On a norm convergence theorem with respect to the Vilenkin system in the Hardy spaces

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Abstract. In 1993. the author proved the $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \|S_k f\|_1 / k = \|f\|_1$ convergence for functions f in $H(G_m)$ (the so-called "atomic" Hardy space) with respect to all G_m Vilenkin group. In this paper we prove that this theorem fails to hold in the case of the so-called unbounded Vilenkin group and the "maximal" Hardy space.

Introduction and Results

First we introduce some necessary definitions and notations of the theory of the Vilenkin systems. The Vilenkin systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [8]). Let $m := (m_k, k \in \mathbf{N})(\mathbf{N} := \{0, 1, \ldots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the m_k-th discrete cyclic group. Z_{m_k} can be represented by the set $\{0, 1, \ldots, m_k - 1\}$, where the group operation is the mod m_k addition and every subset is open. The measure on Z_{m_k} is defined such that the measure of every singleton is $\frac{1}{m_k}(k \in \mathbf{N})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

This gives that every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbf{N})$, where $x_i \in Z_{m_i}, (i \in \mathbf{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ) and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbf{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

A base for the neighborhoods of G_m can be given as follows

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

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for $x \in G_m, n \in \mathbf{P} := \mathbf{N} \setminus \{0\}$. Let $0 = (0, i \in \mathbf{N}) \in G_m$ denote the nullelement of $G_m, I_n := I_n(0)\,(n \in \mathbf{N})$. Furthermore, let $L^p(G_m)\,(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbf{N})\,(f \in L^1.)$

The concept of the maximal Hardy space ([Sch, Sim]) $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f|$ $(f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm

$$\|f\|_{H^1} := \|f^*\|_1.$$

The so-called atomic Hardy space $H(G_m)$ is defined for bounded Vilenkin groups as follows [Sch, Sim]. A function $a \in L^\infty(G_m)$ is called an atom, if either $a = 1$ or a has the following properties: $\operatorname{supp} a \subseteq I_a, \| a \|_\infty \le \frac{1}{\mu(I_a)}, \int_{I_a} a = 0$, where $I_a \in \mathcal{I} := \{I_n(x) : x \in G_m, n \in \mathbf{N}\}$. The elements of $\mathcal{I}$ are called intervals on G_m. We say that the function f belongs to $H(G_m)$, if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i 's are atoms and for the coefficients $\lambda_i\,(i \in \mathbf{N})$ $\sum_{i=0}^\infty |\lambda_i| < \infty$ is true. It is known that $H(G_m)$ is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infinum is taken all over decompositions

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \in H(G_m).$$

If the sequence m is not bounded, then we define the set of intervals in a different way ([5]), that is we have "more" intervals than in the bounded case.

A set $I \subset G_m$ is called an interval if for some $x \in G_m$ and $n \in \mathbf{N}, I$ is of the form $I = \bigcup_{k \in U} I_n(x,k)$ where U is one of the following sets

$$U_1 = \left\{0, \dots, \left[\frac{m_n}{2}\right] - 1\right\}, U_2 = \left\{\left[\frac{m_n}{2}\right], \dots, m_n - 1\right\}$$

$$U_3 = \left\{0, \dots, \left[\frac{[m_n/2] - 1}{2}\right] - 1\right\}, U_4 = \left\{\left[\frac{[m_n/2] - 1}{2}\right] - 1, \dots, \left[\frac{m_n}{2}\right] - 1\right\}, \dots$$

etc., and $I_n(x,k) := \{y \in G_m : y_j = x_j (j < n), y_n = k\}$, $(x \in G_m, k \in Z_{m_n}, n \in \mathbf{N})$. The rest of the definition of the atomic Hardy space H is the same as in the bounded case.

It is known that if the sequence m is bounded, then $H^1 = H$, otherwise H is a proper subset of H^1 [2].

Let $M_0 := 1$, $M_{n+1} := m_n M_n$ $(n \in \mathbf{N})$. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\},\ i \in \mathbf{N}),$$

where only a finite number of n_i's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi \imath \frac{x_n}{m_n}\right) (x \in G_m, n \in \mathbf{N}, \imath := \sqrt{-1})$$

Then

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbf{N})$$

the nth Vilenkin function. The system $\psi := (\psi_n : n \in \mathbf{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbf{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n, \psi_n(x + y) = \psi_n(x)\psi_n(y), \psi_n(-x) = \bar{\psi}_n(x), |\psi_n| = 1 (k, n \in \mathbf{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n, S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y, x) = D_n(y - x) := \sum_{k=0}^{n-1} \psi_n(y)\bar{\psi}_n(x).$$

Then

$$(S_n f)(y) = \int_{G_m} f(x) D_n(y - x) dx \ \ (n \in \mathbf{N},\ y \in G_m,\ f \in L^1(G_m)).$$

It is well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0), \end{cases}$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) \ (f \in L^1(G_m), n \in \mathbf{N})$$

and

$$D_n(x) = \psi_n(x) \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(x)$$

$(x \in G_m, n \in \mathbf{N}, f \in \mathbf{L}^1(G_m))$. For more details on Vilenkin systems see e.g. [1].

In 1983. B. Smith proved ([7]) for the trigonometric system the following convergence theorem

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} = \|f\|_1$$

for functions f in the "classical" Hardy space. In 1987. P. Simon proved [6] this theorem for the Walsh system. (The Walsh system is a Vilenkin system, $m_j = 2$ for all $j \in \mathbf{N}$ in this case.) In 1993. the author improved [3] this result, that is proved this theorem for the Vilenkin systems on bounded Vilenkin groups and in the case of the $H(G_m)$ "atomic" Hardy space — for unbounded ones, too. Does this theorem hold in the case of unbounded Vilenkin groups and the "maximal" Hardy space? (For unbounded Vilenkin groups $H \not\subseteq H^1$.) We give a negative answer for this question.

