On a class of differential equations connected with number-theoretic polynomials

KRYSTYNA GRYTCZUK

Abstract. In this paper we consider the special class of differential equations of second order. For this class we find a general solution which is strictly connected with some number-theoretic polynomials such as Dickson, Chebyschev, Pell and Fibonacci.

1. Introduction

Consider the following class of the polynomials:

$$W_n(x,c) = \left(\frac{x+\sqrt{x^2+c}}{2}\right)^n + \left(\frac{x-\sqrt{x^2+c}}{2}\right)^n \tag{1}$$

with respect to c, where $n \geq 1$ is the degree of the polynomial $W_n(x,c)$. It is known (see[2], p. 94) that the Dickson polynomial $D_n(x,a)$ of degree $n \geq 1$ and integer parameter a can be represent in the form:

$$D_n(x,a) = \left(\frac{x+\sqrt{x^2-4a}}{2}\right)^n + \left(\frac{x-\sqrt{x^2-4a}}{2}\right)^n. \tag{D}$$

We note that the Dickson polynomial belongs to class (1) if we take $c = -4a$. Taking $c = -1$ in (1) we obtain the Chebyschev polynomial of the second kind. For $c = 1$ we get the Pell polynomial and for $c = 4$ the Fibonacci polynomial.

We prove the following:

Theorem. The general solution of the differential equation

$$\left(x^2+c\right)y'' + xy' - n^2y = 0; \quad x^2+c > 0 \tag{*}$$

is of the form

$$y = C_1\left(\frac{x+\sqrt{x^2+c}}{2}\right)^n + C_2\left(\frac{x-\sqrt{x^2+c}}{2}\right)^n, \tag{**}$$

where C_1, C_2 are arbitrary constants.

We remark that the general solution (**) is strictly connected with the polynomials $W_n(x,c)$ defined by (1).

2. Basic Lemmas

Lemma 1. (see [1], Thm. 2.) Let the real-valued functions $s_0, t_0 u, v \in C^2(J)$, where $J \subset \mathbf{R}$ and $u \neq 0, v \neq 0$. Then the functions

$$y_1 = s_0 u^\lambda, \quad y_2 = t_0 v^\lambda, \tag{2}$$

where λ is non-zero real constant, are the particular solutions of the differential equation

$$D_0 y'' + D_1 y' + D_2 y = 0, \tag{3}$$

where

$$D_0 = \det\begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, \quad D_1 = \det\begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, \quad D_2 = \det\begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} \tag{4}$$

and

$$s_1 = s_0' + \lambda s_0 \frac{u'}{u}, \quad t_1 = t_0' + \lambda t_0 \frac{v'}{v} \tag{5}$$

$$s_2 = s_1' + \lambda s_1 \frac{u'}{u}, \quad t_2 = t_1' + \lambda t_1 \frac{v'}{v}\,. \tag{6}$$

Lemma 2. Let λ, s_0, t_0 be non-zero real constants and let non-zero real functions $u, v \in C^2(J)$, $J \subset \mathbf{R}$ be linearly independent over the real number field $\mathbf{R}$. Then the general soltution of the differential equation:

$$\det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} y'' + \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} y' + \lambda \det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} y = 0, \tag{$***$}$$

where

$$g = \frac{u''}{u} - (1-\lambda)\left(\frac{u'}{u}\right)^2, \quad h = \frac{v''}{v} - (1-\lambda)\left(\frac{v'}{v}\right)^2 \tag{7}$$

is of the form

$$y = C_1 s_0 u^\lambda + C_2 t_0 v^\lambda, \tag{8}$$

where C_1, C_2 are arbitrary constants.

