

Representation of integers as terms of a linear recurrence with maximal index

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Abstract. For sequences $H_n(a,b)$ of positive integers, defined by $H_0=a$, $H_1=b$ and $H_n=H_{n-1}+H_{n-2}$, we investigate the problem: for a given positive integer N find positive integers a and b such that $N=H_n(a,b)$ and n is as large as possible. Denoting by $R(N)=r$ the largest integer, for which $N=H_r(a,b)$ for some a and b , we give bounds for $R(N)$ and a polynomial time algorithm for computing it. Some properties of $R(N)$ are also proved.

Introduction

Let $H_n(a,b)$ be a sequence of positive integers defined by $H_0 = a$, $H_1 = b$ and $H_n = H_{n-1} + H_{n-2}$ where a and b are arbitrary positive integers (the parameters). The sequence $H_n(a,b)$ occurs in a problem of COHN [1]: *Given a positive integer N , find positive integers a and b such that $N = H_n(a,b)$ and n is as large as possible.*

COHN [1] actually formulated the problem slightly differently, replacing 'n is as large as possible' by 'a+b is as small as possible'. However this makes little difference. We shall consider the problem as stated above.

Let $R = R(N)$ be the largest integer R such that $N = H_R(a,b)$ for some $a, b \geq 1$. The function R is well defined. For any $N \geq 1$, there exist integers a, b and n such that $N = H_n(a,b)$, $1 \leq a$, $1 \leq b$. Since $N = H_1(1, N)$, we can let $n = 1$, $a = 1$ and $b = N$. If $2 \leq N$, we can also let $a = 1$, $b = N - 1$ and $n = 2$ so $N = H_2(1, N - 1)$. Thus there always exist integers n , a and b such that $N = H_n(a,b)$, $1 \leq a$ and $1 \leq b$.

It is also easy to see that there exist a, b and r such that $N = H_r(a,b)$, $1 \leq a$, $1 \leq b$ and r is maximal. If $N = H_r(a,b)$, $1 \leq a$ and $1 \leq b$, then $r \leq H_r(a,b)$. Hence for all such r, a and b , $r \leq N$. Thus all possible values of r are bounded above by N . In fact this argument shows that $R(N) \leq N$ for all N .

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The first few values of R are given by $R(1) = 1$, $R(2) = 2$, $R(3) = 3$, $R(4) = 3$, $R(5) = 4$, $R(6) = 3$, $R(7) = 4$, $R(8) = 5$, $R(9) = 4$, $R(10) = 4$, $R(11) = 5$, $R(12) = 4$ and $R(13) = 6$. Note that the function R is not increasing. That is, $N \leq M$ does not imply $R(N) \leq R(M)$.

Since $R(N)$ is well defined, Cohn's problem becomes one of giving an algorithm to compute $R(N)$. In this paper we shall give a simple algorithm which solves this problem. We shall also show that this algorithm is polynomial time, that is the time to find $R(N)$ is less than a polynomial in $\ln(N)$. We also prove some theorems about the number of N such that $R(N) = r$ and about the number of pairs (a, b) such that $H_r(a, b) = N$. First we need some lemmas.

1. Representation of N in the form $N = H_r(a, b)$ with r maximal

We use $\lfloor x \rfloor$ to denote the floor of x , (integer part of x). $\lceil x \rceil$ denotes the ceiling of x , $\lceil x \rceil = -\lfloor -x \rfloor$. F_n denotes the n^{th} Fibonacci number, where $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_i + F_{i+1}$. L_n denotes the n^{th} Lucas number, defined by $L_0 = 2$, $L_1 = 1$ and $L_{i+2} = L_i + L_{i+1}$. We define $H_{-n}(a, b)$ by $H_{-n}(a, b) = (-1)^{n+1} H_n(-a, b - a)$.

Below we shall use many elementary identities and inequalities such as $L_{n+1} = 2F_n + F_{n+1}$, $L_n + 1 \leq F_{n+2}$ for $1 \leq n$ and $F_{n+1} < L_n$, for $2 \leq n$. We shall also need the following well known identity due to HORADAM [3].

Lemma 1.1. For all integers n, a and b , $H_n(a, b) = aF_{n-1} + bF_n$.

Proof. By induction on n using $F_{i+2} = F_i + F_{i+1}$. The result can also be seen to hold for negative n since $H_{-n}(a, b) = (-1)^n (aF_{n+1} - bF_n)$.

Lemma 1.2. $H_n(a, b) = H_n(a + F_n, b - F_{n-1})$ and $H_n(a, b) = H_n(a - F_n, b + F_{n-1})$.

Lemma 1.3. For all integers n, k, a and b , we have

- (i) $H_n(a, b) = H_{n-1}(b, a + b)$.
- (ii) $H_n(a, b) = H_{n-k}(H_k(a, b), H_{k+1}(a, b))$,
- (iii) $H_n(a, b) = H_{n+1}(b - a, a)$.
- (iv) $H_n(a, b) = H_{n+k}(H_{-k}(a, b), H_{1-k}(a, b))$.

Proof. They follow from the definitions.

Lemma 1.4. If $N = H_r(a, b)$, $1 \leq a$, $1 \leq b$ and $R(N) = r$, then $b \leq a$.

Proof. Suppose $R(n) = r$ and $N = H_r(a, b)$. If $a < b$, then by Lemma 1.3 we would have $N = H_r(a, b) = H_{r+1}(b - a, a)$ so that $r + 1 \leq R(N)$, contradicting $r = R(N)$.

Earlier we saw that $n = 1$ is realizable as a value of n such that $N = H_n(a, b)$ for $a \geq 1, b \geq 1$. In the next lemma we shall show that all values of $n \leq R(N)$ are realizable as values of n such that $N = H_n(a, b)$. We shall call this the Intermediate Value Lemma (IVL).

