of hyperbolic geometry

LAJOS SZILASSI

Abstract. Teaching non-Euclidean geometries for students at the Juhász Gyula Teachers Training College, seems to be an effective way of developing their visual imagination.

This is also an objective of well-known Euclidean models of hyperbolic geometry. In particular we now deal with the circle-model of Poincare we have thought to be more suggestive because of its property of preserving angles.

At this lecture we are presenting a computer programme which besides illustrating the basic notions of hyperbolic geometry (hyper cycle, paracycle, pencils of lines etc.) also demonstrates the problem described above. It presents a “Cartesian-like” system of co-ordinates on the Poincare model, on which the graphs of some well-know functions can be studied in this system of co-ordinates.

We can see that the smaller the unit is chosen comparing to the radius of the basic circle the more the graph approaches its usual graph.

This is an effective way to make prospective teachers aware that when in the school they say “The graph of the function $y = x$ is a line”, they virtually state an equivalent form of the Euclid’s parallel axiom.

At lectures in Geometry at the Juhász Gyula Teacher Training College, teaching non-Euclidean geometries is an effective way of developing a visual approach to Geometry in students. For prospective teachers, it is especially important that these topics, which require a higher level of abstraction, be treated a visual and suggestive way, as this is how they will be able to make the most use of their studies when teaching. This aim is also served by Euclidean models of hyperbolic geometry. We are now considering the Poincare model (P-model in what follows), which we have found more suggestive than other models, due to its property of preserving angles.

The reason why this model is treated less frequently is probably that figures seem to be somewhat more fastidious to draw than, for instance, in the Cayley–Klein model, which uses methods of projective geometry. Using computer, however, it is not much more difficult to draw an arc, for example, than a segment of line. The analogy between the axial reflection of the Euclidean plane and the inversion on the P-model is also more suggestive than the central collineation on the Cayley–Klein model.

We introduce the notion of congruence axiomatically, using axioms of the reflection, at the Teacher Training College. On the one hand, this is a continuation of the way geometry is taught at primary school, on the other hand it is clearer this way how absolute geometry splits into Euclidean and hyperbolic geometry depending on what axioms we accept.

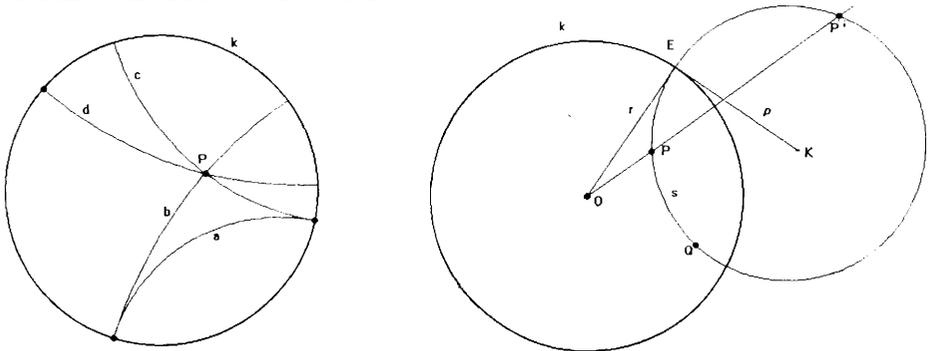
The P-model is also suitable for visualising the most important notions and theorems of absolute geometry in a different way. This way the relation between the geometry developed axiomatically and the geometry based on our "experience" and "intuition" can be seen more clearly.

Any graph—like graphs of geometrical configurations on the P-model—can only become really suggestive if we can direct drawing and see ourselves how the graphs change while changing the parameters.

This aim is served by the interactive computer programme to be presented. It enables the user that he himself can draw the graphs, which are to make notions of hyperbolic geometry more suggestive.