Proof. By the assumptions of Lemma 1 and Lemma 2 it follows that

$$s_1 = \lambda s_0 \frac{u'}{u}, \quad t_1 = \lambda t_0 \frac{v'}{v}. \tag{9}$$

From (9) and (6) we obtain

$$s_2 = s_1' + \lambda s_1 \frac{u'}{u} = \lambda s_0 \left(\frac{u''}{u} - (1-\lambda) \left(\frac{u'}{u} \right)^2 \right) \tag{10}$$

and

$$t_2 = t_1' + \lambda t_1 \frac{v'}{v} = \lambda t_0 \left(\frac{v''}{v} - (1-\lambda) \left(\frac{v'}{v} \right)^2 \right). \tag{11}$$

Let us denote by $g = \frac{u''}{u} - (1-\lambda)\left(\frac{u'}{u}\right)^2$ and by $h = \frac{v''}{v} - (1-\lambda)\left(\frac{v'}{v}\right)^2$. Then the formulae (10) and (11) have the form:

$$s_2 = \lambda s_0 g, \quad t_2 = \lambda t_0 h. \tag{12}$$

By (12), (9) and Lemma 1 it follows that the differential equation (3) reduce to $(***)$. On the other hand from Lemma 1 it follows that the functions $y_1 = s_0 u^\lambda$ and $y_2 = t_0 v^\lambda$ are the particular solutions of $(***)$. Now we observe that the functions u, v are linearly independent over $\mathbf{R}$ if and only if the functions u^λ and v^λ are linearly independent over $\mathbf{R}$. Indeed, denote by $W(u^\lambda, v^\lambda)$ the Wronskian of the functions u^λ and v^λ and let

$$D_0 = \det \begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix}.$$

Then we have

$$D_0 = (uv)^{-1} \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}, \tag{13}$$

and

$$W\left(u^\lambda, v^\lambda\right) = \det \begin{pmatrix} u^\lambda & v^\lambda \\ \left(u^\lambda\right)' & \left(v^\lambda\right)' \end{pmatrix} = \lambda (uv)^\lambda \det \begin{pmatrix} 1 & 1 \\ \frac{u'}{u} & \frac{v'}{v} \end{pmatrix}. \tag{14}$$

Since $\det\begin{pmatrix} 1 & 1 \\ \frac{u'}{u} & \frac{v'}{v} \end{pmatrix} = \det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix}$, from the definition of D_0, (13) and (14) we get

$$W\left(u^{\lambda}, v^{\lambda}\right) = \lambda(uv)^{\lambda} D_0 = \lambda(uv)^{\lambda-1} \det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}. \tag{15}$$

From (15) easily follows that the functions u^{λ}, v^{λ} are linearly independent over $\mathbf{R}$ if and only if the functions u, v have the same property. Using the assumption of Lemma 2 about the functions u, v we obtain that the functions u^{λ}, v^{λ} and also $y_1 = s_0 u^{\lambda}, y_2 = t_0 v^{\lambda}$ are linearly independent over $\mathbf{R}$. Since the functions y_1, y_2 are the particular solutions of $(***)$, the function $y = C_1 y_1 + C_2 y_2 = C_1 s_0 u^{\lambda} + C_2 t_0 v^{\lambda}$ is a general solution of $(***)$. The proof of Lemma 2 is complete.

3. Proof of the Theorem

Let $\lambda = n$ be natural number and let $s_0 = t_0 = 1$. Moreover, let $u = a(x)+b(x)\sqrt{k}$ and $v = a(x)-b(x)\sqrt{k}$, where k is fixed non-zero constant. If the functions u, v are linearly independent over $\mathbf{R}$ then by Lemma 2 it follows that the general solution of the differential equation

$$\det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} y'' + \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} y' + n \det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} y = 0 \tag{16}$$

is of the form

$$y = C_1 \left(a(x) + b(x)\sqrt{k}\right)^n + C_2 \left(a(x) - b(x)\sqrt{k}\right)^n, \tag{17}$$

where $g = \frac{u''}{u} - (1-n)\left(\frac{u'}{u}\right)^2$ and $h = \frac{v''}{v} - (1-n)\left(\frac{v'}{v}\right)^2$ and C_1, C_2 are arbitrary constants. Now, we put $a(x) = \frac{x}{2}$, $b(x) = \frac{\sqrt{x^2+c}}{2}$, $k = 1$, where $x^2 + c > 0$. Then we have

$$u = \frac{x + \sqrt{x^2+c}}{2}, \quad v = \frac{x - \sqrt{x^2+c}}{2}. \tag{18}$$