Lemma 1.5. (I.V.L.) *If $n \leq R(N)$, then there exist a, b such that $N = H_n(a, b)$, $1 \leq a$ and $1 \leq b$.*

Proof. Suppose $r = R(N)$ and $n \leq r$. There exist $a \geq 1, b \geq 1$ such that $N = H_r(a, b)$. Let $k = r - n$. Then $0 \leq k$. By Lemma 1.3 (ii), $N = H_r(a, b) = H_{r-k}(H_k(a, b), H_{k+1}(a, b)) = H_n(H_k(a, b), H_{k+1}(a, b))$ where $1 \leq H_k(a, b)$ and $1 \leq H_{k+1}(a, b)$, since $0 \leq k$ and $a, b \geq 1$.

Lemma 1.6. *If $n \geq 1$ then $R(F_{n+1}) = n$.*

Proof. Let $r = R(F_{n+1})$. Since $F_{n+1} = F_{n-1} + F_n = H_n(1, 1)$, $n \leq r$. Conversely, $F_{n+1} = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r-1} + F_r = F_{r+1}$. Hence $n \geq r$. Therefore $n = r$.

Lemma 1.7. *If $n \geq 2$, then $R(L_{n+1}) = n + 1$.*

Proof. Here we need the inequality $L_{n+1} + 1 \leq F_{n+3}$. Let $r = R(L_{n+1})$. Since L_n may be defined by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_n + L_{n+1}$, we have $L_n = H_n(2, 1)$ and so $L_{n+1} = H_{n+1}(2, 1)$. Hence $n + 1 \leq r$. Conversely, $L_{n+1} = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r-1} + F_r = F_{r+1}$. Hence $F_{r+1} \leq L_{n+1}$. Therefore $F_{r+1} + 1 \leq L_{n+1} + 1 \leq F_{n+3}$ and so $F_{r+1} < F_{n+3}$. Therefore $r + 1 < n + 3$. Hence $r < n + 2$. Therefore $r \leq n + 1$. So $r = n + 1$ and $R(L_{n+1}) = n + 1$.

Lemma 1.8. *If $N < F_{n+1}$, then $R(N) < n$.*

Proof. Let $R(N) = r$. Then there exist $a, b \geq 1$ such that $N = H_r(a, b)$. Hence we have $F_{n+1} > N = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r-1} + F_r = F_{r+1}$. Thus $F_{n+1} > F_{r+1}$. Hence we have $n + 1 > r + 1$ so that $n > r$. In other words $n > R(N)$.

Corollary 1.9. *If $1 \leq n$ and $N \leq F_{n+1}F_{n+2}$, then $R(N) \leq 2n$.*

Proof. If $1 \leq n$, then $F_{n+2} < L_{n+1}$. Hence $F_{n+1}F_{n+2} < F_{n+1}L_{n+1} = F_{2n+2}$. Therefore $N < F_{2n+2}$. Hence by Lemma 1.8, $R(N) < 2n + 1$. Therefore $R(N) \leq 2n$.

Lemma 1.10. *Let A be an arbitrary positive integer and suppose $0 \leq n$. Then*

- (i) $n = R(AF_{n+1})$ if $A \leq F_n$,
(ii) $n < R(AF_{n+1})$ if $F_n < A$.

Proof. $AF_{n+1} = A(F_{n-1} + F_n) = AF_{n-1} + AF_n = H_n(A, A)$ implies $n \leq R(AF_{n+1})$. For (i) suppose $A \leq F_n$ and $n + 1 \leq R(AF_{n+1})$. By the Intermediate Value Lemma there exist $c \geq 1$ and $d \geq 1$ such that $AF_{n+1} = cF_n + dF_{n+1}$. Then $F_{n+1} \mid cF_n$ and $(F_n, F_{n+1}) = 1$ imply $F_{n+1} \mid c$. Hence $F_{n+1} \leq c$ so that $d = 0$. Hence $R(AF_{n+1}) = n$. For (ii) suppose $F_n < A$. Then there exist b and t such that $A = tF_n + b$, $1 \leq b$ and $1 \leq t$. Let $a = tF_{n+1}$. Then $AF_{n+1} = (tF_{n+1})F_n + bF_{n+1} = H_{n+1}(tF_{n+1}b) = H_{n+1}(a, b)$, $1 \leq a$ and $1 \leq b$. Hence $n + 1 \leq R(AF_{n+1})$ so that $n < R(AF_{n+1})$.

Corollary 1.10. For all $n \geq 0$, $R(F_n F_{n+1}) = n$.

Lemma 1.11. If $F_n F_{n+1} < N$, then $n < R(N)$.

Proof. Suppose $F_n F_{n+1} < N$. We shall show that $n + 1 \leq R(N)$ by finding a and b such that $N = H_{n+1}(a, b) = aF_n + bF_{n+1}$, $1 \leq a$ and $1 \leq b$. Let b be the least positive solution to the congruence $N \equiv bF_{n+1} \pmod{F_n}$, (taking $b = F_n$, if $F_n \mid N$, so that $b \geq 1$). We claim

$$(1.12) \quad bF_{n+1} + F_n \leq N.$$

This inequality (1.12) will be proved by considering two cases:

Case 1. $N \equiv 0 \pmod{F_n}$. Then $b = F_n$. Since $F_n \mid N$ and $F_n F_{n+1} < N$, we have $F_n(F_{n+1} + 1) \leq N$. So we have $F_{n+1}b + F_n = F_{n+1}F_n + F_n = F_n(F_{n+1} + 1) \leq N$, and so (1.12) holds.

Case 2. $N \not\equiv 0 \pmod{F_n}$. Then $1 \leq b < F_n$, so $b \leq F_n - 1$. Therefore $bF_{n+1} + F_n \leq (F_n - 1)F_{n+1} + F_n = F_n F_{n+1} + (F_n - F_{n+1}) \leq F_n F_{n+1} < N$ and so again (1.12) holds.

Now that (1.12) is established, let $a = (N - bF_{n+1})/F_n$. Then a is an integer, $N = aF_n + bF_{n+1} = H_{n+1}(a, b)$ and (1.12), implies $1 \leq a$.

