Pure powers in recurrence sequences

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Abstract. Let G be a linear recursive sequence of order k satisfying the recursion $G_n=A_1G_{n-1}+\cdots+A_kG_{n-k}$. In the case $k=2$ it is known that there are only finitely many perfect powers in such a sequence.

Ribenboim and McDaniel proved for sequences with $k=2$, $G_0=0$ and $G_1=1$ that in general for a term G_n there are only finitely many terms G_m such that G_nG_m is a perfect square. P. Kiss proved that for any n there exists a number q_0, depending on G and n, such that the equation $G_nG_x=w^q$ in positive integers x,w,q has no solution with $x>n$ and $q>q_0$. We show that for any n there are only finitely many $x_1,x_2,\ldots,x_k,x,w,q$ positive integers such that $G_nG_{x_1}\cdots G_{x_k}G_x=w^q$ and some conditions hold.

Let $R = R(A,B,R_0,R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1),$$

where A, B, R_0 and R_1 are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. α/β is not a root of unity, where α and β denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1,1,0,1)$ and $R(2,1,0,1)$ of the sequence R is called Fibonacci and Pell sequence, respectively.

Many results are known about relationship of the sequences R and perfect powers. For the Fibonacci sequence Cohn [2] and Wylie [23] showed that a Fibonacci number F_n is a square only when $n = 0,1,2$ or 12. Pethő [12], furthermore London and Finkelstein [9,10] proved that F_n is full cube only if $n = 0,1,2$ or 6. From a result of Ljunggren [8] it follows that a Pell number is a square only if $n = 0,1$ or 7 and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results was showed by McDaniel and Ribenboim [11], Robbins [19,20] Cohn [3,4,5] and Pethő [15]. Shorey and Stewart [21] showed, that any non degenerate binary recurrence sequence contains only finitely many perfect powers which can be effictively determined. This results follows also from a result of Pethő [14].

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Another type of problems was studied by Ribenboim and McDaniel. For a sequence R we say that the terms R_m, R_n are in the same square-class if there exist non zero integers x, y such that

$$R_m x^2 = R_n y^2,$$

or equivalently

$$R_m R_n = t^2,$$

where t is a positive rational integer.

A square-class is called trivial if it contains only one element. Ribenboim [16] proved that in the Fibonacci sequence the square-class of a Fibonacci number F_m is trivial, if $m \neq 1, 2, 3, 6$ or 12 and for the Lucas sequence $L(1,1,2,1)$ the square-class of a Lucas number L_m is trivial if $m \neq 0, 1, 3$ or 6. For more general sequences $R(A, B, 0, 1)$, with $(A, B) = 1$, Ribenboim and McDaniel [17] obtained that each square class is finite and its elements can be effectively computed (see also Ribenboim [18]).

Further on we shall study more general recursive sequences.

Let $G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ be a k^{th} order linear recursive sequence of rational integers defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \qquad (n > k-1),$$

where $A_1, \ldots, A_k$ and $G_0, \ldots, G_{k-1}$ are not all zero integers. Denote by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ the distinct zeros of the polynomial $x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k$. Assume that $\alpha, \alpha_2, \ldots, \alpha_s$ has multiplicity $1, m_2, \ldots, m_s$ respectively and $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, s$. In this case, as it is known, the terms of the sequence can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \cdots + r_s(n)\alpha_s^n \qquad (n \geq 0), \tag{1}$$

where $r_i (i = 2, \ldots, s)$ are polynomials of degree $m_i - 1$ and the coefficients of the polynomials and a are elements of the algebraic number field $\mathbf{Q}(\alpha, \alpha_2, \ldots, \alpha_s)$. Shorey and Stewart [21] prowed that the sequence G does not contain q^{th} powers if q is large enough. This result follows also from [7] and [22], where more general theorems where showed.

Kiss [6] generalized the square-class notion of Ribenboim and McDaniel. For a sequence G we say that the terms G_m and G_n are in the same q^{th}-power class if $G_m G_n = w^q$, where w, q rational integers and $q \geq 2$.

In the above mentioned paper Kiss proved that for any term G_n of the sequence G there is no terms G_m such that $m > n$ and G_n, G_m are elements of the same q^{th}-power class if q sufficiently large.

The purpose of this paper to generalize this result. We show that the under certain conditions the number of the solutions of equation

$$G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q$$

where n is fixed, are finite.

We use a well known result of Baker [1].

