

# Quasi multiplicative functions with congruence property

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**Abstract.** We prove that if an integer-valued quasi multiplicative function  $f$  satisfies the congruence  $f(n+p) \equiv f(n) \pmod{p}$  for all positive integers  $n$  and all primes  $p \neq \pi$ , where  $\pi$  is a given prime, then  $f(n) = n^\alpha$  for some integer  $\alpha \geq 0$ .

An arithmetical function  $f(n) \neq 0$  is said to be multiplicative if  $(n, m) = 1$  implies

$$f(nm) = f(n)f(m)$$

and it is called completely multiplicative if this holds for all pairs of positive integers  $n$  and  $m$ . In the following we denote by  $\mathcal{M}$  and  $\mathcal{M}^*$  the set of all integer-valued multiplicative and completely multiplicative functions, respectively. Let  $\mathbf{N}$  be the set of all positive integers and  $\mathcal{P}$  be the set of all primes.

The problem concerning the characterization of some arithmetical functions by congruence properties was studied by several authors. The first result of this type was found by M. V. Subbarao [7], namely he proved in 1966 that if  $f \in \mathcal{M}$  satisfies

$$(1) \quad f(n+m) \equiv f(m) \pmod{n} \quad \text{for all } n, m \in \mathbf{N},$$

then there is an  $\alpha \in \mathbf{N}$  such that

$$(2) \quad f(n) = n^\alpha \quad \text{for all } n \in \mathbf{N}.$$

A. Iványi [2] extended this result proving that if  $f \in \mathcal{M}^*$  and (1) holds for a fixed  $m \in \mathbf{N}$  and for all  $n \in \mathbf{N}$ , then  $f(n)$  has also the same form (2). It is shown in [4] that the result of Subbarao continues to hold if the relation (1) is valid for  $n \in \mathcal{P}$  instead for all positive integers. In [6] we improved the results of Subbarao and Iványi mentioned above by proving that if  $M \in \mathbf{N}$ ,  $f \in \mathcal{M}$  satisfy  $f(M) \neq 0$  and

$$f(n+M) \equiv f(M) \pmod{n} \quad \text{for all } n \in \mathbf{N},$$

then (2) holds. Later, in the papers [3]–[5] we obtained some generalizations of this result, namely we have shown that if integers  $A > 0$ ,  $B > 0$ ,  $C \neq 0$ ,  $N > 0$  with  $(A, B) = 1$  and  $f \in \mathcal{M}$  satisfy the relation

$$f(An + B) \equiv C \pmod{n} \quad \text{for all } n \geq N,$$

then there are a positive integer  $\alpha$  and a real-valued Dirichlet character  $\chi \pmod{A}$  such that  $f(n) = \chi(n)n^\alpha$  for all  $n \in \mathbf{N}$ ,  $(n, A) = 1$ .

In 1985, Subbarao [8] introduced the concept of weakly multiplicative arithmetic function  $f(n)$  (later renamed quasi multiplicative arithmetic functions) as one for which the property

$$f(np) = f(n)f(p)$$

holds for all primes  $p$  and positive integers  $n$  which are relatively prime to  $p$ . In the following let  $\mathcal{QM}$  denote the set of all integer-valued quasi multiplicative functions. In [1] J. Fabrykowski and M. V. Subbarao proved that if  $f \in \mathcal{QM}$  satisfies

$$(3) \quad f(n + p) \equiv f(n) \pmod{p}$$

for all  $n \in \mathbf{N}$  and all  $p \in \mathcal{P}$ , then  $f(n)$  has the form (2). They also conjectured that this result continues to hold even if the relation (3) is satisfied for an infinity of primes instead of for all primes. This conjecture is still open.

Let  $\mathcal{A} \subset \mathcal{P}$ , and assume that the congruence (3) holds for all  $n \in \mathbf{N}$  and for all  $p \in \mathcal{A}$ . For each positive integer  $n$  let  $H(n)$  denote the product of all prime divisors  $p$  of  $n$  for which  $p \in \mathcal{A}$ . It is obvious from the definition that  $H(n) \mid H(mn)$  holds for all positive integers  $n$  and  $m$ , furthermore one can deduce that if  $f \in \mathcal{QM}$  satisfies the congruence (3) for all  $n \in \mathbf{N}$  and for all  $p \in \mathcal{A}$ , then

$$f(n + m) \equiv f(m) \pmod{H(n)} \quad \text{for all } n, m \in \mathbf{N}.$$

Thus the conjecture of Fabrykowski and Subbarao is contained in the following

**Conjecture.** *Let  $A, B$  be fixed positive integers with the condition  $(A, B) = 1$  and  $\mathcal{A}$  is an infinite subset of  $\mathcal{P}$ . If a function  $f \in \mathcal{QM}$  and integer  $C \neq 0$  satisfy the congruence*

$$f(An + B) \equiv C \pmod{H(n)} \quad \text{for all } n \in \mathbf{N},$$

then there are a positive integer  $\alpha$  and a real-valued Dirichlet character  $\chi$  (mod  $A$ ) such that

$$f(n) = \chi(n)n^\alpha \quad \text{for all } n \in \mathbf{N}, (n, A) = 1.$$

In this note we prove this conjecture for a special case, when  $A = B = 1$  and  $\mathcal{P} = \mathcal{A} \cup \{\pi\}$ , where  $\pi$  is a fixed prime.

**Theorem.** *Let  $\pi$  be a given prime and let  $H(n)$  be the product of all prime divisors  $p$  of  $n$  for which  $p \neq \pi$ . If a function  $f \in \mathcal{QM}$  and an integer  $C \neq 0$  satisfy the congruence*

$$(4) \quad f(n+1) \equiv C \pmod{H(n)}$$

for all  $n \in \mathbf{N}$ , then there is a non-negative integer  $\alpha$  such that

$$f(n) = n^\alpha \quad \text{for all } n \in \mathbf{N}.$$

We shall use some lemmas in the proof of our theorem.