Corollary 1.13. If $1 < n$ and $F_{2n} \leq N$, then $n < R(N)$.

Proof. By Lemma 1.11. If $1 < n$, then $F_{n+1} < L_n$ and $F_n F_{n+1} < F_n L_n = F_{2n} \leq N$.

Lemma 1.14. If $1 \leq N$, then $R(N) \leq ([1 + 2.128 \cdot \ln(N)])$.

Proof. Let $r = R(N)$. The inequality holds for $N = 1$, since $R(1) = 1$. Suppose $N \geq 2$. Then $2 \leq r$. Let $k = r + 1$. Then $3 \leq k$ so that we can use the inequality

$$(1.15) \quad (8/5)^{k-2} < F_k \quad (3 \leq k).$$

(This inequality, which is well known, is easy to prove by induction on $k \geq 3$, using the fact that if $x = 8/5$, then $x^2 < x + 1$). Using the inequality

with $k = r + 1$, by Lemma 1.8 we get $(8/5)^{r-1} < F_{r+1} \leq N$. Taking logs of both sides we have $(r - 1)\ln(8/5) \leq \ln(N)$. Hence we have $r - 1 \leq \ln(N)/\ln(8/5) < \ln(N)/(47/100) < \ln(N) \cdot 2.128$, proving the lemma.

Lemma 1.16. *If $1 \leq N$, then $\lceil 1.5 + .893 \cdot \ln(N) \rceil \leq R(N)$.*

Proof. Let $r = R(N)$. Lemma 1.11 implies $N \leq F_r F_{r+1}$. If $N \leq 6$, the inequality can be checked by cases. Suppose $7 \leq N$. Then $4 \leq r$. We will use the following elementary inequality which is easy to prove using the fact that $x^2 > x + 1$ for $x = 7/4$.

$$(1.17) \quad F_k < (7/4)^{k-2} \quad (3 < k).$$

Using the inequality twice, with $k = r$ and $k = r + 1$, we get

$$(1.18) \quad N \leq F_r F_{r+1} < (7/4)^{r-2} (7/4)^{r-1} = (7/4)^{2r-3}.$$

Taking logs of both sides, $\ln(N) < (2r - 3)\ln(7/4)$. Hence $\ln(N)/\ln(7/4) < 2(r - 1.5)$. Therefore $2^{-1} \cdot \ln(N)/\ln(7/4) < r - 1.5$. Consequently $1.5 + 2^{-1} \cdot \ln(N)/\ln(7/4) < r$. Hence $\lceil 1.5 + 2^{-1} \cdot \ln(N)/\ln(7/4) \rceil \leq r$. Therefore $\lceil 1.5 + .893 \cdot \ln(N) \rceil \leq r$.

Corollary 1.19. *For $N \geq 1$,*

$$\lceil 1.5 + .893 \cdot \ln(N) \rceil \leq R(N) \leq \lfloor 1 + 2.128 \cdot \ln(N) \rfloor.$$

Proof. By Lemma 1.14 and Lemma 1.15.

Corollary 1.20. *If $R(N) = r$, then $F_{r+1} \leq N \leq F_r F_{r+1}$.*

Proof. Suppose $R(N) = r$. By Lemma 1.8, $F_{r+1} \leq N$. By Lemma 1.11, $N \leq F_r F_{r+1}$.

The equation $N = H_r(a, b)$ sometimes has two solutions (a, b) in positive integers with $r = R(N)$. E.g. if $N = 6$, then $R(6) = 3$, $6 = H_3(2, 2)$ and $6 = H_3(4, 1)$.

Definition 1.21. N is called a *double number* if there exist $a, b, c, d \geq 1$ such that $N = H_r(a, b) = H_r(c, d)$, $a \neq c$ or $b \neq d$, (equivalently if $a \neq c$ and $b \neq d$), where $r = R(N)$. If N is not a double number, then N is called a *single number*.

Examples 1.22. Some representations of N in the form $N = H_r(a, b)$ with $R = R(N)$:

$$\begin{array}{llll} N = 1, & R = 1, & a = 1, & b = 1, \\ N = 10, & R = 4, & a = 2, & b = 2, \end{array}$$

$N = 100,$	$R = 7,$	$a = 6,$	$b = 4,$
$N = 1,000,$	$R = 12,$	$a = 8,$	$b = 2,$
$N = 10,000,$	$R = 12,$	$a = 80,$	$b = 20,$
$N = 100,000,$	$R = 14,$	$a = 269,$	$b = 99,$
$N = 1,000,000,$	$R = 19,$	$a = 154,$	$b = 144,$
$N = 10,000,000,$	$R = 19,$	$a = 1540,$	$b = 1440,$
$N = 100,000,000,$	$R = 23,$	$a = 5143,$	$b = 311,$
$N = 1,000,000,000,$	$R = 23,$	$a = 51430,$	$b = 3110.$

$N = 1,000,000,000$ happens to be an example of a double number. For we have $N = H_r(c, d)$ also for $c = 22773$ and $d = 20821$, besides $a = 51430$ and $b = 3110$. Other examples of double numbers are $15 = H_4(6, 1) = H_4(3, 3)$ and $32 = H_5(9, 1) = H_5(4, 4)$.

In the next section we shall prove that the equation $N = H_r(a, b)$ never has three solutions (a, b) in positive integers with $r = R(N)$. (Of course it may have other solutions when $r < R(N)$. E.g. for $N = 6$ and $r = 3$ we have $6 = H_2(1, 5) = H_2(2, 4) = H_2(3, 3)$ where $2 < r$.) Thus there is no concept of a triple number.

2. An algorithm for $R(N)$

In this section we shall show that there exists an algorithm for computing $R(N)$. In fact we shall prove that there is a polynomial-time algorithm for computing $R(N)$. We give a procedure which finds, given N , the value of $R(N)$ and also a and b . Since the number of steps in the procedure will be less than a polynomial in $\ln(N)$, the number of bit operations needed to compute $R(N)$ will be less than a polynomial in $\ln(N)$.