Theorem. If $\sup_n \frac{\log^2 m_n}{\log M_{n+1}} = \infty$, then there exists a function $f \in H^1(G_m)$, such that

$$\sup_n \frac{1}{\log n} \sum_{k=1}^{n} \|S_k f\|_1 / k = \infty.$$

PROOF. Let

$$f_k(x) := \begin{cases} M_{k+1}, & x \in I_k(0,1) \\ -M_{k+1}, & x \in I_k(0,\Delta_k) \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\Delta_k := \begin{cases} m_k/2, & 2|m_k \\ (m_k-1)/2, & 2\not| m_k \end{cases},$$

for $m_k \geq 4$. If $m_k < 4$, then $f_k := 0$. It is easy to see that $n \geq M_{k+1}$ implies $S_n f_k = f_k$ and $n < M_k$ implies $S_n f_k = 0$.

If $M_k \leq n < M_{k+1}$, then let $y \in I_k(0,l), l \neq 1, \Delta_k$ and $x \in I_k(0,1) \cup I_k(0,\Delta_k)$. Consequently, $y - x \in I_k \setminus I_{k+1}$,

$$D_n(y-x) = \left(\sum_{j=0}^{k-1} n_j M_j\right) r_k^{n_k}(y-x) + M_k \sum_{p=0}^{n_k-1} r_k^p(y-x),$$

$(\psi_n(y-x) = r_k^{n_k}(y-x))$ thus

$$\int\limits_{I_k(0,l)} |S_n f_k| \geq \frac{-1}{M_{k+1}} \left(\sum_{j=0}^{k-1} n_j M_j\right) \int\limits_{G_m} |f_k| + \frac{1}{m_k} \left| \int\limits_{G_m} f_k(x) \frac{r_k^{n_k}(y-x)-1}{r_k(y-x)-1} dx \right|$$

$$=: (1) + (2).$$

$|(1)| \leq \frac{c}{m_k}$ is trivial.

$$(2) = \frac{1}{m_k} \left| \frac{r_k^{n_k}((l-1)e_k)-1}{r_k((l-1)e_k)-1} - \frac{r_k^{n_k}((l-\Delta_k)e_k)-1}{r_k((l-\Delta_k)e_k)-1} \right|.$$

For $l \leq \left[\frac{m_k}{4}\right]$

$$\left| \frac{r_k^{n_k}((l-\Delta_k)e_k)-1}{r_k((l-\Delta_k)e_k)-1} \right| \leq \frac{2}{\sin \pi \frac{|l-\Delta_k|}{m_k}} \leq c.$$

That is,

$$(2) \geq \frac{1}{m_k} \frac{\left|\sin \pi \frac{n_k(l-1)}{m_k}\right|}{\left|\sin \pi \frac{l-1}{m_k}\right|} - c.$$

These estimations give

$$\|S_n f_k\|_1 \geq \sum_{l \neq 1, \Delta_k} \int_{I_k(0,l)} |S_n f_k| \geq \frac{c}{m_k} \sum_{l=2}^{[m_k/4]} \left(\frac{\left|\sin \pi \frac{n_k(l-1)}{m_k}\right|}{\left|\sin \pi \frac{l-1}{m_k}\right|} - c \right)$$

$$\geq c \log n_k - c.$$

This follows

$$\frac{1}{\log M_{k+1}} \sum_{n=M_k}^{M_{k+1}-1} \frac{\|S_n f_k\|_1}{n} \geq \frac{c}{\log M_{k+1}} \sum_{n=M_k}^{M_{k+1}-1} \frac{\log n_k}{n_k M_k}$$

$$= \frac{c}{\log M_{k+1}} \sum_{n_k=1}^{m_k-1} \frac{\log n_k}{n_k} \geq c\frac{\log^2 m_k}{\log M_{k+1}}.$$

Define the sequences of indices ν_i $i \in \mathbf{P}$ such that

$$\frac{\log^2 m_{v_k}}{\log M_{v_k}+1} > k^3 \quad (k \in \mathbf{P}).$$

Set $f := \sum_{i=1}^{\infty} \frac{1}{i^2} f_{\nu_i}$. Then $\|f\|_{H^1} \leq \sum_{i=1}^{\infty} \frac{1}{i^2}\|f_{\nu_i}\|_{H^1} \leq c$.

$$\frac{1}{\log M_{\nu_k+1}} \sum_{n=M_{\nu_k}}^{M_{\nu_k+1}-1} \frac{\|S_n f\|_1}{n} \geq \frac{1}{\log M_{\nu_k+1}} \sum_{n=M_{\nu_k}}^{M_{\nu_k+1}-1} \|S_n\left(\frac{1}{k^2} f_{\nu_k}\right)\|_1/n$$

$$-\frac{1}{\log M_{\nu_k+1}} \sum_{n=M_{\nu_k}}^{M_{\nu_k+1}-1} \|S_n(\sum_{i=0}^{k-1} \frac{1}{i^2} f_{\nu_i})\|_1/n$$

$$\geq ck - \frac{1}{\log M_{\nu_k+1}} \sum_{n=M_{\nu_k}}^{M_{\nu_k+1}-1} \left(\sum_{i=1}^{k-1} \frac{1}{i^2}\|f_{\nu_i}\|_1\right)/n = ck - c,$$

for all $k \in \mathbf{P}$. That is

$$\sup_n \frac{1}{\log n} \sum_{k=1}^{n} \|S_k f\|_1/k = \infty.$$

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