Relation between Dirichlet kernels with respect to Vilenkin-like systems

ISTVÁN BLAHOTA*

Abstract. In this paper we discuss the relation between Dirrichlet-kernels with respect to Vilenkin an Vilenkin-like systems. This relation gives a useful tool in field of approximation theory on compact totally disconnected Abelian groups.

Introduction

Let $m := (m_0, m_1, \ldots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_j} := \{0, 1, \ldots, m_j - 1\}$ the additive group of integers modulo m_j $(j \in \mathbf{N})$. Define the group G_m as the cartesian product of the discrete cyclic groups Z_{m_j},

$$G_m := \mathop{\times}_{j=0}^{\infty} Z_{m_j}.$$

The elements of G_m can be represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$ $(x_j \in Z_{m_j})$. It easy to give a base the neighborhoods of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m | y_0 := x_0, \ldots, y_{n-1} := x_{n-1}\}$$

for $x \in G_m, n \in \mathbf{N}$, $k = 0, 1, \ldots, m_n - 1$. Define $I_n := I_n(0)$ for $n \in \mathbf{P}$ $(\mathbf{P} := \mathbf{N}\{0\})$. Then I_n is a subgroup of G_m $(n \in \mathbf{N})$. The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k}, k \in \mathbf{N})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

---

* Research supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. F007347 and the National Scientific Foundation of the Hungarian Credit Bank (Alapítvány a Magyar Felsőoktatásért és Tudományért, MHB)

If $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbf{N})$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j} (j \in \mathbf{N})$ and only a finite number of n_j's differ from zero.

Define on G_m the *generalized Rademacher functions* in the following way:

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \quad \imath^2 := x \in G_m, \quad k \in \mathbf{N}.$$

It is known that the functions

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbf{N})$$

on G_m are elements of the character group of G_m, and all the elements of the character group are of this form. If $x, y \in G_m, n, m \in \mathbf{N}$ then it is easy to see that

$$\psi_n(x + y) = \psi_n(x)\psi_n(y),$$

and

$$\psi_{n+m}(x) = \psi_n(x)\psi_m(x).$$

The system $(\psi_n | n \in \mathbf{N})$ is called a *Vilenkin system* and G_m a *Vilenkin group*.

The Dirichlet kernels are

$$D_n^{\psi}(x) := \sum_{k=0}^{n-1} \psi_k(x) \quad (n \in \mathbf{N})$$

with respect to the Vilenkin system for which it is known (see [4]) that:

Theorem A.

$$D_{M_n}^{\psi}(x) = \begin{cases} M_n, & x \in I_n \\ 0, & x \notin I_n \end{cases} \quad (n \in \mathbf{N}).$$

Let $\mathcal{A}_n$ be the σ-algebra generated by cosets $I_n(z)$, where $(n \in \mathbf{N})(z \in G_m)$. Let $\alpha_j^k, \alpha_n (k, j, n \in \mathbf{N})$ be functions satisfying the following conditions:

(i) $\alpha_j^k : G_m \to \mathbf{Cis}\mathcal{A}_j$ - measurable $(k, j \in \mathbf{N})$,

(ii) $|\alpha_j^k| := \alpha_0^k := \alpha_j^0 := \alpha_j^k(0) := 1 (k, j \in \mathbf{N})$

(iii) $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{j(n)} (n \in \mathbf{N}, j(n) := \sum_{k=j}^{\infty} n_k M_k)$.

Let $\chi_n := \psi_n \alpha_n \quad (n \in \mathbf{N})$. A function system $\{\chi_n | n \in \mathbf{N}\}$ of this type is called a $\psi\alpha$ *(Vilenkin-like) system* on Vilenkin group G_m.

The ψ and $\psi\alpha$ systems are orthonormal and complete in $L^1(G_m)$.

The Dirichlet kernels with respect to $\psi\alpha$ system are

$$D_n^{\chi}(x,y) := \sum_{k=0}^{n-1} \chi_k(x)\overline{\chi_k(y)} \quad (n \in \mathbf{N})$$

The subsequence $D_{M_n}^{\chi}$ has a closed form

$$D_{M_n}^{\chi}(x,y) = \begin{cases} M_n, & x-y \in I_n \\ 0, & x-y \notin I_n \end{cases} (n \in \mathbf{N}),$$

(see [2]). We will use the following theorem, too (see [1]):

Theorem B.

$$D_{jM_t}^{\chi}(x,y) = \alpha_{jM_t}(x)\overline{\alpha}_{jM_t}(y) D_{jM_t}^{\psi}(x-y) \quad (n \in \mathbf{N}, x, y \in G_m).$$

Lemma. Let $x \in G_m$, $j, t \in \mathbf{N}$. Then $D_{jM_t}^{\psi}(x) = D_{M_t}^{\psi}(x) \sum_{k=0}^{j-1} \psi_{kM_t}(x)$.