Suppose N is given. To compute $R(N)$, begin with any sufficiently large value of r , satisfying $r \geq R(N)$. For example by Corollary 1.19 we can take $r = \lceil 1 + 2.128 \cdot \ln(N) \rceil$. Then proceed as follows.

Step 1: Find a positive solution b to the congruence

$$(2.1) \quad N \equiv bF_r \pmod{F_{r-1}}, \quad (1 \leq b).$$

This congruence is solvable in natural numbers since $(F_r, F_{r-1}) = 1$. Hence there is a solution b in the range $1 \leq b \leq F_{r-1}$. Take the least such b in this range.

Step 2: Check whether

$$(2.2) \quad bF_r < N.$$

If this is the case, put $a = (N - bF_r)/F_{r-1}$. Then a is an integer by (2.1). Also we have $N = aF_{r-1} + bF_r$ and condition (2.2) implies $1 \leq a$. In this

case the algorithm terminates and $R(N) = r$. If (2.2) does not hold, then we decrease r by 1 and return to Step 1. We iterate steps 1 and 2, decreasing r until (2.2) holds. Since initially $R(N) \leq \lfloor 1 + 2.128 \cdot \ln(N) \rfloor \leq r$, the algorithm must terminate after at most $\lfloor 1 + 2.128 \cdot \ln(N) \rfloor$ iterations.

We claim that this computation is polynomial time. Certainly the number of operations needed at each step is less than or equal to a polynomial in $\ln(N)$. Calculation of F_r requires time exponential in $\ln(r)$, i.e. proportional to a polynomial in r . However r is less than or equal to a polynomial in $\ln(N)$, since $F_r \leq N$. So this is polynomial time.

In addition to finding r , the algorithm also finds (a, b) such that $H_r(a, b) = N$. The pair (a, b) is not uniquely dependent upon N . There is sometimes a second pair (c, d) such that $H_r(c, d) = N$. As sketched above the algorithm finds the pair (a, b) with least b . It can easily be extended also to find the second pair (c, d) , when that exists. After (a, b) has been found, let $d = b + F_{r-1}$ and $c = a - F_r$. Then $N = H_r(c, d)$ by Lemma 1.2. d is positive. If $dF_r < N$, then c will also be positive and (c, d) will be a second pair. If not, then there is no second pair, i.e. N is a single.

The algorithm can be simplified to yield a more explicit formula for $r = R(N)$ and explicit formulas for a, b, c and d . For this we shall use an old identity of LUCAS [4]:

$$(2.3) \quad F_{r-1}^2 - F_{r-2} \cdot F_r = (-1)^r.$$

Multiplying both sides of (2.3) by $(-1)^r N$ and rearranging terms we get

$$(2.4) \quad (-1)^r F_{r-1} N \cdot F_{r-1} - (-1)^r F_{r-2} N \cdot F_r = N.$$

Equation (2.4) provides a solution to the linear diophantine equation $aF_{r-1} + bF_r = N$. It shows that $AF_{r-1} + BF_r = N$ will hold if we put $A = A_r(N)$ and $B = B_r(N)$, where

$$(2.5) \quad A_r(N) = (-1)^r F_{r-1} N \quad \text{and} \quad B_r(N) = -(-1)^r F_{r-2} N.$$

Thus $a = A_r(N)$ and $b = B_r(N)$ is a particular solution of the equation $aF_{r-1} + bF_r = N$. Since $(F_r, F_{r+1}) = 1$, from a particular solution we may obtain all solutions (a, b) by

$$(2.6) \quad a = A_r(N) - tF_r, \quad b = B_r(N) + tF_{r-1}, \quad (t = 0, \pm 1, \pm 2, \pm 3, \dots).$$

Then by Lemma 1.1 $H_r(a, b) = N$ for all integers t . Now define $g_r(N)$ and $h_r(N)$ by

$$(2.7) \quad g_r(N) = \frac{(-1)^r F_{r-2} N + 1}{F_{r-1}}$$

and

$$(2.8) \quad h_r(N) = \frac{(-1)^r F_{r-1} N - 1}{F_r}.$$

Then $g_r(N)$ and $h_r(N)$ are reals. For a and b as in (2.6), we have $1 \leq a$ iff $t \leq h_r(N)$ and $1 \leq b$ iff $g_r(N) \leq t$. Hence (a, b) is a positive solution of $aF_{r-1} + bF_r = N$ iff

$$(2.9) \quad g_r(N) \leq t \leq h_r(N).$$

Since t is integer valued, condition (2.9) is equivalent to

$$(2.10) \quad g_r(N) \leq [g_r(N)] \leq t \leq [h_r(N)] \leq h_r(N).$$

Condition (2.9) is in turn equivalent to $[g_r(N)] \leq h_r(N)$ and also to $g_r(N) \leq [h_r(N)]$.

From (2.3), (2.7) and (2.8), it is easy to see that

$$(2.11) \quad h_r(N) - g_r(N) = \frac{N - F_{r+1}}{F_{r-1}F_r}.$$

The functions $g_r(N)$ and $h_r(N)$ give us a new algorithm to compute $R(N)$. We have

Theorem 2.12. *Suppose $N > 1$. Then $R(N)$ is the largest integer $r > 1$ such that*

$$(2.12) \quad \left\lceil \frac{(-1)^r F_{r-2} N + 1}{F_{r-1}} \right\rceil \leq \left\lfloor \frac{(-1)^r F_{r-1} N - 1}{F_r} \right\rfloor.$$

Furthermore, the set of $r > 1$ satisfying (2.12) is the set $\{2, 3, \dots, R(N)\}$. Hence (2.12) can be used as an algorithm to calculate $R(N)$.

