

EXPONENTIAL STABILITY OF REGULAR LINEAR SYSTEMS ON BANACH SPACES

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Abstract: The article deals with the vd-transformation in Banach space and its application in studying the stability of trivial solution of differential equations. A sufficient condition of exponential stability of regular linear systems with burfication on Banach space will be proved.

vd-transformation and it's properties

In this section we shall give the definition, examples and some properties of a vd-transformation on Banach spaces. It is an expansion of a vd-transformation on finite dimension spaces given by Yu. S. Bogdanov ([2]–[6]). From that, we shall give the definition of regular linear equations which are applied to study the stability of regular linear equations with burfication on Banach spaces.

Let E be a Banach space and G be an open simple connected domain containing the origin O of E .

We define H as follows $H = G \times \mathbf{R} = \{\eta = (x, t) : x \in G, \quad t \in \mathbf{R}\}$.

Let $v_0: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be a function which is continuous, monotone strictly increasing and satisfies the following conditions:

$$v_0(0) = 0; \quad v_0(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$

Let $d: \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}$ be a given real function of two variables: and d satisfies the following conditions for all $\gamma > 0, \gamma_3 > \gamma_2 > \gamma_1 > 0$;

$$(d_1) \quad d(\gamma_2, \gamma_1) = -d(\gamma_1, \gamma_2),$$

$$(d_2) \quad d(\gamma_2, \gamma) > d(\gamma_1, \gamma),$$

$$(d_3) \quad d(\gamma_3, \gamma_2) + d(\gamma_2, \gamma_1) \geq d(\gamma_3, \gamma_1),$$

$$(d_4) \quad \cup_{\gamma \in \mathbf{R}^+} \{d(\gamma, \gamma_1)\} = \mathbf{R}.$$

Suppose that, $l: H \rightarrow H$ is a diffeomorphism,

$$\eta = (x, t) \mapsto \eta' = (x', t')$$

satisfying the following equalities:

$$l(0, t) = (0, t), \quad l(x, t) = (x', t')$$

for all $t \in \mathbf{R}$. It is easy to prove that the set L of all those transformations $L = \{l\}$ is a group with the composition of maps.

Let v be a real function

$$v: H^* \rightarrow \mathbf{R}_+, \quad \eta = (x, t) \rightarrow v(\eta) = v_0(\|x\|)$$

where $H^* = G^* \times \mathbf{R} = (G \setminus \{0\}) \times \mathbf{R}$.

Since the function $v: H^* \rightarrow \mathbf{R}_+$ is independent of t , that is, $v(x, t) = v(x, t')$ for all $t, t' \in \mathbf{R}$, we can denote by $v(x)$ the value of $v(x, t)$ for any $x \in G^*$ and $t \in \mathbf{R}$.

Definition. The transformation $l \in L$ is called vd-transformation iff

$$(1) \quad \sup_{\eta \in H^*} |d\{v(\eta), v[l(\eta)]\}| < +\infty$$

From the definition of function d , we also have

$$\sup_{\eta' \in H^*} |d\{v(\eta'), v[l^{-1}(\eta')]\}| < +\infty.$$

Consequently, if we denote by L_{vd} the set of vd-transformation then it is a subgroup of L .

Examples

1. Suppose $v_0(x, t) = \|x\|$, $d_0(\gamma_1, \gamma_2) = \ln \frac{\gamma_1}{\gamma_2}$, and $l(x, t)$ (with a fixed t) is a linear transformation having bounded partial derivation with respect to t . Then, l is $v_0 d_0$ -transformation if and only if it's a Lyapunov transformation ([1]).
2. If $v(x, t) = |x|^2$; $E = \mathbf{R}$

$$d(\gamma_1, \gamma_2) = \begin{cases} \sqrt{\gamma_1} - \sqrt{\gamma_2} & \text{if } \gamma_1 \cdot \gamma_2 \geq 1 \\ \frac{1}{\sqrt{\gamma_2}} - \frac{1}{\sqrt{\gamma_1}} & \text{if } \gamma_1 \cdot \gamma_2 < 1, \end{cases}$$

then all conditions from d_1 are satisfied. So

$$l(x, t) = (x + \frac{1}{2} \sin t \sin^2 x, t)$$

is a vd-transformation.

