A note on the products of the terms of linear recurrences

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Abstract. For an integer $\nu>1$ let $G^{(i)}$ $(i=1,\dots,\nu)$ be linear recurrences defined by

$$G_n^{(i)}=A_1^{(i)}G_{n-1}^{(i)}+\dots+A_{k_i}^{(i)}G_{n-k_i} \quad (n\geq k_i).$$

In the paper we show that the equation

$$dG_{x_1}^{(1)}\cdots G_{x_\nu}^{(\nu)}=sw^q,$$

where d,s,w,q,x_i are positive integers satisfying some conditions, implies the inequality $q<q_0$ with some effectively computable constant q_0. This result generalizes some earlier results of Kiss, Pethő, Shorey and Stewart.

1. Introduction

Let $G^{(i)}=\{G_n^{(i)}\}_{n=0}^{\infty}$ $(i=1,2,\dots,\nu)$ be linear recurrences of order k_i $(k_i\geq 2)$ defined by

$$G_n^{(i)}=A_1^{(i)}G_{n-1}^{(i)}+\dots+A_{k_i}^{(i)}G_{n-k_i}^{(i)} \quad (n\geq k_i), \tag{1}$$

where the initial values $G_j^{(i)}$ $(j=0,1,\dots,k_i-1)$ and the coefficients $A_l^{(i)}$ $(l=1,2,\dots,k_i)$ of the sequences are rational integers. We suppose, that $A_{k_i}^{(i)}\neq 0$ and there is at least one non-zero initial value for any recurrences.

By $\alpha_1^{(i)}=\gamma_i,\alpha_2^{(i)},\dots,\alpha_{t_i}^{(i)}$ we denote the distinct roots of the characteristic polynomial

$$p_i(x)=x^{k_i}-A_1^{(i)}x^{k_i-1}-\dots-A_{k_i}^{(i)}$$

of the sequence $G^{(i)}$, and we assume that $t_i>1$ and $|\gamma_i|>|\alpha_j^{(i)}|$ for $j>1$. Consequently $|\gamma_i|>1$. Suppose that the multiplicity of the roots γ_i are 1. Then the terms of the sequences $G^{(i)}$ $(i=1,2,\dots,\nu)$ can be written in the form

$$G_n^{(i)}=a_i\gamma_i^n+p_2^{(i)}(n)\left(\alpha_2^{(i)}\right)^n+\dots+p_{t_i}^{(i)}(n)\left(\alpha_{t_i}^{(i)}\right)^n \quad (n\geq 0), \tag{2}$$

where $a_i \neq 0$ are fixed numbers and $p_j^{(i)}$ $(j = 1, 2 \dots, t_i)$ are polynomials of

$$\mathbf{Q}(\gamma_i, \alpha_2^{(i)}, \dots, \alpha_{t_i}^{(i)})[x]$$

(see e.g. [8]).

A. Pethő [4,5,6], T. N. Shorey and C. L. Stewart [7] showed that a sequence $G(= G^{(i)})$ does not contain q-th powers if q is large enough. Similar result was obtained by P. Kiss in [2]. In [3] we investigated the equation

$$G_x H_y = w^q \tag{3}$$

where G and H are linear recurrences satisfying some condititons, and showed that if x and y are not too far from each other then q is (effectively computable) upper bounded: $q < q_0$.

2. Theorem

Now we shall investigate the generalization of equation (3). Let $d \in \mathbf{Z}$ be a fixed non-zero rational integer, and let $p_1, \dots, p_t$ be given rational primes. Denote by S the set of all rational integers composed of $p_1, \dots, p_t$:

$$S = \{s \in \mathbf{Z} \colon s = \pm p_1^{e_1} \cdots p_t^{e_t},\ e_i \in \mathbf{N}\}\,. \tag{4}$$

In particular $1 \in S$ $(e_1 = \dots = e_t = 0)$. Let

$$\mathcal{G}(x_1, \dots, x_\nu) = G_{x_1}^{(1)} \dots G_{x_\nu}^{(\nu)} \tag{5}$$

be a function defined on the set $\mathbf{N}^\nu$. By the definitions of the sequences $G^{(i)}$'s $\mathcal{G}$ takes integer values. With a given d let us consider the equation

$$d\mathcal{G}(x_1, \dots, x_\nu) = sw^q$$

in positive integers $w > 1$, q, x_i $(i = 1, 2, \dots, \nu)$ and $s \in S$. We will show under some conditions for $\mathcal{G}$ that $q < q_0$ is also fulfilled if q satisfies the equation above. Exactly, using the Baker-method, we will prove the following