Proof. By Lemma 1.8, if $r \leq R(N)$, then $F_{r+1} < N$ and hence by (2.11), $g_r(N) \leq h_r(N)$. Thus

$$(2.13) \quad r \leq R(N) \Rightarrow g_r(N) \leq h_r(N).$$

By (2.9) and the IVL, for all $r \leq R(N)$, there exist t ($g_r(N) \leq t \leq h_r(N)$), and this implies $[g_r(N)] \leq [h_r(N)]$. On the other hand, by (2.9), when $R(N) < r$, there is no integer t such that $g_r(N) \leq t \leq h_r(N)$ and so we have not $[g_r(N)] \leq [h_r(N)]$.

This shows that the set of $r > 1$ satisfying (2.12) is an interval.

This approach to $R(N)$, thru $g_r(N)$ and $h_r(N)$, also gives a new algorithm to decide whether N is a single or a double. From (2.9) and (2.10) we have

$$(2.14) \quad N \text{ is a single iff } [g_r(N)] = [h_r(N)].$$

Also

$$(2.15) \quad N \text{ is a double iff } [g_r(N)] < [h_r(N)].$$

From (2.5), (2.6), (2.7) and (2.8) we can obtain explicit formulas for a, b, c and d :

$$(2.16) \quad a = A_r(N) - [g_r(N)]F_r \quad \text{and} \quad b = B_r(N) + [g_r(N)]F_{r-1},$$

$$(2.17) \quad c = A_r(N) - [h_r(N)]F_r \quad \text{and} \quad d = B_r(N) + [h_r(N)]F_{r-1}.$$

If N is a single, $c = a$ and $d = b$. If N is a double, $c = a - F_r$ and $d = b + F_{r-1}$. Thus when $r = R(N)$, formulas (2.16) and (2.17) can be used as definitions of a, b, c and d . The ratio on the right side of (2.11) is not always less than 2 however, even when $r = R(N)$. In this case, when $r = R(N)$, we have only the weak inequality

$$(2.18) \quad R(N) \leq r \implies \frac{N - F_{r+1}}{F_{r-1}F_r} < \alpha + 1.$$

Here $\alpha = (1 + \sqrt{5})/2 = 1.61803\dots$ so that $\alpha + 1 = 2.61803\dots$. The idea of the proof is the following: From Lemma 1.11 we see that $R(N) \leq r$ implies $N \leq F_r F_{r+1}$. Then $(F_r F_{r+1} - F_{r-1})F_{r-1}F_r < \alpha + 1$ can be shown using $F_r^2 < \alpha F_1 F_{r-1} + F_{r+1}$.

Next we shall prove that there are no triples. The following lemmas will be used.

Lemma 2.19. *If $1 < r$, then $F_r F_{r+1} < (1 + 2F_{r+1})F_{r-1} + F_r$.*

Proof. If $1 < r$, then $F_r < 1 + 2F_{r-1}$. Hence

$$\begin{aligned} F_r F_{r+1} &< (1 + 2F_{r-1})F_{r+1} = F_{r+1} + 2F_{r-1} \cdot F_{r+1} \\ &= F_{r-1} + 2F_{r-1} \cdot F_{r+1} + F_r = (1 + 2F_{r+1})F_{r-1} + F_r. \end{aligned}$$

Lemma 2.20. *Suppose $1 < N$, $N = H_r(a, b)$, $R(N) = r$ and $1 \leq b$. Then $a \leq 2F_r$.*

Proof. Let $r = R(N)$ and $N = H_r(a, b)$. Since $1 < N$ and $r = R(N)$, we have $1 < r$. We claim

$$(2.21) \quad a < b + 2F_{r+1}.$$

If not, then $b + 2F_{r+1} \leq a$. Since $1 \leq b$ and $N = H_r(a, b)$, by Lemma 2.19 and Lemma 1.11 we have

$$N = aF_{r-1} + bF_r \geq (b + 2F_{r+1})F_{r-1} + bF_r \geq (1 + 2F_{r+1})F_{r-1} + F_r > F_r F_{r+1}.$$

But this contradicts Lemma 1.11 which says that $N \leq F_r F_{r+1}$, since $r = R(N)$. Hence (2.21) holds. Now it is easy to see that

$$N = aF_{r-1} + bF_r = (b + 2F_{r+1} - a)F_r + (a - 2F_r)F_{r+1}.$$

Supposing $2F_r < a$ and using (2.21), we get the contradiction $R(N) \geq r + 1$. So $a \leq 2F_r$.

Theorem 2.22. *If $R(N) = r$, then the equation $N = H_r(a, b)$ has at most two solutions in positive integers a, b . There are no triples.*

Proof. Suppose the equation $N = H_r(a, b)$ has three solutions in positive integers, say (a, b) , (c, d) and a third solution (x, y) . Then $c = a - F_r$, $d = b + F_{r-1}$, $x = a - 2F_r$ and $y = b + 2F_{r-1}$. But by Lemma 2.20, $a \leq 2F_r$. Hence $x \leq 0$, a contradiction.

From Theorem 2.22, if $r = R(N)$, then $\lfloor h_r(N) \rfloor \leq \lceil g_r(N) \rceil + 1$. And so in (2.15), when $\lceil g_r(N) \rceil < \lfloor h_r(N) \rfloor$, we have $\lceil g_r(N) \rceil + 1 = \lfloor h_r(N) \rfloor$.

Following $F_n F_{n+1}$ there is a very long interval consisting entirely of singles.

Suppose $R(N) = r$. Recall from Corollary 1.20 that if $R(N) = r$, then N must lie in the interval $F_{r+1} \leq N \leq F_r F_{r+1}$. We can show that most N in this interval are singles.

Theorem 2.23. *If $F_n F_{n+1} < N < F_n F_{n+1} + F_n^2 + F_{n+2}$, then N is a single.*

We won't prove this result, (Theorem 2.23.). However it will be clear how to do so after we have proved Lemma 3.1 in the next section.

Taking a limit as $n \rightarrow \infty$, one finds that the interval $[F_n F_{n+1}, F_n F_{n+1} + F_n^2 + F_{n+2}]$ occupies some 38% of the interval $[F_n F_{n+1}, F_{n+1} F_{n+2}]$. ($\beta^2 = ((1 - \sqrt{5})/2)^2 = (-.61803)^2 = .381966\dots$) Thus on average more than 28% of N are singles. Actually, in the next section, we shall prove that 92.7% of N are singles.