From example 1, we can see that a vd-transformation is an expansion of Lyapunov transformation, but it still keeps an important property, stability of the trivial solution of a following differential equation on the Banach space E :

$$(2) \quad \begin{cases} \frac{dx}{dt} = f(x, t) \\ f(0, t) \equiv 0 \end{cases}$$

We denote by $x(t; \xi)$ the solution of equation (2) which satisfies the initial condition $x(t_0, \xi) = \xi$ and suppose that

$$\lambda = \lim_{\varepsilon \rightarrow 0^+} \sup_{\substack{\|\xi\| \leq \varepsilon \\ t \geq t_0}} \|x(t; \xi)\|; \lambda_1 = \lim_{\varepsilon \rightarrow 0^+} \sup_{\varepsilon \rightarrow 0^+} \sup_{\substack{v(\xi) \leq \varepsilon \\ t \geq t_0}} v(x(t; \xi)).$$

Definition. ([7]). The solution $x = 0$ of differential equation (2) is said to be Lyapunov stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for each solution $x(t)$ of (2), with its initial value $x(t_0) = \xi$ satisfying the condition $\|\xi\| < \delta(\varepsilon)$

then the inequality $\|x(t, \xi)\| < \varepsilon$ holds for all $t \geq t_0$.

From the definition we can see that the solution $x = 0$ of differential equation (2) is stable iff $\lambda = 0$.

Proposition 1. $\lambda = 0$ if and only if $\lambda_1 = 0$.

Proof. By the continuity of the function v we immediately have $\lim_{\xi \rightarrow 0} v(\xi) = 0$. Since $v(\|x\|)$ is monotone strictly increasing we can deduce $\lim_{v(\xi) \rightarrow 0} \xi = 0$.

Therefore

$$(3) \quad \lim_{k \rightarrow \infty} \xi_k = 0 \Leftrightarrow \lim_{k \rightarrow \infty} v(\xi_k) = 0.$$

We assume that $\lambda = 0$, then:

$$\lim_{k \rightarrow \infty} \|x(t_k, \xi_k)\| = 0$$

for all sequences $\{\varepsilon_k\} \subset \mathbf{R}_+ : \varepsilon_k \rightarrow 0; \{\xi_k\} \subset E : \|\xi_k\| < \varepsilon_k$ and $\{t_k\} \subset \mathbf{R}, t - k \geq t_0$. Because of (3), we have

$$\lim_{k \rightarrow \infty} \|x(t_k, \xi_k)\| = 0 \Leftrightarrow \lim_{k \rightarrow \infty} v(x(t_k, \xi_k)) = 0.$$

It follows that $\lambda = 0 \Leftrightarrow \lambda_1 = 0$.

Proposition 2. A vd -transformation conserves the stability of trivial solution $x = 0$ of differential equation (2).

Proof. By the vd -transformation

$$(x, t) \rightarrow l(x, t) = (y, t),$$

the equation (2) is transformed to the following one:

$$(4) \quad \frac{dy}{dt} = g(y, t)$$

By assumption, the solution $x = 0$ of equation (2) is stable, that means:

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{\substack{\|x_0\| \leq \varepsilon \\ t \geq t_0}} \|x(t; x_0)\| = 0 \Leftrightarrow \lim_{\varepsilon \rightarrow 0^+} \sup_{\substack{v(x_0) \leq \varepsilon \\ t \geq t_0}} v[x(t; x_0)] = 0.$$

If the solution $y = 0$ of (4) is unstable, then

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{\substack{v(y_0) \leq \varepsilon \\ t \geq t_0}} v[y(t; y_0)] > 0.$$

It means that there exists a positive number δ such that

$$(5) \quad \exists \{\eta_n\} \subset E: \eta_n \rightarrow 0; \exists \{t_n\} \subset \mathbf{R}_+: t_n \geq t_0; \forall n \in N: v[y(t_n; \eta_n)] \geq \delta.$$