1. General description of the programme

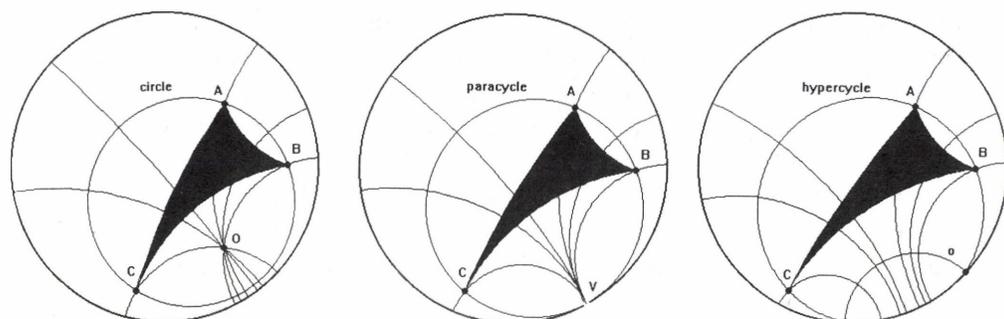
In the first part of the programme we can draw some basic geometrical configurations on the P-model on the screen, e.g. a line determined by two of its points, a segment with its perpendicular bisector, a regular curve (i.e. a circle, a paracycle, a hypercycle or a line) passing through three points, etc. It suffices to make a procedure which draws the line of the hyperbolic plane through two points on the P-model. Using this interactively, we can easily visualise the relation of two lines.



Creating such a subroutine is only a problem of calculation and programming. From the parameters of the circle of inversion $k(O, r)$ and the points P and Q (with respect to the Cartesian system of coordinates on the screen) first we had to determine the inverse $P' = \varphi_k(P)$ of the point P with respect to the circle k , then the parameters of the circle s passing through P, P', Q , finally the arc of s contained in k (or, if s happens to be a line, a diameter of k). We have chosen to use the polar coordinates (with respect to O) of

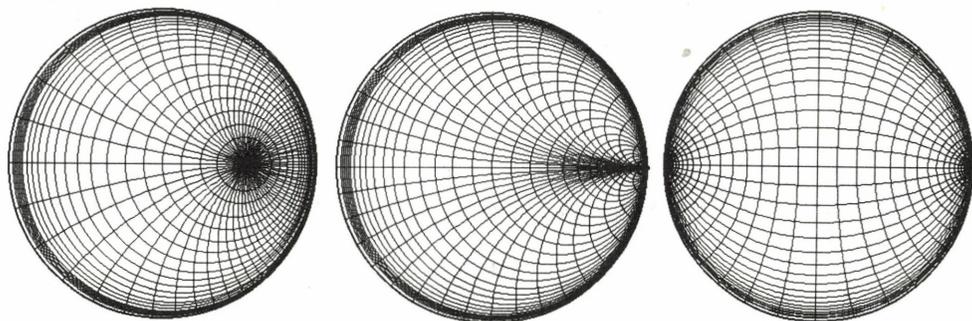
that point of the line (represented as an arc on the P-model), which is the closest to the origin O , lines passing through O were given by their normal vector. Using these parameters has proved to be advantageous especially when drawing pencils.

Another procedure used many times is drawing the perpendicular bisector of a segment determined by two points of the hyperbolic plane on the P-model. It can easily be used to visualise the relation of the perp. bisectors of the sides of a triangle, about which, in absolute geometry, we can only prove that they belong to the same type of pencil. By demonstrating this, we can make students aware that when they prove in the school the theorem about the centre of the circle circumscribed to a triangle, they in fact accept the existence of the intersection of the two perp. bisectors as an equivalent form of the parallel axiom.



On this figure we have drawn the perp. Bisectors of the sides of the triangle ABC , the regular curve passing through its vertices, and the lines perpendicular to the curve and passing through the vertices. These six lines belong to the same pencil in each of the three cases. The carrier of this pencil is either a point O , or a direction V (point at infinity), or a line o .

The second part of the programme demonstrates the three different types of pencils of the hyperbolic plane and the family of regular curves corresponding to them.

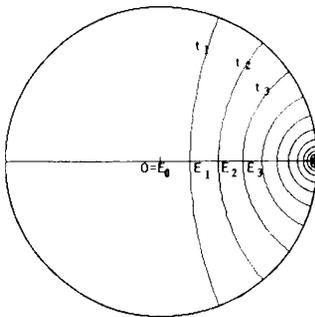


In each of the three cases those regular curves of the pencil were drawn, which intersect the lines of the pencil in points at a constant distance from each other. The user can modify both this distance—which was chosen of unit length—and other parameters of the model, within certain limits, in order to realise connections between parameters and the graph thus obtained. Finally we constructed a system of coordinates similar to that of the Cartesian system of the Euclidean plane, and drew the graphs of a few functions in this system.