From (18) we obtain

$$u' = \frac{1}{2}\left(\frac{x + \sqrt{x^2+c}}{\sqrt{x^2+c}}\right), \quad v' = -\frac{1}{2}\left(\frac{x - \sqrt{x^2+c}}{\sqrt{x^2+c}}\right). \tag{19}$$

By (18) and (19) easily follows that the functions u, v are linearly independent over $\mathbf{R}$, because the Wronskian $W(u,v) \neq 0$. On the other hand from (19) we obtain

$$u'' = \frac{1}{2}\frac{c}{(x^2+c)\sqrt{x^2+c}}, \quad v'' = -\frac{1}{2}\frac{c}{(x^2+c)\sqrt{x^2+c}}. \tag{20}$$

From (19) and (18) we get

$$\frac{u'}{u} = \frac{1}{\sqrt{x^2+c}}, \quad \frac{v'}{v} = -\frac{1}{\sqrt{x^2+c}}, \tag{21}$$

hence by (21) it follows that

$$\left(\frac{u'}{u}\right)^2 = \left(\frac{v'}{v}\right)^2 = \frac{1}{x^2+c}. \tag{22}$$

Simlarly from (20) and (18) we obtain

$$\begin{aligned} \frac{u''}{u} &= \frac{c}{(x^2+c)\left(x+\sqrt{x^2+c}\right)\sqrt{x^2+c}}, \\ \frac{v''}{v} &= -\frac{c}{(x^2+c)\left(x-\sqrt{x^2+c}\right)\sqrt{x^2+c}}. \end{aligned} \tag{23}$$

From (21) we calculate that

$$D_0 = \det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} = \frac{v'}{v} - \frac{u'}{u} = -\frac{2}{\sqrt{x^2+c}}. \tag{24}$$

In similar way from (22) and (23) we get

$$D_1 = \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} = g - h = -\frac{2x}{(x^2+c)\sqrt{x^2+c}}. \tag{25}$$

On the other hand by (21) and (23) it follows that

$$D_2 = \det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} = h\frac{u'}{u} - g\frac{v'}{v} = \frac{2n}{(x^2+c)\sqrt{x^2+c}}. \tag{26}$$

Now, we see that from (24), (25) and (26) the differential equation (16) has the following form:

$$(27)\qquad \left(x^2+c\right)y''+xy'-n^2y=0,$$

so denote that (27) is the same equation as in our Theorem. Thus, by Lemma 2 it follows that the general solution of (27) is given by the formula

$$y=C_1\left(\frac{x+\sqrt{x^2+c}}{2}\right)^n+C_2\left(\frac{x-\sqrt{x^2+c}}{2}\right)^n$$

and the proof of the Theorem is complete.

Remark. Consider the following functional matrix;

$$M(x)=\frac{1}{2}\begin{pmatrix} x & \sqrt{x^2+c} \\ \sqrt{x^2+c} & x \end{pmatrix}.$$

Then we can calculate that the functions $u=\frac{x+\sqrt{x^2+c}}{2}$ and $v=\frac{x-\sqrt{x^2+c}}{2}$ are the characteristic roots of this matrix. Hence, we observe that the general solution of the differential equation (16) is linear combination of the powers such roots.

References

[1] A. Grytczuk and K. Grytczuk, *Functional recurrences, Applications of Fibonacci Numbers*, Ed. by G. E. Bergum et al., Kluwer Acad. Publ., Dordrecht, 1990, 115–121.

[2] P. Moree and G. L. Mullen, Diskson polynomial discriminators, *J. Number Theory*, **59** (1996), 88–105.

Institute of Mathematics
Technical University
Zielona Góra, ul. Podgórna 50
Poland