Since $v[x(t_n, \xi_n)] \rightarrow 0$ as $n \rightarrow \infty$, where $\xi_n, t_n = l^{-1}(\eta_n, t_n)$ one could say

$$(6) \quad v[x(t_n; \xi_n)] < \delta, \forall n \in N.$$

From (5), (6) and d_4) we deduce:

$$\begin{aligned} |d\{v[x(t_n; \xi_n)], v[y(t_n; \eta_n)]\}| &= d\{v[y(t_n; \eta_n)], v[x(t_n; \xi_n)]\} \\ &> d\{\delta, v[x(t_n; \xi_n)]\} \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consequently

$$\sup_{n \in N} |d\{v[x(t_n; \xi_n)], v[l(x(t_n; \xi_n))]\}| = +\infty$$

that contradicts the definition of 1.

Regular system

Definition. A transformation $l \in L$, satisfying the following condition for all $\eta \in H^*$:

$$d\{v(\eta), v[l(\eta)]\} = o(t) \quad \text{as } t \rightarrow \pm\infty,$$

is called a generalized vd-transformation.

Definition. A transformation $y = L(t)x$ is a generalized Lyapunov one if:

$$(7) \quad \chi[L(t)] = \chi[L^{-1}(t)] = 0$$

where $\chi[L(t)] := \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln \|L(t)\|$ is called characteristic exponent of $L(t)$.

By definition we immediately have following remarks:

Remark 1. Generalized Lyapunov transformations conserve Lyapunov exponents [1].

Remark 2. A generalized Lyapunov transformation is generalized vd-transformation when

$$v(x) = \|x\|, d(\gamma_1, \gamma_2) = \ln \frac{\gamma_1}{\gamma_2},$$

and l is homogeneously linear for x (where $l(x, t) = (L(t)x, t)$).

Now we shall prove a necessary and sufficient condition for which a differential system on finite dimension spaces are regular. Since this condition plays an important role for the conception of a regular differential equations on Banach spaces and we could not find it in literature, we shall formulate it as a lemma.

We consider the following linear differential system:

$$(8) \quad \frac{dx}{dt} = A(t)x$$

where $x \in \mathbf{R}^n$, $A(t) \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and is real continuous for all $t \in \mathbf{R}$ and $\sup \|A(t)\| < \infty$.

Let $X(t)$ be a normal fundamental matrix of (8) and $\sigma_x = \sum_{k=1}^m n_k \alpha_k$ be the sum of all its exponent numbers ([1]).

Definition. ([1]) The linear system (8) is said to be regular iff

$$\sigma_x = \varliminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \text{Sp } A(\tau) d\tau.$$

Lemma. *A necessary and sufficient condition that the system (8) to be regular one is there exists a generalized Lyapunov transformation carrying the system (8) to the system with constant matrix $B \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$:*

$$(9) \quad \frac{dy}{dt} = By$$

Proof. Let $y = L(t)x$ be a generalized Lyapunov transformation, $X(t)$ be a normal fundamental matrix of system (8). It follows that $Y(t) = L(t)X(t)$ is a fundamental matrix of system (9). Since

$$\det Y(t) = \det L(t) \det X(t),$$

we have

$$\begin{aligned}
 \det Y(t_0) \exp(t - t_0) \operatorname{Sp} B &= \det L(t) \det X(t_0) \exp \int_{t_0}^t \operatorname{Sp} A(t_1) dt_1 \\
 \exp \int_{t_0}^t \operatorname{Sp} A(t_1) dt_1 &= |c(t_0)| |\det L^{-1}(t)| \exp[(t - t_0) \operatorname{Sp} B], \\
 \text{where } c(t_0) &= \det [Y(t_0) X^{-1}(t_0)], \\
 \Rightarrow \frac{1}{t} \int_{t_0}^t \operatorname{Sp} A(t_1) dt_1 &= \frac{1}{t} \ln |c(t_0)| + \frac{1}{t} \ln |\det L^{-1}(t)| + \left(1 - \frac{t_0}{t}\right) \operatorname{Sp} B \\
 \Rightarrow \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \operatorname{Sp} A(t_1) dt_1 &= \operatorname{Sp} B + \chi [\det L^{-1}(t)].
 \end{aligned}$$