3. The number of N such that $R(N) = r$

In this section we consider the problem of the number of N such that $R(N) = r$. Here r is a fixed positive integer. The number of such N must

be finite. By Lemma 1.11, the number of such N must be less than or equal to $F_r F_{r+1}$. We shall give an exact formula for this number. First we need some lemmas.

Lemma 3.1. *Suppose $R(N) = r$ and a, b, c, d are as defined in (2.16) and (2.17). Then $N = H_r(a, b) = H_r(c, d)$. If N is a single, then $c = a, d = b$,*

$$(i) \quad 1 \leq b \leq a \leq F_r \quad \text{and} \quad 1 \leq b \leq F_{r-1}.$$

If N is a double, then we have $c = a - F_r, d = b + F_{r-1}, b \leq a$,

$$(ii) \quad F_{r-1} < d \leq c \leq F_r, \quad F_{r+1} < a \leq 2F_r \quad \text{and} \quad 1 \leq b \leq F_{r-2}.$$

Proof. Suppose a, b, c and d are as above and $N > 1$. Let $r = R(N)$. Then $N = H_r(a, b) = H_r(c, d)$. Suppose first N is a single. By (2.16) and (2.17), $c = a, d = b, 1 \leq a$ and $1 \leq b$. By Lemma 1.2, $N = H_r(a, b) = H_r(a + F_r, b - F_{r-1})$. Hence $b \leq F_{r-1}$, else N would be a double. By Lemma 1.4, $b \leq a$. By Lemma 1.2 we know $N = H_r(a, b) = H_r(a - F_r, b + F_{r-1})$. Hence $a \leq F_r$, else N would be a double. Therefore (i) holds.

Next suppose N is a double. Then by (2.15), (2.16) and (2.17), $c = a - F_r, d = b + F_{r-1}, 1 \leq a, b, c, d$. By Lemma 2.20, $a \leq 2F_r$. Since $c = a - F_r$, this implies $c \leq F_r$. By Lemma 1.4, since $N = H_r(c, d), d \leq c$. Hence $d \leq F_r$. Since $d \leq F_r$ and $d = b + F_{r-1}, b + F_{r-1} \leq F_r$, so that $b \leq F_r - F_{r-1} = F_{r-2}$, i.e. $b \leq F_{r-2}$. Since $0 < b$ and $d = b + F_{r-1}, F_{r-1} < d$. Since $F_{r-1} < d$ and $d \leq c, F_{r-1} < c$. Since $a = c + F_r$, this implies that $F_{r+1} < a$. Hence statement (ii) holds.

Lemma 3.2. *If $R(N) = r$, then there exist unique positive integers x and y satisfying*

$$(3.2) \quad N = H_r(x, y) \quad \text{and} \quad 1 \leq y \leq x \leq F_r.$$

Proof. By Lemma 3.1. If N is a single, then we can let $x = a$ and $y = b$. If N is a double, then we can let $x = c$ and $y = d$ and we will have $\leq y \leq x \leq F_r$. x and y are unique by Theorem 2.22, to the effect that $N = H_r(x, y)$ has at most two solutions. Every N is either a single or a double. Note that if N is a double, then $x = a$ and $y = b$ won't satisfy $1 \leq y \leq x \leq F_r$ since $F_{r+1} < a$.

Lemma 3.3. *Suppose $R(N) = r$. Then all solutions (x, y) of $N = H_r(x, y)$ in positive integers satisfy either*

$$(3.3.1) \quad 1 \leq y \leq x \leq F_r$$

or

$$(3.3.2) \quad F_{r+1} < x \leq 2F_r, \quad 1 \leq y \leq F_{r-2} \quad \text{and} \quad y \leq x.$$

But not both.

Proof. By Theorem 2.22, N is either a double or a single. Hence there are only two cases to consider. If N is a single, then $(x, y) = (a, b)$ and condition (3.3.1) holds by Lemma 3.1. (i). If N is a double, then $(x, y) = (a, b)$ or $(x, y) = (c, d)$. In the first case, by Lemma 3.1 (ii) (3.3.2) holds. In the second case, by Lemma 3.1 (ii) (3.3.1) holds.

Lemma 3.4. *Suppose $R(N) = r$. Then all solutions of $N = H_r(x, y)$ in positive integers (x, y) satisfy the conditions $x \leq 2F_r$ and $y \leq F_r$.*

Proof. By Lemma 3.3, either (3.3.1) holds or (3.3.2) holds. (3.3.1) implies $x \leq F_r \leq 2F_r$ and $y \leq F_r$. (3.3.2) implies $x \leq 2F_r$ and $y \leq F_{r-2} \leq F_r$. Hence $x \leq 2F_r$ and $y \leq F_r$.

Lemma 3.5. *If $0 < k$, then for all positive integers a and b ,*

$$0 < H_k(a, b) < H_{k+1}(a, b).$$

Proof. From the definition it follows that $H_n(a, b)$ is a strictly increasing sequence of positive integers.

Theorem 3.6. *There exist integers x and y such that*

$$(3.6) \quad N = H_n(x, y) \quad \text{and} \quad 1 \leq y \leq x \leq F_n$$

iff $n = R(N)$. Furthermore x and y are unique.

Proof. To prove the first part of the theorem suppose $R(N) = n$. Then by Lemma 3.2 there exist unique integers x and y such that $N = H_n(x, y)$ and $1 \leq y \leq x \leq F_n$, i.e. (3.6). To prove the second part suppose x and y are integers satisfying (3.6). Then $n > 0$. Let $R(N) = r$. Then $n \leq r$. Let $k = r - n$. By definition of $R(N)$ there are positive integers a and b such that $N = H_r(a, b)$. By Lemma 1.3 (ii), since $n = r - k$, we have $N = H_r(a, b) = H_n(H_k(a, b), H_{k+1}(a, b))$ so that $N = H_n(H_k(a, b), H_{k+1}(a, b))$.