Theorem. *Let $\mathcal{G}(x_1, \dots, x_\nu)$ be the function defined in (5). Futher let $0 \neq d \in \mathbf{Z}$ be a fixed integer, and let δ be a real number with $0 < \delta < 1$. Assume that $G(x_1, \dots, x_\nu) \neq \prod_{i=1}^{\nu} a_i \gamma_i^{x_i}$ if $x_i > n_0$ $(i = 1, 2, \dots, \nu)$. Then the equation*

$$d\mathcal{G}(x_1, \dots, x_\nu) = sw^q \tag{6}$$

in positive integers $w > 1, q, x_1, \ldots, x_\nu$ *and* $s \in S$ *for which* $x_j > \delta \max_i\{x_i\}$ $(j = 1, 2, \ldots, \nu)$, *implies that* $q < q_0$, *where* q_0 *is an effectively computable number depending on* $n_0, \delta, G^{(1)}, \ldots, G^{(\nu)}$.

3. Lemmas

In the proof of our Theorem we need a result due to A. Baker [1].

Lemma 1. *Let* $\pi_1, \pi_2, \ldots, \pi_r$ *be non-zero algebraic numbers of heights not exceeding* $M_1, M_2, \ldots, M_r$ *respectively* $(M_r \geq 4)$. *Further let* $b_1, b_2, \ldots, b_{r-1}$ *be rational integers with absolute values at most* B *and let* b_r *be a non-zero rational integer with absolute value at most* B' $(B' \geq 3)$. *Suppose, that* $\sum_{i=1}^{r} b_i \log \pi_i \neq 0$. *Then there exists an effectively computable constant* $C = C(r, M_1, \ldots, M_{r-1}, \pi_1, \ldots, \pi_r)$ *such that*

$$\left|\sum_{i=1}^{r} b_i \log \pi_i\right| > e^{-C\left(\log M_r \log B' + \frac{B}{B'}\right)}, \tag{7}$$

where logarithms have their principal values.

We need the following auxiliary result.

Lemma 2. *Let* $c_1, \ldots, c_k$ *be positive real numbers and* $0 < \delta < 1$ *be an arbitrary real number. Further let* $x_1, \ldots, x_k$ *be natural numbers with maximum value* $x_m = \max_i\{x_i\}$ $(m \in \{1, \ldots, k\})$. *If* $x_j > \delta x_m$ $(j = 1, \ldots, k)$ *and* $x_m > x_0$ *then there exists a real number* $c > 0$, *which depends on* $k, \delta, \max_i\{c_i\}$ *and* x_0, *for which*

$$\sum_{i=1}^{k} e^{-c_i x_i} < e^{-c(x_1 + \cdots + x_k)} = e^{-cx}, \tag{8}$$

where $x = x_1 + \cdots + x_k$.

Proof of Lemma 2. Using the conditions of the lemma we have

$$\sum_{i=1}^{k} e^{-c_i x_i} < \sum_{i=1}^{k} e^{-c_i \delta x_m} = \sum_{i=1}^{k} e^{-d_i x_m},$$

where $d_i = \delta c_i$. If $d_m = \min_i\{d_i\}$ then

$$\sum_{i=1}^{k} e^{-d_i x_m} \leq k e^{-d_m x_m} = e^{\log k - d_m x_m}.$$

Since $x_m \geq x_0$, it follows that

$$e^{\log k - d_m x_m} \leq e^{-d_m^* x_m} = e^{-ckx_m} \leq e^{-cx}$$

with a suitable constant d_m^* and $c = \frac{d_m^*}{k}$.

4. Proof of the Theorem

By $c_1, c_2, \ldots$ we denote positive real numbers which are effectively computable. We may assert, without loss of generality, that the terms of the recurrences $G^{(i)}$ are positive, $d > 0$, $s > 0$ and the inequality

$$|\gamma_1| \geq |\gamma_2| \geq \cdots \geq |\gamma_\nu| \tag{9}$$

also holds.

Let us observe that it is sufficient to consider the case $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$. Otherwise, if we suppose that some $x_j \leq n_0$ $(j \in \{1, 2, \ldots, \nu\})$ then $x_m = \max_i\{x_i\}$ cannot be arbitrary large because of the assertion $x_j > \delta x_m$. It means that we have finitely many possibilities to choose the ν-tuples $(x_1, \ldots, x_\nu)$, and the range of $\mathcal{G}(x_1, \ldots, x_\nu)$ is finite. So with a fixed d, if inequality (6) is satisfied then q must be bounded.