This lemma is needed in the proof of the theorem.

Theorem Let $x, y \in G_m, n \in \mathbf{N}$. Then

$$D_n^{\chi}(x,y) = D_n^{\psi}(x-y)$$

holds if and only if $n \in \{jM_t | 0 \le j < m_t; t, j \in \mathbf{N}\}$.

PROOF of the lemma. Using the statements and theorems mentioned above we have the following equations:

$$D_{jM_t}^{\psi}(x) = \sum_{l=0}^{jM_t-1} \psi_l(x) = \sum_{h=0}^{j-1} \left(\sum_{l=hM_t}^{(h+1)M_t-1} \psi_l(x) \right) =$$

$$\sum_{h=0}^{j-1} \left(\sum_{l=hM_t}^{(h+1)M_t-1} \psi_{l_0 M_0 + \cdots + l_{t-1} M_{t-1} + hM_t}(x) \right) =$$

$$\sum_{h=0}^{j-1} \left(\sum_{l=hM_t}^{(h+1)M_t-1} \psi_{l_0 M_0}(x) \cdots \psi_{l_{t-1} M_{t-1}}(x) \psi_{hM_t}(x) \right) =$$

$$\sum_{h=0}^{j-1}\sum_{l_{t-1}=0}^{m_t-1}\cdots\sum_{l_0=0}^{m_0-1}\psi_{l_0M_0}(x)\cdots\psi_{l_{t-1}M_{t-1}}(x)\psi_{hM_t}(x)=$$

$$\sum_{h=0}^{j-1}\psi_{hM_t}(x)\sum_{l_{t-1}=0}^{m_t-1}\psi_{l_{t-1}M_{t-1}}(x)\cdots\sum_{l_0=0}^{m_0-1}\psi_{l_0M_0}(x)=$$

$$\left(\sum_{h=0}^{j-1}\psi_{hM_t}(x)\right)\prod_{k=0}^{t-1}\sum_{l_k=0}^{m_k-1}\psi_{l_kM_k}(x)=$$

$$\left(\sum_{h=0}^{j-1}\psi_{hM_t}(x)\right)\sum_{l_{t-1}=0}^{m_t-1}\psi_{l_{t-1}M_{t-1}}(x)\cdots\sum_{l_0=0}^{m_0-1}\psi_{l_0M_0}(x)=$$

$$\left(\sum_{h=0}^{j-1}\psi_{hM_t}(x)\right)\sum_{l_{t-1}=0}^{m_n-1}\cdots\sum_{l_0=0}^{m_0-1}\psi_{l_0M_0}(x)\cdots\psi_{l_{t-1}M_{t-1}}(x)=$$

$$\left(\sum_{h=0}^{j-1}\psi_{hM_t}(x)\right)\sum_{l_{t-1}=0}^{m_t-1}\cdots\sum_{l_0=0}^{m_0-1}\psi_{l_0M_0+\cdots+l_{t-1}M_{t-1}}(x)=$$

$$\left(\sum_{h=0}^{j-1}\psi_{hM_t}(x)\right)\sum_{l=0}^{M_t-1}\psi_l(x)=D^{\psi}_{M_t}(x)\sum_{k=0}^{j-1}\psi_{kM_t}(x).$$

This completes the proof of the Lemma.

PROOF of the theorem The form $n=jM_t$ is not unique. (For example $jM_{t+1}=(jm_t)M_t$.) In our presentation let $n=jM_t$ be that expression, in which j is the least.

1. *Sufficiency.* Suppose that $n\in\{jM_t|t,j\in\mathbf{N};0\le j<m_t\}$, and $x,y\in G_m$.

1.1. Let $x-y\notin I_t$. In this case by the theorem A. we have $D^{\psi}_{M_t}(x-y)=0$. By lemma $D^{\psi}_{jM_t}(x-y)=0$. The theorem B. shows that

$$D^{\chi}_{jM_t}(x,y)=\alpha_{jM_t}(x)\overline{\alpha}_{jM_t}(y)D^{\psi}_{jM_t}(x-y).$$

So $D^{\chi}_{jM_t}(x,y)=0$, too. Consequently if $x-y\notin I_t,t,j\in\mathbf{N}$, then

$$D^{\chi}_{jM_t}(x,y)=D^{\psi}_{jM_t}(x-y)=0.$$

1.2. Let $x-y \in I_t, t, j \in \mathbf{N}, \quad 0 \leq j < m_t$. Then $x_0 - y_0 = 0, \ldots, x_{t-1} - y_{t-1} = 0$,

$$\alpha_{jM_t}(x)\overline{\alpha}_{jM_t}(y) =$$
$$\alpha_1^{1(jM_t)}(x_0, n_1, \cdots, n_{t-1}, n_t, n_{t+1}, \cdots) \cdots$$
$$\alpha_t^{t(jM_t)}(x_0, x_1, \cdots, x_{t-1}, n_t, n_{t+1}, \cdots)$$
$$\overline{\alpha}_1^{1(jM_t)}(x_0, n_1, \cdots, n_{t-1}, n_t, n_{n+1}, \cdots) \cdots$$
$$\overline{\alpha}_t^{t(jM_t)}(x_0, x_1, \cdots, x_{t-1}, n_t, n_{t+1}, \cdots) =$$
$$\alpha_{jM_t}(x)\overline{\alpha}_{jM_t}(x) = |\alpha_{jM_t}(x)| = 1.$$