Without describing the technical details of programming, we are presenting some—mostly mathematical—problems, which can lead to the graphs of the figures, including the graphs of functions.

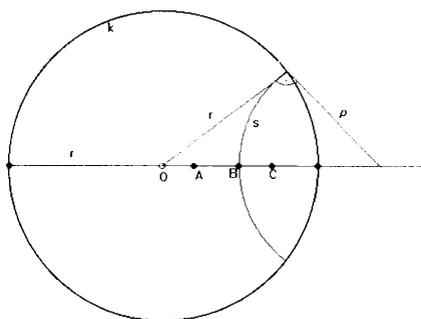
2. Number line on the P-model

First we want to see how to draw a number line of the hyperbolic plane on the P-model. Let this “line” be the diameter e of the circle k . We assign 0 to the centre $O = E_0$, 1 to an arbitrary point E_1 of the line e . The point E_2 corresponding to 2 can be constructed the following way. We draw the “line” perpendicular to e and containing E_1 — call this line t_1 —, then invert the point E_0 to this “line”: $E_2 = \varphi_{t_1}(E_0)$. Similarly, $E_3 = \varphi_{t_2}(E_1)$, where $t_2 \perp e$ and $t_2 \in E_2$, $t_2 \in E_2$, and so on. This way we have obtained lines on the P-model perpendicular to a given line and intersecting it in points corresponding to integers.



Obviously, we will need the “screen-coordinates” of these points for drawing on the screen. However, we are now going to use a Cartesian (i.e. Euclidean) system of coordinates with the centre of the circle k as origin. (The transformation taking these systems into each other is a problem of programming.) In what follows, $h(\overline{AB})$ stands for the distance of the points A and B on the hyperbolic plane, $d(\overline{AB})$ for their distance on the P-model (i.e. on the Cartesian system mentioned above.)

So we must find a sequence a_n for which $h(\overline{OE_1}) = 1$, $d(\overline{OE_1}) = k = a_1$, $h(\overline{OE_n}) = n$, and $d(\overline{OE_n}) = a_n$ where $k < r$ are arbitrary real numbers.



Take the points A and B on the half line with starting point O . Let $a = d(\overline{OA})$, $b = d(\overline{OB})$, and ρ the radius of the circle s perpendicular to $\{OB\}$ (and to k) and passing through B . Further, let $\varphi_s(A) = C'$ and $c = d(\overline{OC'})$. (C' is the image of the point A under the reflection with respect to the line s on the hyperbolic plane.) From $\varphi_s(A) = C'$ it follows that $\rho^2 = (\rho + b - a)(\rho + b - c)$. Also, $s \perp k$, whence $(b + \rho)^2 = r^2 + \rho^2$.

Thus $c = \frac{2br^2 - a(r^2 + b^2)}{r^2 + b^2 - 2ab}$. Thus applying the sequence of inversion mentioned above, we obtain the following recursive formula for the sequence a_n :

$$a_0 = 0,$$

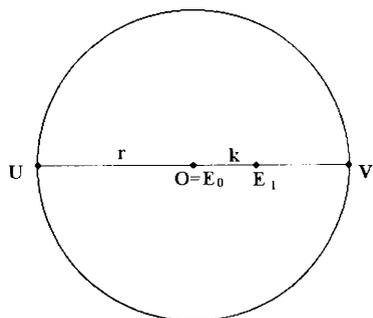
$$a_1 = k \text{ (where } k < r \text{ is an arbitrary real number) ,}$$

$$a_n = \frac{2a_{n-1} \cdot r^2 - a_{n-2} \cdot (r^2 + a_{n-1}^2)}{r^2 + a_{n-1}^2 - 2a_{n-1} \cdot a_{n-2}} \text{ if } n \geq 2 \text{ is an integer. It can be shown}$$

that $\lim_{n \rightarrow \infty} a_n = r$.

Thus far we can only construct points corresponding to natural numbers on those number lines of the hyperbolic plane, which appear as diameters on the P-model.