In the sequel we suppose that $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$. Let $x_1, \ldots, x_\nu$, w, q and $s \in S$ be integers satisfying (6). We may assume that if

$$s = p_1^{e_1} \cdots p_t^{e_t} \tag{10}$$

then $e_j < q$, else a part of s can be joined to w^q. Using (2), from (6) we have

$$sw^q = d \prod_{i=1}^{\nu} a_i (\gamma_i)^{x_i} \left(1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \cdots\right). \tag{11}$$

A consequence of the assumptions $|\gamma_i| > |\alpha_j^{(i)}|$ $(1 < j \leq t_i)$ is that

$$\left(1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \cdots\right) \longrightarrow 1 \quad \text{whenever} \quad x_i \longrightarrow \infty. \tag{12}$$

Hence there exist real constants $0 < \varepsilon_1, \ldots, \varepsilon_\nu < 1$ such that

$$d \prod_{i=1}^{\nu} |a_i||\gamma_i|^{x_i}(1 - \varepsilon_i) < sw^q < d \prod_{i=1}^{\nu} |a_i||\gamma_i|^{x_i}(1 + \varepsilon_i),$$

and

$$c_1 \prod_{i=1}^{\nu} |\gamma_i|^{x_i} < sw^q < c_2 \prod_{i=1}^{\nu} |\gamma_i|^{x_i}.$$

As before, let $x = x_1 + \cdots + x_\nu$ and applying (9) we may write

$$\log c_1 + x \log |\gamma_\nu| < \log s + q \log w < \log c_2 + x \log |\gamma_1|.$$

Since $\log s \geq 0$, we have

$$\log c_3 + x \log |\gamma_\nu| < q \log w < \log c_2 + x \log |\gamma_1| \tag{13}$$

with $c_3 = \frac{c_1}{s}$. From (13) it follows that

$$c_4 \frac{x}{q} < \log w < c_5 \frac{x}{q} \tag{14}$$

with some positive constants c_4, c_5. Ordering the equality (11) and taking logarithms, by the definition of ε_i we obtain

$$Q = \left| \log \frac{sw^q}{d \prod_{i=1}^{\nu} |a_i| |\gamma_i|^{x_i}} \right| = \left| \log \prod_{i=1}^{\nu} \left| 1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i} \right)^{x_i} + \cdots \right| \right| <$$

$$< \sum_{i=1}^{\nu} \log |1 + \varepsilon_i| \leq \sum_{i=1}^{\nu} e^{-c_i^\star x_i},$$

where $Q \neq 0$ if we assume, that $x_i > n_0$ for every $i = 1, 2, \ldots, \nu$, and $c_i^\star$ is a suitable positive constant $(i = 1, 2, \ldots, \nu)$. Applying Lemma 2 and using the notation $x = x_1 + \cdots + x_\nu$, it yields that

$$Q < e^{-c_6(x_1 + \cdots + x_\nu)} = e^{-c_6 x}. \tag{15}$$

On the other hand

$$Q = \left| \log s + q \log w - \log d - \log \prod_{i=1}^{\nu} |a_i| - x_1 \log |\gamma_1| - \cdots - x_\nu \log |\gamma_\nu| \right|, \tag{16}$$

where $\log s = e_1 \log p_1 + \cdots + e_t \log p_t$ (see (10)). Now we may use Lemma 1 with $\pi_r = w = M_r$, since the ordinary heights of p_j $(j = 1, 2, \ldots, t)$, d, $\prod_{i=1}^{\nu} |a_i|$ and $|\gamma_i|$ $(i = 1, 2, \ldots, \nu)$ are constants. So $B' = q$. In comparison the absolute values of the integer coefficients of the logarithms in (16), we can choose B as $B = x$. So by (16) and Lemma 1 it follows that

$$Q > e^{-c_7\left(\log w \log q + \frac{x}{q}\right)}. \tag{17}$$

Combining (15) and (17) it yields the following inequality:

$$c_6 x < c_7 \left(\log w \log q + \frac{x}{q}\right), \tag{18}$$

and by (14) it follows that

$$c_6 x < c_7 \left(\log w \log q + \frac{1}{c_4} \log w\right) < c_8 \log w \log q \tag{19}$$

with some $c_8 > 0$. Applying (14) again, we conclude that $\frac{1}{c_5} q \log w < x$ and so by (19)

$$c_9 q < \log q \tag{20}$$

follows. But (20) implies that $q < q_0$, which proves the theorem.

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