If $x - y \in I_t, t, j \in \mathbf{N}, \quad 0 \leq j < m_t$, then

$$D^{\chi}_{jM_t}(x, y) = D^{\psi}_{jM_t}(x - y).$$

This completes the proof of the first part of the theorem.

2. *Necessity.* If $n = \sum_{k=0}^{s} n_k M_k$, then let

$$\alpha_k^{k(n)}(x_0, \ldots, x_{k-1}, n_k, \ldots, n_s) := \begin{cases} \exp(i) & \text{if } x_0 \ldots x_{k-1} \neq 0 \text{ and } k \leq s \\ 1 & \text{else.} \end{cases}$$

In this case does not exist such $j \in \mathbf{P}$ that $\alpha_j(x) = 1$.

2.1. Now we suppose that $n \notin \{jM_h | h, j \in \mathbf{N}\}$. Let t be defined in the following way

$$t := \min\{k| \quad n < M_k; k, n \in \mathbf{N}\}.$$

Let $x' := (0, 0, \ldots, 0, x_t := 1, 0, \ldots), t \in \mathbf{N}$ and $y' := (0, 0, 0, \ldots)$. In this case

$$D^{\chi}_n(x', y') = \sum_{k=0}^{n-1} \chi_k(x') =$$

$$c\sum_{k=0}^{n-1} \psi_k(x') = c\sum_{k=0}^{n-1} \psi_k(x' - y') = cD^{\psi}_n(x' - y'),$$

where $c \neq 1$. We prove that $D^{\psi}_n(x' - y') \neq 0$, thus in this case $D^{\psi}_n(x' - y') \neq D^{\chi}_n(x', y')$. But

$$D^{\psi}_n(x' - y') = \sum_{k=0}^{n-1} \exp\left(\frac{2\pi \imath k_t}{m_t}\right) = \sum_{k=0}^{n-1} \exp\left(\frac{2\pi \imath O}{m_t}\right) = n \neq 0.$$

2.2 Let $n \in \{jM_t | t, j \in \mathbf{N}, \quad m_t < j\}$. It is easy to see that $x' - y' \in I_t$, but $x' - y' \notin I_{t+1}$. Then

$$D^{\chi}_{jM_t}(x', y') = cD^{\psi}_{jM_t}(x' - y'),$$

where $c \neq 1$. We will prove that $D^{\psi}_{jM_t}(x' - y') \neq 0$, thus

$$D^{\chi}_{jM_t}(x', y') \neq D^{\psi}_{jM_t}(x' - y').$$

We have by lemma

$$D^{\psi}_{jM_t}(x' - y') = D^{\psi}_{M_t}(x' - y') \sum_{k=0}^{j-1} \psi_{kM_t}(x' - y') = M_t \sum_{k=0}^{j-1} \psi_{kM_t}(x' - y') =$$

$$M_t \sum_{k=0}^{j-1} \exp\left(\frac{2\pi\imath}{m_t}\right)^2 = M_t \frac{1 - \exp\left(\frac{2\pi\imath}{m_t}\right)^2}{1 - \exp\left(\frac{2\pi\imath}{m_t}\right)} \neq 0,$$

since $m_t \not| j$.

The proof of theorem is complete.

Acknowledgement

The author wishes to thank to Professor G. Gát for setting the problem.

References

[1] G. Gát, Vilenkin, Fourier Series and Limit Peridic Arithmetic Functions, *Colloquia Mathematica Societatis János Bolyai*, **58** (1990), 315–332.

[2] G. Gát, Orthonormal systems on Vilenkin groups, *Acta Mathematica Hungarica* **58 (1—2)** (1991), 193–198

[3] F. Schipp, W. R. Wade, P. Simon and J. Pál, Walsh Series, An Introduction to Dyadic Harmonic Analysis, *Akadémiai Kiadó, Budapest, and Adam Hilger, Bristol and New York* (1990).

[4] G. H. Agajev, N. Ya. Vilenkin, G. M. Dzsafarli, A. I. Rubinstein, Multiplicative systems of functyions and harmonic analysis on 0-dimensional groups, Izd. "ELM" (Baku, SSSR) (1981).