The distance between the hyperbolic points A and B (the hyperbolic measure of the segment \overline{AB}) is obtained by $h(\overline{AB}) = \frac{c}{2} |\ln(UVAB)|$ on the Cayley-Klein model (where collinear points appear as collinear points), where c is an arbitrary constant, U and V are the end points of the chord (or diameter) containing A and B .



Transformations between the two models fix points of the diameters of the (common) circle of inversion, so this formula can be used also in this case, if the constant c is chosen so that whenever, $d(\overline{OE_1}) = k = a_1$ $h(\overline{OE_1}) = 1$ also holds.

As $(UVOE_1) = \frac{\overline{UO}}{\overline{UV}} \div \frac{\overline{VO}}{\overline{VE_1}} = \frac{r}{r} \div \frac{r+k}{r-k} = \frac{r-k}{r+k} < 1$ from the equation $1 = \frac{c}{2} |\ln(UVOE_1)|$ we have $1 = c \cdot \ln \sqrt{\frac{r+k}{r-k}}$, thus $c = \frac{1}{\ln \sqrt{\frac{r+k}{r-k}}}$. So for any

point X on a line passing through the centre O of the P-model, for which

$$d(\overline{OX}) = x, \text{ we have } h(\overline{OX}) = h(x) = \frac{\ln \sqrt{\frac{r+k}{r-k}}}{\ln \sqrt{\frac{r+k}{r-k}}}$$

It can be shown that $h(-x) = -h(x)$, hence the point X can indeed be any point on a diameter of the circle of inversion, that is $-r < x < r$.

Now we need the inverse of this function, which maps a number line of the hyperbolic plane with origin O to the Cartesian system of the P-model.

The inverse of the function $h(x)$ is $d(x) = r \cdot \operatorname{th} \left(x \cdot \ln \sqrt{\frac{r+k}{r-k}} \right)$, and its domain is the set of real numbers, its range is the open interval $(-r; r)$. It can be shown that for any natural number n we have $a_n = d(n)$, so that on the P-model we can construct points of the hyperbolic number line corresponding to any (not only natural) number.

3. An “orthogonal” system of coordinates on the hyperbolic plane

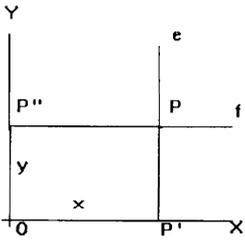
The question that is arising now is how to construct a system of coordinates on the hyperbolic plane similar to that of the Cartesian system of Euclidean geometry.

The Cartesian system assigns bijectively a point of the Euclidean plane to every pair of real numbers. When drawing the graph of a function, we in fact draw the set of points corresponding to pairs of numbers assigned to each other by the function.

In hyperbolic geometry, this is somewhat more complex. First we assign (a suitable way) a point of the plane to every pair (x, y) , then assign the point corresponding to it on the P-model (i.e. its coordinates (x_k, y_k) in the Cartesian system with origin O), and then, if we want to present it on the computer, it should be changed to the coordinates of the screen. The latter, however, is more a problem of programming than Mathematics.

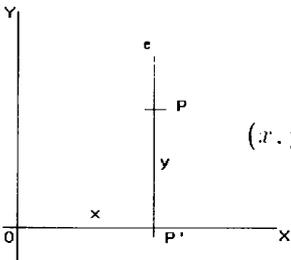
The Cartesian system consists of two perpendicular number lines, usually with the same unit. The bijection between the pairs (x, y) and the points of the plane can be realised two different ways.

The first one: We determine the points P' and P'' on the axes X and Y corresponding to the numbers x and y , then construct the point P as the intersection of the lines e and f perpendicular to X and Y and passing through P' and P'' , respectively.



$$(x, y) \Rightarrow \left\langle \begin{array}{l} x \Rightarrow P' \in X \Rightarrow e \\ y \Rightarrow P'' \in Y \Rightarrow f \end{array} \middle| \begin{array}{l} \{ e \perp X \\ e \in P' \} \\ \{ f \perp Y \\ f \in P'' \} \end{array} \right\rangle \Rightarrow P = e \cap f$$

