

# Functions having quadratic differences in a given class

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**Abstract.** Starting from a problem of Z. Daróczy we define the quadratic difference property and show that the class of all real-valued continuous functions on  $\mathbf{R}$  and some of its subclasses have this property while the class of all bounded functions does not have.

## 1. Introduction

For a function  $f: \mathbf{R} \rightarrow \mathbf{R}$  (the reals) and for a fixed  $y \in \mathbf{R}$  define the function  $\Delta_y f$  on  $\mathbf{R}$  by  $\Delta_y f(x) = f(x + y) - f(x)$ ,  $x \in \mathbf{R}$ . The functions  $A, N: \mathbf{R} \rightarrow \mathbf{R}$  are said to be additive and quadratic if

$$A(x + y) = A(x) + A(y) \quad x, y \in \mathbf{R}$$

and

$$N(x + y) + N(x - y) = 2N(x) + 2N(y) \quad x, y \in \mathbf{R},$$

respectively. It is well-known (see [1], [5], [2]) that, if an additive function is bounded from one side on an interval of positive length then  $A(x) = cx$ ,  $x \in \mathbf{R}$  for some  $c \in \mathbf{R}$  and there are discontinuous additive functions. Similarly, if a quadratic function is bounded on an interval of positive length then  $N(x) = dx^2$ ,  $x \in \mathbf{R}$  for some  $d \in \mathbf{R}$  and there are discontinuous quadratic functions.

In [4] Z. DARÓCZY asked that for which properties  $T$  the following statement is true:

(\*) Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function such that for all fixed  $y \in \mathbf{R}$  the function  $\Delta_y \Delta_{-y} f$  has the property  $T$ . Then

$$(1) \quad f = f^* + N + A \quad \text{on } \mathbf{R}$$

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where  $f^*$  has the property  $T$ ,  $N$  is a quadratic function and  $A$  is an additive function.

In this note we prove that, if  $T$  is the  $k$ -times continuously differentiability ( $k \geq 0$  integer) or  $k$ -times differentiability ( $k > 0$  integer or  $k = +\infty$ ) or being polynomial then the statement  $(*)$  is true while if  $T$  is the boundedness then  $(*)$  is not true.

## 2. Preliminary results

The following lemma will play an important role in our investigations.

**Lemma 1.** *For all functions  $f: \mathbf{R} \rightarrow \mathbf{R}$  and for all  $u, v, x \in \mathbf{R}$  we have*

$$(2) \quad \begin{aligned} \Delta_u \Delta_v f(x) &= \Delta_{\frac{u-v}{2}} \Delta_{-\frac{u-v}{2}} f \left( x + \frac{u+v}{2} \right) \\ &\quad - \Delta_{\frac{u+v}{2}} \Delta_{-\frac{u+v}{2}} f \left( x + \frac{u+v}{2} \right). \end{aligned}$$

The proof is a simple computation therefore it is omitted.

Another basic tool we will use is the following result of DE BRUIJN ([3] Theorem 1.1.)

**Theorem 1.** *Suppose that  $f: \mathbf{R} \rightarrow \mathbf{R}$  is a function such that the function  $\Delta_y f$  is continuous for all fixed  $y \in \mathbf{R}$ . Then  $f = f^* + A$  on  $\mathbf{R}$  with some continuous  $f^*: \mathbf{R} \rightarrow \mathbf{R}$  and additive  $A: \mathbf{R} \rightarrow \mathbf{R}$ .*

Finally we will need the following two lemmata.

**Lemma 2.** *Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function such that  $\Delta_u \Delta_v f$  is continuous for all fixed  $u, v \in \mathbf{R}$ . Define*

$$(3) \quad H(x, u, v) = \Delta_u \Delta_v f(x) - f(u+v) + f(u) + f(v) \quad x, u, v \in \mathbf{R}.$$

*Then the function  $(x, u) \rightarrow H(x, u, v)$ ,  $(x, u) \in \mathbf{R}^2$  is continuous for all fixed  $v \in \mathbf{R}$ .*

**Proof.** Let  $v \in \mathbf{R}$  be fixed. Since  $\Delta_u(\Delta_v f)$  is continuous for all fixed  $u \in \mathbf{R}$ , Theorem 1 implies that  $\Delta_v f = f_v^* + A_v$  on  $\mathbf{R}$  where  $f_v^*: \mathbf{R} \rightarrow \mathbf{R}$  is continuous and  $A_v$  is additive. Thus, by (3),

$$\begin{aligned} H(x, u, v) &= \Delta_v f(x+u) - \Delta_v f(x) - \Delta_v f(u) + f(v) \\ &= f_v^*(x+u) - f_v^*(x) - f_v^*(u) + f(v) \end{aligned}$$

whence the continuity of  $(x, u) \rightarrow H(x, u, v)$ ,  $(x, u) \in \mathbf{R}^2$  follows.