Because of $\chi [L^{-1}(t)] = 0$ we have

$$\chi [\det L^{-1}(t)] \leq n \chi [L^{-1}(t)] = 0$$

Analogously, from $\chi [L(t)] = 0$ it follows that

$$\chi [\det L(t)] \leq 0$$

On the other hand, since

$$\det L(t) \cdot \det L^{-1}(t) = 0,$$

the following is held: $\chi [\det L(t)] + \chi [\det L^{-1}(t)] \geq 0$.

$$\text{Therefore } \chi [\det L(t)] = \chi [\det L^{-1}(t)] = 0$$

It follows from these equalities that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln |\det L^{-1}(t)| = 0$$

and finally

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \operatorname{Sp} A(t_1) dt_1 = \operatorname{Sp} B.$$

Since the Lyapunov transformation conserves Lyapunov exponents and the X is normal, Y is normal too and

$$\sigma_x = \sigma_y = \operatorname{Sp} B,$$

we have

$$\sigma_x = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \operatorname{Sp} A(t_1) dt_1,$$

i.e. the system (8) is regular.

Conversely, let the system (8) be regular. We will denote by $X(t)$ the fundamental normal matrix of (8), which has the exponent numbers: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Consider the Jordan matrix B , in which $\lambda_1, \dots, \lambda_n$ are elements on the diagonal.

Denoting $Y(t)$ the fundamental normal matrix of the system (9), we constate that the column of which has the same exponent numbers (with the same order): $\lambda_1, \lambda_2, \dots, \lambda_n$.

Putting $L(t) = Y(t)X^{-1}(t)$ we will prove that $y = L(t)x$ is a generalized Lyapunov transformation.

Suppose that

$$Y(t) = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \dots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \dots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \dots & y_{nn}(t) \end{pmatrix}$$

$$X^{-1}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{pmatrix}.$$

Tdilehen $\chi[y^{(k)}] = \lambda_k$, where $y^{(k)}(t) = \text{colon}(y_{1k}(t) \dots y_{nk}(t))$.

Because of the regularity of (8), we have $\chi[x^{(k)}(t)] = -\lambda_k$, where $x^{(k)}(t) = (x_{k1}(t) \dots x_{kn}(t))$.

We consider now the diagonal matrix

$$\Delta = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

We find then

$$L(t) = Y(t)e^{-t\Delta}e^{t\Delta}X^{-1}(t) = \phi(t)\Psi(t)$$

in which $\phi(t) = Y(t)e^{-t\Delta}$, $\Psi(t) = e^{t\Delta}X^{-1}(t)$. It follows that

$$\chi[\phi(t)] = \max_{j,k} \chi[y_{jk}(t)e^{-\lambda_k t}] = 0$$

$$\chi[\Psi(t)] = \max_{j,k} \chi[x_{j,k}(t)e^{\lambda_j t}] = 0.$$

Consequently,

$$\chi[L(t)] \leq \chi[\phi(t)] + \chi[\Psi(t)] = 0$$

Analogously we can prove that $\chi[L^{-1}(t)] \leq 0$.

However, from $L(t) \cdot L^{-1}(t) = E$, we immediately find that $\chi[L(t)] + \chi[L^{-1}(t)] \geq 0$, i.e. $\chi[L(t)] = \chi[L^{-1}(t)] = 0$. The lemma is proved.

Definition. A linear differential equation:

$$(10) \quad \frac{dx}{dt} = A(t)x,$$

where $A(t) \in \mathcal{L}(E, E)$ and is continuous for all $t \in \mathbf{R}$ and $\sup_t \|A(t)\| < \infty$, is said to be regular one iff there is a generalized Lyapunov transformation $y = L(t)x$ carrying which to the linear differential equation with constant operator:

$$(11) \quad \frac{dy}{dt} = By.$$

Now we shall give a main theorem to regular differential equations on Banach spaces.