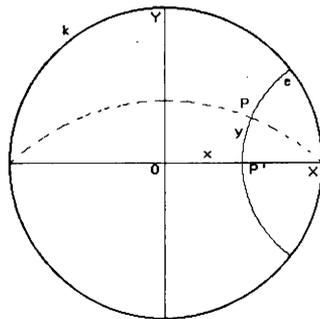
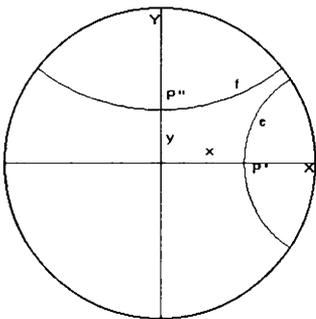
The other one: we determine the point P' corresponding to x on the axis X , then we choose the point P on the line containing P' and perpendicular to X (the locus of points with the same abscissa) for which the length of $P'P$ is y .



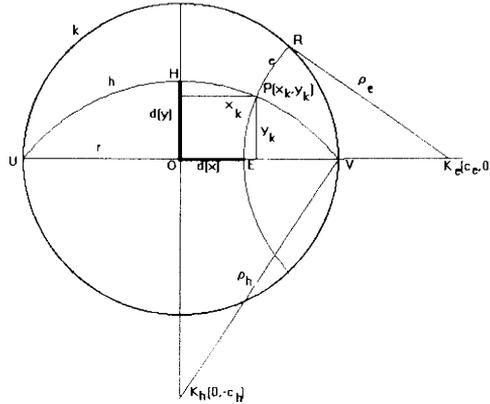
$$(x, y) \Rightarrow \left\langle \begin{array}{l} x \Rightarrow P' \in X \Rightarrow e \\ y \Rightarrow \end{array} \middle| \begin{array}{l} \{ e \perp X \\ e \in P' \} \end{array} \right\rangle \Rightarrow P = \left\{ \begin{array}{l} P \in e \\ P'P = y \end{array} \right.$$

In Euclidean geometry the rectangle $OP'PP''$ has the properties which make these two constructions equivalent.

However, this is different in hyperbolic geometry. If we chose the first way, it may very well happen that the lines perpendicular to the axes do not intersect, if the reals x, y are big enough. Choosing the other way, on the other hand, the locus of points with given ordinate—i.e. points from the same distance from the axis X —will be a hypercycle. This is how we obtain the lattice we saw when demonstrating ultraparallel.



The loci of both points with the same abscissa (line) and points with the same ordinate (hypercycle) will be an arc ι representing the line or arc h representing the hypercycle, respectively, on the P-model. The intersection of these arcs will be the point corresponding to (x, y) , i.e. the coordinates (x_k, y_k) in the Cartesian system with origin O , which we had to determine from the parameters $d(x)$, $d(y)$ and r .



Using the notation of the figure we get the equations of circles ι and h :

$$(x - c_\iota)^2 + y^2 = \rho_\iota^2 \text{ where } c_\iota = \rho_\iota + d(x) = \overline{OK_\iota},$$

$$x^2 + (y + c_h)^2 = \rho_h^2 \text{ where } c_h = \rho_h - d(y) = \overline{HK_h}.$$

It also holds that $c_\iota^2 = \rho_\iota^2 + r^2$ because s is perpendicular to the circle of inversion k , and $c_h^2 = \rho_h^2 - r^2$ as h intersects k in opposite points.

From the above we have $c_\iota x + c_h y = r^2$ for the radical axis of circles ι and h .

Also, as $e \perp h$, the points O and P both lie on the Thales' circle of the segment $K_\iota K_h$, so the angles $OK_\iota P <$ and $OK_h P <$ are equal. Hence:

$$\frac{x_k}{\rho_h} = \frac{y_k}{\rho_\iota}.$$

Calculation shows that

$$x_k = \frac{r^2 \cdot d(x) \cdot (r^2 + d^2(y))}{r^4 + d^2(x) \cdot d^2(y)} \text{ and } y_k = \frac{r^2 \cdot d(y) \cdot ((r^2 + d^2(x)))}{r^4 + d^2(x) \cdot d^2(y)}.$$

As both $d(x)$ and $d(y)$ contain r as coefficient, these formulae can be simplified:

$$x_k = \frac{th(q \cdot x) \cdot (1 + th^2(q \cdot y))}{1 + th^2(q \cdot x) \cdot th^2(q \cdot y)} r, \quad y_k = \frac{th(q \cdot y) \cdot (1 + th^2(q \cdot x))}{1 + th^2(q \cdot x) \cdot th^2(q \cdot y)} r$$