**Lemma 1.** *Assume that the conditions of the theorem are satisfied. Then  $f \in \mathcal{M}^*$ , i.e*

$$f(ab) = f(a)f(b)$$

holds for all  $a, b \in \mathbf{N}$ . Furthermore  $C = 1$ .

**Proof.** Assume that  $a$  and  $b$  are fixed positive integers. Let  $q$  be a prime with the condition

$$(5) \quad q > \max(a, b, |C|, |Cf(ab) - f(a)f(b)|) \quad \text{and} \quad q \neq \pi.$$

Since  $(ab, q) = 1$ , one can deduce from Dirichlet's theorem that there are positive integers  $x, y, u$  and  $v$  such that

$$ax = qy + 1, \quad (x, ab) = 1, \quad x \in \mathcal{P}$$

and

$$bu = qv + 1, \quad (u, abx) = 1, \quad u \in \mathcal{P}.$$

Then we have

$$abxu = qT + 1,$$

where  $T := y + v + qyv$ . Thus, we infer from (4) and the fact  $f \in \mathcal{QM}$ , that

$$f(a)f(x) = f(ax) = f(qy + 1) \equiv C \pmod{q},$$

$$f(b)f(u) = f(bu) = f(qv + 1) \equiv C \pmod{q}$$

and

$$f(ab)f(x)f(u) = f(abxu) = f(qT + 1) \equiv C \pmod{q}.$$

These and (5) show that  $f(x)f(u) \not\equiv 0 \pmod{q}$ , consequently

$$f(a)f(b) \equiv Cf(ab) \pmod{q}.$$

Hence, we infer from the last relation together and the fact  $q > |Cf(ab) - f(a)f(b)|$  that

$$(6) \quad Cf(ab) = f(a)f(b).$$

Thus, we have proved that (6) holds for all positive integers  $a$  and  $b$ . By applying (6) with  $a = b = 1$ , we have  $C = 1$  and so the proof of Lemma 1 is finished.

**Lemma 2.** *Assume that the conditions of the theorem are satisfied. Let  $Q$  be a positive integer. Then for each prime divisor  $q$  of  $f(Q)$  we have  $q|\pi Q$ .*

**Proof.** Let  $Q$  be a positive integer and assume on the contrary that there exists a prime  $q$  such that  $q|f(Q)$  and  $(q, \pi Q) = 1$ .

Since  $(Q, q) = 1$ , we infer that there are positive integers  $x$  and  $y$  such that

$$Qx = qy + 1.$$

By using Lemma 1, it follows from (4) and the fact  $q \neq \pi$  that

$$0 \equiv f(Q)f(x) = f(Qx) = f(qy + 1) \equiv 1 \pmod{q},$$

which is a contradiction. Thus the proof of Lemma 2 is finished.

Lemma 2 shows that for each prime  $p$ , we can write  $f(p)$  as follows:

$$|f(p)| = p^{\alpha(p)}\pi^{\beta(p)},$$

consequently

$$(7) \quad |f(\pi)| = \pi^{\alpha},$$

for some non-negative integer  $\alpha$ .

Now we can prove our theorem.

**Proof of the theorem.** We shall prove that  $f(n) = n^\alpha$  is satisfied for all  $n \in \mathbf{N}$ , where  $\alpha \geq 0$  is given in (7).

Let  $n, s$  be positive integers. By (4), we have

$$f(n\pi^{2s}) = f((n\pi^{2s} - 1) + 1) \equiv 1 \pmod{H(n\pi^{2s} - 1)}.$$

On the other hand, it follows from Lemma 1 and (7) that

$$n^\alpha f(n\pi^{2s}) = n^\alpha f(n)\pi^{2\alpha s} = f(n)(n\pi^{2s})^\alpha \equiv f(n) \pmod{H(n\pi^{2s} - 1)}.$$

These imply

$$f(n) \equiv n^\alpha \pmod{H(n\pi^{2s} - 1)},$$

therefore, by setting  $s \rightarrow \infty$ , we have  $H(n\pi^{2s} - 1) \rightarrow \infty$  and so  $f(n) = n^\alpha$ . This holds for each positive integer  $n$ , consequently it also holds for all  $n \in \mathbf{N}$ . The theorem is proved.

## References

- [1] J. FABRYKOWSKI and M. V. SUBBARAO, A class of arithmetic functions satisfying a congruence property, *Journal Madras University, Section B* **51** (1988), 48–51.
- [2] A. IVÁNYI, On multiplicative functions with congruence property, *Ann. Univ. Sci. Budapest, Sect. Math.* **15** (1972), 133–137.
- [3] I. JOÓ and B. M. PHONG, Arithmetical functions with congruence properties, *Ann. Univ. Sci. Budapest, Sect. Math.* **35** (1992), 151–155.
- [4] B. M. PHONG, Multiplicative functions satisfying a congruence property, *Periodica Math. Hungar.* **26** (1991), 123–128.
- [5] B. M. PHONG, Multiplicative functions satisfying a congruence property V, *Acta Math. Hungar.* **62** (1993), 81–87.
- [6] B. M. PHONG and J. FEHÉR, Note on multiplicative functions satisfying congruence property, *Ann. Univ. Sci. Budapest, Sect. Math.* **33** (1990), 261–265.
- [7] M. V. SUBBARAO, Arithmetic functions satisfying congruence property, *Canad. Math. Bull.*, **9** (1966), 143–146.
- [8] M. V. SUBBARAO, *Amer. Math. Soc. Abstract*, **86** # T-11-15, p. 324.

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