Let consider differential equation

$$(12) \quad \frac{dx}{dt} = A(t)x + f(x, t),$$

where $A(t) \in \mathcal{L}(E, E)$ and $\sup_{t \in \mathbf{R}} \|A(t)\| < \infty$, $f \in C^{(1,0)}(E \times \mathbf{R})$, $f(0, t) \equiv 0$,

$$\|f(x, t)\| \leq \Psi(t)\|x\|^m \quad (m > 1); \quad \chi[\Psi(t)] = 0.$$

Under these conditions, we show the following theorem:

Theorem. *If the equation (10) is regular and all its characteristic exponents are not larger than $-\lambda < 0$, the trivial solution $x = 0$ of the equation (10) is exponential stability ([7]). I.e there exist $N > 0, A > 0$ such that*

$$\|x(t)\| \leq Ae^{-N(t-t_0)}\|x(t_0)\|$$

for all solutions $x(t)$ of (12).

Proof. We denote by $X(t)(X(t_0) = Id_E)$ its Cauchy operator of equation (10) ([7], p. 147).

1. First we will estimate the resolvent operator $K(t, \tau) = X(t)X^{-1}(\tau)$ ($t_0 \leq \tau \leq t$).

Because of the regularity of the equation (10), there is a generalized Lyapunov transformation $y = L(t)x$ carrying equation (10) to equation (11).

We have $Y(t) = L(t)X(t)$ is resolvent operator of the equation (11).

If we put $H(t, \tau) = Y(t)Y^{-1}(\tau)$ then $K(t, \tau) = L(t)H(t, \tau)L^{-1}(\tau)$.

Suppose that all characteristic exponents of the equation (10) are not larger than α .

Hence all those of the equation (11) are not too than α , that is for every solution $y(t) = Y(t)y_0$ and $\varepsilon > 0$ there exists $c > 0$ we have

$$\|y(t)\| \leq ce^{(\alpha+\varepsilon/2)t}, \quad \forall t \geq t_0.$$

Then, the operator's family $\{e^{-(\alpha+\varepsilon/2)t}Y(t), t \geq t_0\}$ is point-bounded.

By virtue of the Banach–Steinhaus there exists $c_1 > 0$ such that:

$$\|e^{-(\alpha+\varepsilon/2)t}Y(t)\| \leq c_1 \Leftrightarrow \|Y(t)\| \leq c_1 e^{(\alpha+\varepsilon/2)t}.$$

Therefore $\|H(t, \tau)\| = \|Y(t-\tau)\| \leq c_1 e^{(\alpha_\varepsilon/2)(t-\tau)}$ for the equation with constant operator (11)

On the other hand

$$\chi[L(t)] = \chi[L^{-1}(t)] = 0 \Leftrightarrow \begin{cases} \|L(t)\| \leq c_2 e^{\frac{\varepsilon}{2}t} \\ \|L^{-1}(\tau)\| \leq c_3 e^{\frac{\varepsilon}{2}\tau}. \end{cases}$$

It follows that

$$\begin{aligned} \|K(t, \tau)\| &\leq \|L(t)\| \|H(t, \tau)\| \|L^{-1}(\tau)\| \\ (13) \quad &\leq c_1 c_2 c_3 e^{(\alpha+\varepsilon)(t-\tau)} e^{\varepsilon\tau} = c(\varepsilon, t_0) e^{(\alpha+\varepsilon)(t-\tau)} \end{aligned}$$

where $c = c_1 c_2 c_3 e^{-(\alpha+\varepsilon)\tau}$.

Since $K(t, t_0) = X(t)$ we have

$$(14) \quad \|X(t)\| \leq ce^{(\alpha+\varepsilon)t}$$

In the case, when $\alpha < 0$, there exists a positive number ε such that $\alpha + \varepsilon \leq 0$, whence

$$(15) \quad \|K(t, \tau)\| \leq ce^{\varepsilon t}, \quad \|X(t)\| \leq c.$$

2. We will now prove the theorem. Denoting

$$(16) \quad y = \chi e^{\gamma(t-t_0)}$$

where γ is a positive number such that $0 < \gamma < \lambda$, the equation (12) will be transformed to:

$$(17) \quad \frac{dy}{dt} = B(t)y + g(t, y)$$

with $B(t) = A(t) + \gamma Id_E$

$$(18) \quad g(t, y) = \exp(\gamma(t - t_0))f\left(t, ye^{-\gamma(t-t_0)}\right).$$

Now we show that the equation

$$(19) \quad \frac{d\eta}{dt} = B(t)\eta$$

is regular. Indeed, by the regularity of (10) there is a generalized Lyapunov transformation $z = L(t)x$ carrying (10) to the equation with constant operator:

$$\frac{dz}{dt} = Cz$$

where

$$C = L'(t)L^{-1}(t) + L(t)A(t)L^{-1}(t).$$

The transformation $\xi = L(t)\eta$ implies the following:

$$\frac{d\xi}{dt} = [L'(t)L^{-1}(t) + L(t)B(t)L^{-1}(t)]\xi = (C + \gamma Id_E)\xi.$$

The regularity of (19) is proved.