**Lemma 3.** Suppose that  $L$  is one of the classes of the real-valued functions defined on  $\mathbf{R}$  which are  $k$ -times continuously differentiable for some  $k \geq 0$  integer or  $k$ -times differentiable for some  $1 \leq k \leq +\infty$  or polynomials. If  $f: \mathbf{R} \rightarrow \mathbf{R}$  is continuous and  $\Delta_y f \in L$  for all fixed  $y \in \mathbf{R}$  then  $f \in L$ .

**Proof.** If  $L$  is the class of the continuous functions ( $k = 0$ ) or of the polynomials, furthermore  $f$  is continuous and  $\Delta_y f \in L$  for all fixed  $y \in \mathbf{R}$  then, by Theorem 1 and by [3] page 203, respectively,  $f = f^* + A$  for some  $f^* \in L$  and additive function  $A$ . Therefore, by continuity of  $f$ ,  $A(x) = cx$ ,  $x \in \mathbf{R}$  with some  $c \in \mathbf{R}$  whence  $f \in L$  follows.

The remaining statement of Lemma 3 is just Lemma 3.1. in [3].

### 3. The main results

For the formulation of our main results let us begin with the following

**Definition.** A subset  $E$  of the set of all functions  $f: \mathbf{R} \rightarrow \mathbf{R}$  is said to have the quadratic difference property if for all  $f: \mathbf{R} \rightarrow \mathbf{R}$ , with  $\Delta_y \Delta_{-y} f \in E$  for all  $y \in \mathbf{R}$ , the decomposition (1) holds true on  $\mathbf{R}$  where  $f^* \in E$ ,  $N$  is a quadratic function and  $A$  is an additive function.

First we prove the following

**Theorem 2.** The class of all continuous functions  $f: \mathbf{R} \rightarrow \mathbf{R}$  has the quadratic difference property.

**Proof.** By (2) in Lemma 1 we have that  $\Delta_u \Delta_v f$  is continuous for all fixed  $u, v \in \mathbf{R}$ . In particular,  $\Delta_u(\Delta_1 f)$  is continuous for all fixed  $u \in \mathbf{R}$ . Applying Theorem 1 to  $\Delta_1 f$  we have

$$(4) \quad \Delta_1 f = f_0 + a \quad \text{on } \mathbf{R}$$

with some continuous  $f_0: \mathbf{R} \rightarrow \mathbf{R}$  and additive  $a: \mathbf{R} \rightarrow \mathbf{R}$ . Define the function  $B$  on  $\mathbf{R}^2$  by

$$(5) \quad B(u, v) = \int_0^1 \Delta_u \Delta_v f - \int_0^{u+v} f_0 + \int_0^u f_0 + \int_0^v f_0, \quad (u, v) \in \mathbf{R}^2.$$

Obviously,  $B$  is symmetric. Now we show that  $B$  is additive in its first variable. For all  $u, t$  and  $v$ , we have

$$B(u+t, v) - B(u, v) = \int_0^1 \Delta_{u+t} \Delta_v f - \int_0^{u+t+v} f_0 + \int_0^{u+t} f_0 + \int_0^v f_0$$

$$\begin{aligned}
& - \int_0^1 \Delta_u \Delta_v f + \int_0^{u+v} f_0 - \int_0^u f_0 - \int_0^v f_0 \\
& = \int_u^{u+1} \Delta_t \Delta_v f - \int_0^{u+t+v} f_0 + \int_0^{u+t} f_0 + \int_0^{u+v} f_0 - \int_0^u f_0.
\end{aligned}$$

Since  $\Delta_t \Delta_v f$  and  $f_0$  are continuous functions, the right hand side is continuously differentiable with respect to  $u$  then so is the left hand side. Differentiating both sides with respect to  $u$  and taking into consideration (4) we obtain that

$$\begin{aligned}
\frac{\partial}{\partial u} [B(u+t, v) - B(u, v)] &= \Delta_t \Delta_v \Delta_1 f(u) - f_0(u+t+v) + f_0(u+t) \\
&\quad + f_0(u+v) - f_0(u) \\
&= \Delta_t \Delta_v (f_0 + a)(u) - \Delta_t \Delta_v f_0(u) \\
&= \Delta_t \Delta_v a(u) = 0 \quad (a \text{ being additive}).
\end{aligned}$$