Lemma. *Let $\gamma_1, \ldots, \gamma_v$ be non-zero algebraic numbers. Let $M_1, \ldots, M_v$ be upper bounds for the heights of $\gamma_1, \ldots, \gamma_v$, respectively. We assume that M_v is at least 4. Further let $b_1, \ldots, b_{v-1}$ be rational integers with absolute values at most B and let b_v be a non-zero rational integer with absolute value at most B'. We assume that B' is at least three. Let L defined by*

$$L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v,$$

where the logarithms are assumed to have their principal values. If $L \neq 0$, then

$$|L| > \exp(-C(\log B' \log M_v + B/B')),$$

where C is an effectively computable positive number depending on only the numbers $M_1, \ldots, M_{v-1}$, $\gamma_1, \ldots, \gamma_v$ and v (see Theorem 1 of [1] with $\delta = 1/B'$).

Theorem. *Let G be a k^{th} order linear recursive sequence satisfying the above conditions. Assume that $a \neq 0$ and $G_i \neq a\alpha^i$ for $i > n_0$. Then for any positive integer n, k and K there exists a number q_0, depending on n, G, K and k, such that the equation*

$$(2) \qquad G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q \qquad (n \leq x_1 \leq \cdots \leq x_k < x)$$

in positive integer $x_1, x_2, \ldots, x_k, x, w, q$ has no solution with $x_k < Kn$ and $q > q_0$.

Proof of the theorem. We can assume, without loss of generality, that the terms of the sequence G are positive. We can also suppose that $n > n_0$ and n sufficiently large since otherwise our result follows from [20] and [7].

Let $x_1, x_2, \ldots, x_k, x, w, q$ positive integers satisfying (2) with the above conditions. Let ε_m be defined by

$$\varepsilon_m := \frac{1}{a} r_2(m)\Big(\frac{\alpha_2}{\alpha}\Big)^m + \frac{1}{a} r_3(m)\Big(\frac{\alpha_3}{\alpha}\Big)^m + \cdots + \frac{1}{a} r_s(m)\Big(\frac{\alpha_s}{\alpha}\Big)^m \qquad (m \geq 0).$$

By (1) we have

$$(1+\varepsilon_n)(1+\varepsilon_x)\prod_{i=1}^{k}(1+\varepsilon_{x_i})\,a^{k+2}\alpha^{n+x+x_1+\cdots+x_k}=w^q$$

from which

$$\begin{aligned} q\log w &= (k+2)\log a + \left(n+x+\sum_{i=1}^{k}x_i\right)\log\alpha + \log(1+\varepsilon_n) \\ &\quad + \log(1+\varepsilon_x) + \sum_{i=1}^{k}\log(1+\varepsilon_{x_i}) \end{aligned} \tag{3}$$

follows. It is obvious that $x < n+x+\sum_{i=1}^{k} x_i < (k+2)x$. Using that $\log|1+\varepsilon_m|$ is bounded and $\lim_{m\to\infty}\frac{1}{a}r_i(m)\left(\frac{\alpha_i}{\alpha}\right)^m = 0 \quad (i=2,\ldots,s)$, we have

$$c_1\frac{x}{q} < \log w < c_2\frac{x}{q} \tag{4}$$

where c_1 and c_2 are constants.

Let L be defined by

$$L := \left|\log\frac{w^q}{G_nG_{x_1}G_{x_2}\cdots G_{x_k}a\alpha^x}\right| = \left|\log(1+\varepsilon_x)\right|.$$

By the definition of ε_x and the properties of logarithm function there exists a constant c_3 that

$$L < e^{-c_3x}. \tag{5}$$

On the other hand, by the Lemma with $v = k+4, M_{k+4} = w, B' = q$ and $B = x$ we obtain the estimation

$$L = \left|q\log w - \log G_n - \sum_{i=1}^{k}\log G_{x_i} - \log a - x\log\alpha\right| > e^{-C(\log q\log w + x/q)} \tag{6}$$

where C depends on heights. By $x_k < Kn$ heights depend on $G_n,\ldots,G_{Kn}$, i.e. on n, K, k and on the parameters of the recurrence. By (4), (5) and (6) we have $c_3x < C(\log q\log w + x/q) < c_4\log q\log w$, i.e.

$$x < c_5\log q\log w \tag{7}$$

with some c_3, c_4, c_5. Using (4) and (7) we get $c_6 q \log w < x < c_5 \log q \log w$, i.e. $q < c_7 \log q$, where c_6 and c_7 are constants. But this inequality does not hold if $q > q_0 = q_0(G, n, K, k)$, which proves the theorem.

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