Thus $x = H_k(a, b)$ and $y = H_{k+1}(a, b)$ are particular solutions to the linear diophantine equation $N = xF_{n-1} + yF_n$. Since $(F_n, F_{n+1}) = 1$, all solutions to the equation are given by

$$x = H_k(a, b) - tF_n \quad \text{and} \quad y = H_{k+1}(a, b) + tF_{n-1},$$

where t is an integer. Since $y \leq x$, we have for some t the inequality $H_{k+1}(a, b) + tF_{n-1} \leq H_k(a, b) - tF_n$. This implies

$$t \leq (H_k(a, b) - H_{k+1}(a, b))/F_{n+1},$$

so that

$$t \leq H_k(a, b) - H_{k+1}(a, b).$$

Since $x \leq F_n$, we also have the inequality $H_k(a, b) - tF_n \leq F_n$, which implies

$$H_k(a, b)/F_n \leq t + 1.$$

If $0 < k$, then by Lemma 3.5 we have $t < 0$ and $0 < t + 1$ so that $-1 < t < 0$. This is a contradiction since t is an integer. Hence $k = 0$. Thus $r = n$ and hence $R(N) = n$.

Remark. Condition (3.6) cannot be replaced by the weaker condition $N = H_n(x, y)$ and $1 \leq y \leq x$. This condition is not strong enough to imply $n = R(N)$. For example if $N = 96$, then $R(N) = 6$ but $N = H_5(17, 9)$ and $9 \leq 17$. Also $N = H_5(12, 12)$ and $12 \leq 12$.

Theorem 3.7. *Let r be fixed nonnegative integer. Then the number of N such that $R(N) = r$ is exactly*

$$\frac{F_r (F_r + 1)}{2}.$$

Proof. Let r be fixed nonnegative integer. We will use Theorem 3.6 to count the number of N such that $R(N) = r$. We will count pairs (x, y) such that $1 \leq y \leq x \leq F_r$. For each such pair, we put $N = H_r(x, y)$. For each N there is only one pair (x, y) satisfying $N = H_r(x, y)$ and $1 \leq y \leq x \leq F_r$, by Theorem 2.6. How many pairs (x, y) are there such that $1 \leq x \leq F_r$? For each such x , there are x choices of y such that $1 \leq y \leq x$. Hence the number of N such that $R(N) = r$ is given by the sum

$$\sum_{x=1}^{F_r} x = \frac{F_r (F_r + 1)}{2}.$$

Example 3.7. The number of N such that $R(N) = 5$ is $F_5(F_5 + 1)/2 = 5 \cdot 6/2 = 15$. By Corollary 1.20, these 15 N all lie in the interval $8 = F_6 \leq N \leq F_5 F_6 = 40$. They are the 15 values $N = 8, 11, 14, 16, 17, 19, 20, 22, 24, 25, 27, 30, 32, 35$ and 40.

4. Double numbers

In this section we first prove that there are infinitely many double numbers. Then we give a combinatorial formula for the number of double numbers N having a fixed value of R . Last we give an asymptotic estimate for the number of double numbers up to $F_n F_{n+1}$.

Lemma 4.1. *For all $n > 2$, $F_n F_{n+1}$ is a double number.*

Proof. Suppose $2 < n$. Recall that by Corollary 1.10, $R(F_n F_{n+1}) = n$. We have $F_n F_{n+1} = F_n (F_{n-1} + F_n) = F_n F_{n-1} + F_n F_n = H_n(F_n, F_n)$. On the other hand,

$$\begin{aligned} F_n F_{n+1} &= (F_n + F_n)F_{n-1} + (F_n - F_{n-1})F_n \\ &= H_n(2F_n, F_n - F_{n-1}) = H_n(2F_n, F_{n-2}). \end{aligned}$$

$0 < F_{n-2}$ since $n > 2$. The two representations of $F_n F_{n+1}$ are distinct since $F_n \neq F_{n-2}$.

Lemma 4.2. *For $n > 4$, if $N = F_n(F_{n+1} - 1)$, then $R(N) = n$ and N is a double number.*

Proof. By an argument similar to that in the proof of Lemma 4.1 it is easy to see that

$$(4.2) \quad N = H_n(F_n, F_n - 1) = H_n(2F_n, F_{n-2} - 1).$$

To prove that $R(N) = n$ we will use the IVL. Obviously $n \leq R(N)$. Suppose that $n + 1 \leq R(N)$. Then by the IVL there exist $a \geq 1$ and $b \geq 1$ such that $N = H_{n+1}(a, b)$. Hence $F_n(F_{n+1} - 1) = aF_n + bF_{n+1}$. Then $F_n \mid b$, since $(F_n, F_{n+1}) = 1$. Let $b = eF_n$, where $1 \leq e$. Then we have $a + (e - 1)F_{n+1} < 0$, a contradiction. Thus $R(N) = n$.

We give next a formula for the number of double numbers N with a fixed R value r . For this it is necessary first to characterise double numbers. From section 2 we have the following result.

Lemma 4.3. *Suppose $R(N) = r$. Then N is a double number iff*

$$\left\lceil \frac{(-1)^r F_{r-2} N + 1}{F_{r-1}} + 1 \right\rceil = \left\lfloor \frac{(-1)^r F_{r-1} N - 1}{F_r} \right\rfloor.$$

Proof. See the remark following Theorem 2.22 that N is a double iff $\lceil g_r(N) \rceil + 1 = \lfloor h_r(N) \rfloor$.