Therefore

$$B(u+t, v) - B(u, v) = B(t, v) - B(0, v) = B(t, v),$$

that is,  $B$  is additive in its first (and by the symmetry also in its second) variable. Thus, it is well-known (see [2]) and easy to see that, the function  $N: \mathbf{R} \rightarrow \mathbf{R}$  defined by  $N(u) = \frac{1}{2} B(u, u)$ ,  $u \in \mathbf{R}$  is quadratic and

$$(6) \quad B(u, v) = N(u+v) - N(u) - N(v) \quad u, v \in \mathbf{R}.$$

Define the function  $H: \mathbf{R}^3 \rightarrow \mathbf{R}$  by (3) and apply Lemma 2 to get the continuity of the function  $(x, u) \mapsto H(x, u, v)$ ,  $(x, u) \in \mathbf{R}^2$  for all fixed  $v \in \mathbf{R}$ . This implies that the function  $s: \mathbf{R}^2 \rightarrow \mathbf{R}$  defined by

$$s(u, v) = \int_0^1 H(x, u, v) dx \quad (u, v) \in \mathbf{R}^2$$

is continuous in its first variable (for all fixed  $v \in \mathbf{R}$ ). Therefore, by (3), (5) and (6) we have

$$s(u, v) = \int_0^1 \Delta_u \Delta_v f - f(u+v) + f(u) + f(v)$$

$$\begin{aligned}
 &= B(u, v) + \int_0^{u+v} f_0 - \int_0^u f_0 - \int_0^v f_0 - f(u+v) + f(u) + f(v) \\
 &= N(u+v) - f(u+v) - (N(u) - f(u)) - (N(v) - f(v)) \\
 &\quad + \int_0^{u+v} f_0 - \int_0^u f_0 - \int_0^v f_0 \\
 &= -\Delta_v(f - N)(u) - (N(v) - f(v)) + \int_0^{u+v} f_0 - \int_0^u f_0 - \int_0^v f_0.
 \end{aligned}$$

This implies that  $\Delta_v(f - N)$  is continuous for all fixed  $v \in \mathbf{R}$  and Theorem 1 can be applied again to get the decomposition  $f - N = f^* + A$  on  $\mathbf{R}$  with some continuous  $f^*: \mathbf{R} \rightarrow \mathbf{R}$  and additive function  $A$ , that is, (1) holds and the proof is complete.

**Theorem 3.** *Let  $L$  be as in Lemma 3. Then  $L$  has the quadratic difference property.*

**Proof.** If  $L$  is the class of all continuous functions then the statement is proved by Theorem 2. In the remaining cases, since all functions in  $L$  are continuous, Theorem 2 implies the decomposition (1) with continuous  $f^*$ , quadratic  $N$  and additive  $A$ . We now prove that  $f^* \in L$ . For all  $y \in \mathbf{R}$  we get from (1) that

$$(7) \quad \Delta_y \Delta_{-y} f = \Delta_y \Delta_{-y} f^* + 2N(y).$$

Therefore  $\Delta_y \Delta_{-y} f^* \in L$  for all fixed  $y \in \mathbf{R}$ . Applying (2) in Lemma 1 we obtain that  $\Delta_u(\Delta_v f^*) \in L$  for all fixed  $u, v \in \mathbf{R}$ . Obviously  $\Delta_v f^*$  is continuous thus, by Lemma 3,  $\Delta_v f^* \in L$ . Since  $f^*$  is continuous, Lemma 3 can be applied again to get  $f^* \in L$ .

**Remark.** The set of all bounded functions  $f: \mathbf{R} \rightarrow \mathbf{R}$  does not have the quadratic difference property. Indeed, let

$$f(x) = x \ln(x^2 + 1) + 2 \operatorname{arctg} x - 2x, \quad x \in \mathbf{R}.$$

Applying the Lagrangian mean value theorem with fixed  $u, v, x \in \mathbf{R}$  we have

$$(8) \quad \Delta_u \Delta_v f(x) = u \Delta_v f'(\xi) = uv f''(\eta)$$

for some  $\xi, \eta \in \mathbf{R}$ . Since  $|f''(\eta)| = \frac{2|\eta|}{\eta^2+1} \leq 1$ , (8) implies that  $|\Delta_y \Delta_{-y} f(x)| \leq y^2$  for all  $x, y \in \mathbf{R}$ , that is,  $\Delta_y \Delta_{-y} f$  is bounded for all fixed  $y \in \mathbf{R}$ . Suppose that  $f$  has the decomposition (1) for some bounded  $f^*: \mathbf{R} \rightarrow \mathbf{R}$ , quadratic  $N$  and additive  $A$ . Then  $N + A$  must be bounded on any bounded interval. Thus  $N(x) + A(x) = \alpha x^2 + \beta x$ ,  $x \in \mathbf{R}$  for some  $\alpha, \beta \in \mathbf{R}$ . This and (1) imply that

$$(9) \quad f^*(x) = x \ln(x^2 + 1) + 2 \arctg x - \alpha x^2 - (2 + \beta)x, \quad x \in \mathbf{R}.$$

Since  $f^*$  is bounded,  $0 = \lim_{x \rightarrow +\infty} \frac{f^*(x)}{x^2} = -\alpha$  and thus

$$0 = \lim_{x \rightarrow +\infty} \frac{f^*(x) - 2 \arctg x}{x} = \lim_{x \rightarrow +\infty} (\ln(x^2 + 1) - (2 + \beta)),$$

which is a contradiction. This shows that the set of all bounded functions does not have the quadratic difference property.

## References

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