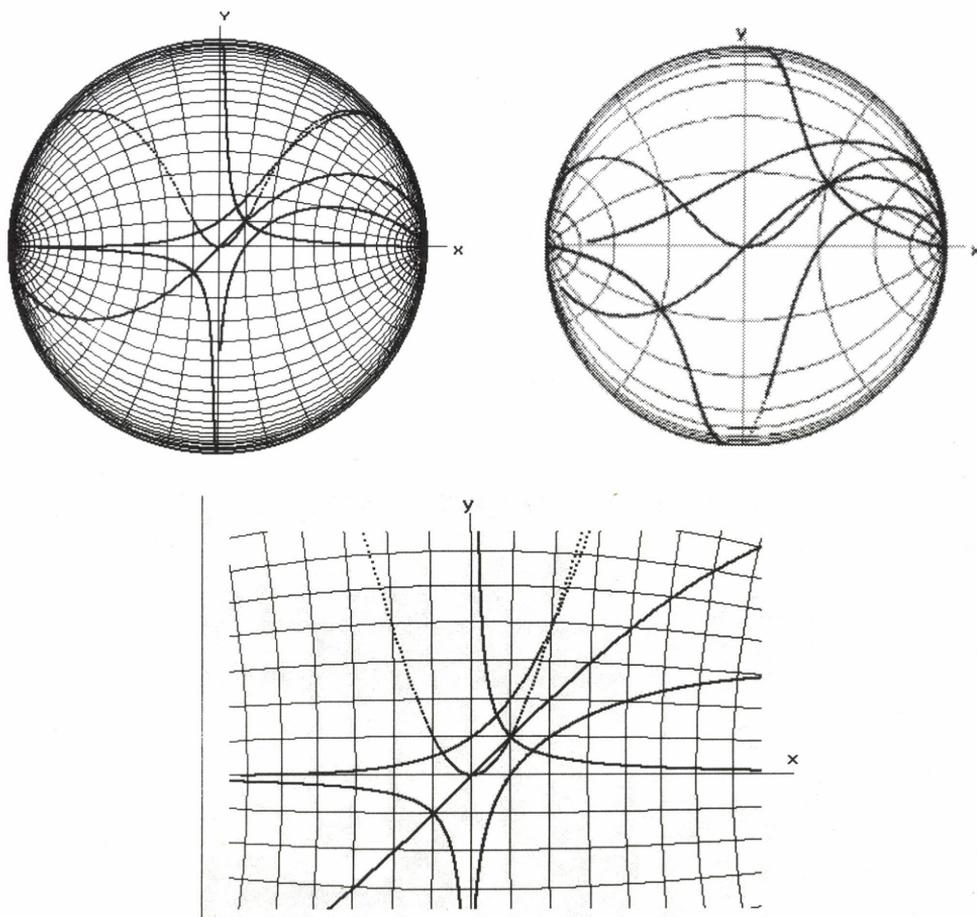
where $q = \ln \sqrt{\frac{r+k}{r-k}}$.

4. The graph of functions on the P-model

As a result, we have drawn the graph of a few well known functions ($y = x$; $y = x^2$; $y = 2^x$; $y = \log_2(x)$; $y = \frac{1}{x}$) in this system of coordinates. We realise that the extent to which these graphs approach their usual shape depends on the ratio of the radius of the circle of inversion and the unit.

The figures show the cases $\frac{k}{r} \approx 0.1$, $\frac{k}{r} \approx 0.3$, $\frac{k}{r} \approx 0.04$. In the last one the graph is very close to its usual form near the origin.

The user of the programme can realise that fixing k and increasing r , the model approaches the Cartesian system of Euclidean geometry. Choosing r big enough, we cannot feel the difference on the screen, just as we cannot feel sitting in a room that the Earth is a sphere.



We hope that this programme will help students understand the basic notions of hyperbolic geometry more easily, and make it clearer for them which are the theorems holding in absolute geometry, and which are true only in the Euclidean geometry. This way we may make prospective teachers aware that when in the school they say “The graph of the function $y = x$ is a line.”, they virtually state an equivalent form of the parallel axiom.