Theorem 4.4. *N is a double number and $R(N) = r$ iff there exist unique positive integers x and y such that*

$$(4.4) \quad N = H_r(x, y) \quad \text{and} \quad F_{r-1} < y \leq x \leq F_r.$$

Proof. For the proof of one part of the theorem, suppose N is a double number and $R(N) = r$. By Lemma 3.1, there exist positive integers c and

d such that $N = H_r(c, d)$ and $F_{r-1} < d \leq c \leq F_r$. Let $x = c$ and $y = d$. Then (4.4) holds. Also since the condition $F_{r-1} < y \leq x \leq F_r$ implies $1 < y \leq x \leq F_r$, x and y are unique by Lemma 3.1. For the proof of the second part, suppose (4.4) for some positive integers x and y . Then since $1 \leq r$, $1 \leq y \leq x \leq F_r$. Hence $R(N) = r$ by Theorem 3.6. N cannot be a single since in that case, by Lemma 3.1, we would have $x = a, y = b$ and $b \leq F_{r-1}$. Hence N is a double.

Note that if (x, y) satisfies $F_{r-1} < y \leq x \leq F_r$, then $(x + F_r, y - F_{r-1})$ satisfies $F_{r+1} < x \leq 2F_r$ and $1 \leq y \leq F_{r-2}$. Also if (x, y) satisfies $F_{r+1} < x \leq 2F_r$ and $1 \leq y \leq F_{r-2}$, then $(x - F_r, y + F_{r-1})$ satisfies $F_{r-1} < y \leq x \leq F_r$. So one could also prove a version of Theorem 4.4, with condition (4.4) replaced by

$$N = H_r(x, y), \quad F_{r+1} < x \leq 2F_r \quad \text{and} \quad 1 \leq y \leq F_{r-2}.$$

Theorem 4.5. *Let $r \geq 3$. The number of N such that N is a double number and $R(N) = r$ is exactly*

$$\frac{F_{r-2}(F_{r-2} + 1)}{2}.$$

Proof. Suppose r is a fixed positive integer. To count the number of double numbers N such that $R(N) = r$ we will use representation (4.4) of Theorem 4.4. We can determine the number of double numbers N such that $R(N) = r$ by counting pairs of integers (x, y) such that $F_{r-1} < y \leq x \leq F_r$. For each such pair (x, y) we can let $N = H_r(x, y)$ since N depends uniquely on (x, y) . How many pairs of integers (x, y) are there such that $F_{r-1} < y \leq x \leq F_r$? Since $F_r - F_{r-1} = F_{r-2}$, there are F_{r-2} choices for x such that $F_{r-1} < x \leq F_r$. For each choice of x , there are x choices for y such that $F_{r-1} < y \leq x$. Therefore the numbers N such that $R(N) = r$ is given by the sum

$$\sum_{x=1}^{F_{r-2}} x = \frac{F_{r-2}(F_{r-2} + 1)}{2}.$$

Example. The number of N such that N is a double and $R(N) = 6$ is $F_4(F_4 + 1)/2 = 3 \cdot 4/2 = 6$. By Corollary 1.20 and Theorem 2.23 with $n = 5$ these N lie in the interval $18 = 5 \cdot 8 + 5^2 + 13 = F_5 F_6 + F_5^2 + F_1 \leq N \leq F_6 F_7 = 8 \cdot 13 = 104$. They are $N = 78, 83, 88, 91, 96$ and 104 .

Lemma 4.6. *For all double numbers $N, N \leq F_n F_{n+1}$ iff $R(N) \leq n$.*

Proof. The first part of the lemma is the contrapositive of Lemma 1.11, if $R(N) \leq n$ then $N \leq F_n F_{n+1}$. For the proof of the second part

suppose N is a double and $N \leq F_n F_{n+1}$. Let $r = R(N)$. We will that $r \leq n$. Suppose not. Suppose $n < r$. Let $N = H_r(a, b)$ where a are as in (2.16). By Lemma 3.1 (ii), since N is a double, $F_{r+1} < a$. $N = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r+1}F_{r-1} + F_r > F_{n+2}F_n \geq F_n$ contradicting $N \leq F_n F_{n+1}$. Therefore $r \leq n$.

Theorem 4.7. *For $n \geq 1$, the number of double numbers $N \leq F_r$ is equal to*

$$\frac{F_{n-1}F_{n-2} + F_n - 1}{2}.$$

Proof. By Lemma 4.6 and Theorem 4.5, the number of double numbers $N \leq F_n F_{n+1}$ is

$$\begin{aligned} \sum_{r=3}^n \frac{F_{r-2}(F_{r-2} + 1)}{2} &= \frac{1}{2} \sum_{r=3}^n (F_{r-2}^2 + F_{r-2}) \\ &= \frac{1}{2} \left(\sum_{i=1}^{n-2} F_i^2 + \sum_{i=1}^{n-2} F_i \right) = \frac{1}{2} (F_{n-2}F_{n-1} + F_n - 1). \end{aligned}$$

What proportion of integers N are double numbers? We shall show on average approximately 7.3% of numbers are doubles. We shall show by proving that for n sufficiently large, approximately $\beta^4/2$ of the numbers N up to $F_n F_{n+1}$ are doubles. Here $\beta = (1 - \sqrt{5})/2 = -61803 \dots$ so $\beta^4/2 = .072949016 \dots$

Theorem 4.8. *The probability that N is a double number is asymptotic to $\beta^4/2$.*

Proof. Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Then $\alpha\beta = -1$. It is known that F_n is asymptotic to $\alpha^n/\sqrt{5}$, i.e. that $\lim F_n/\alpha^n \approx 1/\sqrt{5}$. By Lemma 4.6 and Theorem 4.7, the number of double numbers $N \leq F_n F_{n+1}$, divided by the number of N up to $F_n F_{n+1}$ is equal to

$$\begin{aligned} (F_{n-1}F_{n-2} + F_n - 1)/2F_n F_{n+1} &\approx F_{n-1}F_{n-2}/2F_n F_{n+1} \\ &\approx \left((\alpha^{n-1}/\sqrt{5})(\alpha^{n-2}/\sqrt{5}) \right) / \left(2(\alpha^n/\sqrt{5})(\alpha^{n+1}/\sqrt{5}) \right) \\ &= \alpha^{n-1}\alpha^{n-2}/2\alpha^n\alpha^{n+1} = 1/2\alpha^4 = \beta^4/2. \end{aligned}$$

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