We denote by $\eta(t)$ the solution of (19) and then $e^{-\gamma(t-t_0)}\eta(t)$ is the solution of (10).

This implies:

$$\begin{aligned} & \chi \left[\eta(t)e^{-\gamma(t-t_0)} \right] \leq -\lambda \\ \Rightarrow & \chi [\eta(t)] \leq \chi \left[e^{\gamma(t-t_0)} \right] + \chi \left[\eta(t)e^{-\gamma(t-t_0)} \right] \leq -\lambda + \gamma < 0. \end{aligned}$$

By virtue of the estimation of the resolvent operator the following inequality is true:

$$\|K(t, \tau)\| \leq Ne^{\varepsilon\tau}; \quad t_0 \leq \tau < \infty,$$

where $K(t, \tau)$ is the resolvent operator of (10).

Now consider the solution of (17)

$$y(t) = K(t, t_0)y(t_0) + \int_{t_0}^t K(t, \tau)g(\tau, y(\tau))d\tau,$$

we have

$$\begin{aligned} \|y(t)\| &\leq \|K(t, t_0)\| \cdot \|y(t_0)\| + \int_{t_0}^t \|K(t, \tau)\| \cdot \|g(\tau, y(\tau))\| d\tau \\ &\leq Ne^{\varepsilon(t_0)} \|y(t_0)\| + \int_{t_0}^t Ne^{\varepsilon\tau} e^{\gamma(\tau-t_0)} \Psi(\tau) \|y(\tau)\|^m e^{-m\gamma(\tau-t_0)} d\tau \\ &\leq Ne^{\varepsilon t_0} \|y(t_0)\| + \int_{t_0}^t Ne^{\varepsilon\tau} e^{(1-m)\gamma(\tau-t_0)} c e^{\varepsilon\tau} \|y(\tau)\|^m d\tau \\ &= c_1 \|y(t_0)\| + \int_{t_0}^t c_2 e^{[2\varepsilon-(m-1)\gamma](\tau-t_0)} \|y(\tau)\|^m d\tau \end{aligned}$$

where $c_1 = Ne^{\varepsilon t_0}$, $c_2 = cNe^{-2\varepsilon t_0}$.

Hence

$$(20) \quad \|y(t)\| \leq c_1 \|y(t_0)\| + \int_{t_0}^t c_2 e^{-\delta(\tau-t_0)} \|y(\tau)\|^m d\tau,$$

where $\delta = (m-1)\gamma - 2\varepsilon$.

We will find the positive number ε such that $\delta > 0$.

Since

$$\int_{t_0}^t e^{-\delta(\tau-t_0)} d\tau = \frac{1}{\delta} - \frac{1}{\delta} e^{-\delta(t-t_0)} < \frac{1}{\delta}$$

there is $\Delta > 0$ such that

$$N = (m-1)c_1^{m-1}\|y(t_0)\|^{m-1} \int_{t_0}^t c_2 e^{-\delta(\tau-t_0)} d\tau < 1$$

provided that

$$\|y(t_0)\| < \Delta.$$

We apply here the lemma of Bihari [8] and find

$$\|y(t)\| \leq \frac{c_1\|y(t_0)\|}{[1-N]^{\frac{1}{m-1}}} = A\|y(t_0)\|, A = \frac{c_1}{[1-N]^{\frac{1}{m-1}}}$$

$$\Rightarrow \|x(t)\| \leq Ae^{-\gamma(t-t_0)}\|x(t_0)\|, (x(t_0) = y(t_0))$$

that is the exponential stability of the solution $x = 0$ of (12), and the proof of the theorem is finished.

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