Effective integrability of the differential equation $P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$, II.

KRYSTYNA GRYTCZUK

ABSTRACT. In the present paper we give an application of our result given in [1] to the classical Euler's differential equation.

1. Introduction

Consider the classical Euler's differential equation

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y^{(1)} + a_n y = 0 \tag{1}$$

where $a_i \in \mathbf{R}$.

In the paper [1] it was shown (see Th.2) that the necessary and suffcient condition for the functions

$$y = s_{0,k}(x) u_k^{\lambda}(x) , \quad k = 1, 2 \cdots, n \tag{2}$$

to be the particular solutions of the differential equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{3}$$

is that

$$\sum_{j=0}^{n} P_j(x) s_{n-j,k}(x) = 0 , \quad k = 1, 2, \cdots, n , \tag{4}$$

where $s_{m,k}(x) = s'_{m-1,k}(x) + s_{m-1,k}(x) \frac{u'_k(x)}{u_k(x)}$ and $s_{m,k}(x), u_k(x) \in \mathbf{C}^{(n)}(J)$; $J = (x_1, x_2) \subset \mathbf{R}$, $u_k(x) \neq 0$ for $x \in J$.

In the present note by using this result we prove the following theorem.

Theorem. The necessary and sufficient condition for the function $y_0 = x^{\lambda}$ to be a particular solution of (1) is that the λ satisfies the following algebraic equation:

$$\begin{aligned} F(\lambda) = {} & \lambda(\lambda - 1) \cdots (\lambda - (n-1)) + \\ & a_1 \lambda(\lambda - 1) \cdots (\lambda - (n-2)) + \cdots + a_{n-1}\lambda + a_n = 0. \end{aligned} \tag{5}$$

2. Proof of the Theorem.

Putting in (2) $u_k(x) = x$, $s_{0,k}(x) \equiv 1$ for $k = 1, 2, \cdots, n$ we get

$$y_k = y_0 = x^\lambda . \tag{6}$$

By the definition of the functions $s_{m,k}(x)$ and (6) we obtain

$$\begin{cases} s_j(x)\lambda(\lambda-1)\cdots(\lambda-(j-1))\, x^{\lambda-j}, & j = 1, 2, \cdots, n \\ s_0 \equiv 1 . \end{cases} \tag{7}$$

By (4) and (7) is follows that

$$P_0(x)s_n(x) + P_1(x)s_{n-1}(x) + \cdots + P_n(x)s_0(x) = 0 . \tag{8}$$

On the other hand from (1) we have

$$P_0(x) = x^n , P_1(x) = a_1x^{n-1}, \cdots, P_{n-1}(x) = x^n , P_n(x) = a_n . \tag{9}$$

From (7), (8) and (9) we obtain

$$\begin{aligned} x^n\lambda(\lambda-1)\cdots(\lambda-(n-1))\, x^{\lambda-n} + a_1x^{n-1}\lambda(\lambda-1)\cdots \\ \cdots(\lambda-(n-2))\, x^{\lambda-(n-1)} + \cdots + a_nx^\lambda = 0 . \end{aligned} \tag{10}$$

Let

$$\begin{aligned} F(\lambda) = \lambda(\lambda-1)\cdots(\lambda-(n-1)) + \\ + a_1\lambda(\lambda-1)\cdots(\lambda-(\lambda-2)) + \cdots a_{n-1}\lambda + a_n . \end{aligned} \tag{11}$$

Then by (10) and (11) we have

$$F(\lambda)x^\lambda = 0 . \tag{12}$$

Since $x \neq 0$ on J then by (12) we get $F(\lambda) = 0$, so denote that λ must be a root of the algebraic equation

$$\begin{aligned} \lambda(\lambda-1)\cdots(\lambda-(n-1)) + a_1\lambda(\lambda-1)\cdots(\lambda-(\lambda-2)) + \\ \cdots + a_{n-1}\lambda + a_n = 0 \end{aligned}$$

and the proof is complete.

3. Remarks.

Suppose that all roots of the equation $F(\lambda) = 0$ are distinct and real. Then the differential equation of Euler (1) has n-particular solutions of the form:

$$y_1 = x^{\lambda_1}, \quad y_2 = x^{\lambda_2}, \cdots, y_n = x^{\lambda_n}.$$

It is easy to see that those solutions are linear independent over $J = (0, +\infty)$. Hence in this case we obtain that the general solution of (1) has the form:

$$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2} + \cdots + c_n = x^{\lambda_n}.$$

Now we can assume that all roots of the equation $F(\lambda) = 0$ are distinct but they can be complex numbers. If $\lambda = a + bi$ is a root of F then we have

$$x^\lambda = x^{a+bi} = x^a x^{bi} = x^a e^{ib\ln x} = x\left(\cos(b\ln x) + i\sin(b\ln x)\right)$$

and we see that functions

$$x^a\cos(b\ln x) \quad \text{and} \quad x^a\sin(b\ln x) \tag{13}$$

are real solutions of (1) and linear independent over J. Since $a_i \in \mathbf{R}$ then exists the conjugate complex root to λ, namely $\overline{\lambda} = a - bi$. In similar way we obtain (13). If λ_1 is k-multiple root of $F(\lambda)$ then we have

$$F(\lambda_1) = F'(\lambda_1) = \cdots = F^{(k-1)}(\lambda_1) = 0, \quad F^{(k)}(\lambda_1) \neq 0. \tag{14}$$

Then by differentiation m-time the expression $F(\lambda)x^\lambda$ with respect to λ we obtain

$$L\left(x^\lambda(\ln x)^m\right) = \sum_{j=0}^{m}\binom{m}{j}F^{(j)}(\lambda)x^\lambda(\ln x)^{m-j}. \tag{15}$$

From (14) and (15) it follows that the functions

$$y_m = x^{\lambda_1}(\ln x)^m, \quad m = 0, 1, \cdots, k-1$$

are particular solutions of (1).

Reference

[1] K. Grytczuk, Effective integrability of the differential equation $P_0(x)y^{(n)} + \cdots + P_n(x)y = 0$, *Acta Acad. Paed. Agriensis, Sect. Mat.*, **21** (1993), 95–103.