Acta Acad. Paed. Agriensis, Sectio Mathematicae **26** *(1999) 81–86*

ON THE SHAPE MODIFICATION OF PARAMETRIC CUBIC ARCS

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Abstract: A standard specification of cubic parametric arcs is the Hermite form, when the arc is given by its end-points and tangent vectors (derivatives with respect to the parameter) at them. At first, we examine how the points of an arc change their positions if we scale the end-tangents, then we show how one can achieve prescribed shape modification by means of the alteration of the length of the end-tangents or the parameter range. By prescribed shape modification we mean such an alteration when a chosen point of the arc is carried into a predefined point.

1. Introduction

By cubic parametric arcs we mean arcs of the form

$$\mathbf{r}(u) = \mathbf{r}_3 u^3 + \mathbf{r}_2 u^2 + \mathbf{r}_1 u + \mathbf{r}_0,\ u \in [u_0, u_1] \subset \Re,$$
$$\mathbf{r}_i \in \Re^d,\ (d = 2, 3;\ i = 0, \dots, 3).$$

Cubic parametric curves play an important role in computer aided geometric design (CAGD), since this is the lowest degree polynomial curve by means of which one can describe twisted curves and plane curves with singularity, such as inflection, cusp or loop. There are numerous publications on parametric cubics. In [1] and [3] there is a summary of their different representations and properties, whereas [4] deals with the detection of singularity.

One of the most well-known specifications of parametric cubic arcs is the so called Hermite form, when the arc is specified by its end-points and tangent vectors (derivatives with respect to the parameter) at them. In general, the parameter range is $[0, 1]$. The change of the parametrization $t \in [0, 1]$ to $v \in [a, b]$, i.e. the affine parameter transformation $t(v) = (v - a) / (b - a)$, modifies the shape of the arc, if the end-conditions are unchanged. This shape modification is equivalent to the one, gained by the uniform scaling of end-tangents by the factor $1/(b - a)$. In the forthcoming sections we study shape modifications obtained by the scaling of the end-tangents.

2. Changing the length of end-tangents

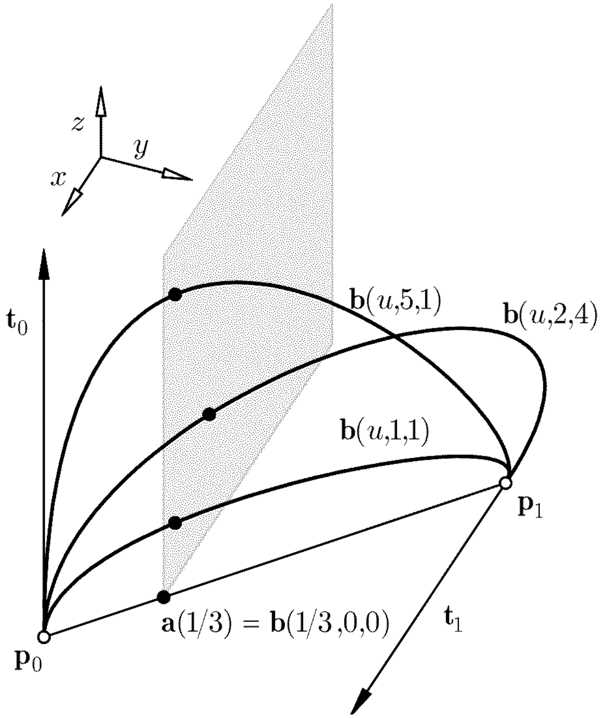

Figure 1: The $\mathbf{b}(u, \lambda_0, \lambda_1)$ arcs

We assume that the arc is given by its end-points $\mathbf{p}_0, \mathbf{p}_1$ and end-tangents $\mathbf{t}_0, \mathbf{t}_1$ moreover, the parameter range is $[0, 1]$. The Bézier points (cf. [2]) of the arcs to be studied are

$$\mathbf{b}_0 = \mathbf{p}_0,\ \mathbf{b}_1 = \mathbf{p}_0 + \frac{\lambda_0}{3}\mathbf{t}_0,\ \mathbf{b}_2 = \mathbf{p}_1 - \frac{\lambda_1}{3}\mathbf{t}_1,\ \mathbf{b}_3 = \mathbf{p}_1,$$
$$\lambda_0, \lambda_1 \in \Re$$

and their Bézier representation is

$$\mathbf{b}(u, \lambda_0, \lambda_1) = \sum_{i=0}^{3} \mathbf{b}_i B_i^3(u) = \mathbf{p}_0 \left(B_0^3(u) + B_1^3(u)\right) +$$
$$\mathbf{p}_1 \left(B_2^3(u) + B_3^3(u)\right) + \frac{\lambda_0}{3}\mathbf{t}_0 B_1^3(u) - \frac{\lambda_1}{3}\mathbf{t}_1 B_2^3(u),$$
$$u \in [0, 1],\ \lambda_0, \lambda_1 \in \Re,$$

where $B_i^3(u) = \binom{3}{i} u^i (1-u)^{3-i}, (i = 0, \ldots, 3)$ is the i^{th} cubic Bernstein polynomial. Introducing the notation

$$\mathbf{a}(u) = \mathbf{p}_0 \left(B_0^3(u) + B_1^3(u)\right) + \mathbf{p}_1 \left(B_2^3(u) + B_3^3(u)\right)$$

we obtain

$$\mathbf{b}(u, \lambda_0, \lambda_1) = \mathbf{a}(u) + \frac{\lambda_0}{3} \mathbf{t}_0 B_1^3(u) - \frac{\lambda_1}{3} \mathbf{t}_1 B_2^3(u) \tag{1}$$

where $\mathbf{a}(u)$ is the convex linear combination of the points $\mathbf{p}_0, \mathbf{p}_1$, since

$$B_0^3(u) + B_1^3(u) = 1 - \left(B_2^3(u) + B_3^3(u)\right)$$

consequently, $\mathbf{a}(u)$ describes the straight line segment bounded by $\mathbf{p}_0$ and $\mathbf{p}_1$.

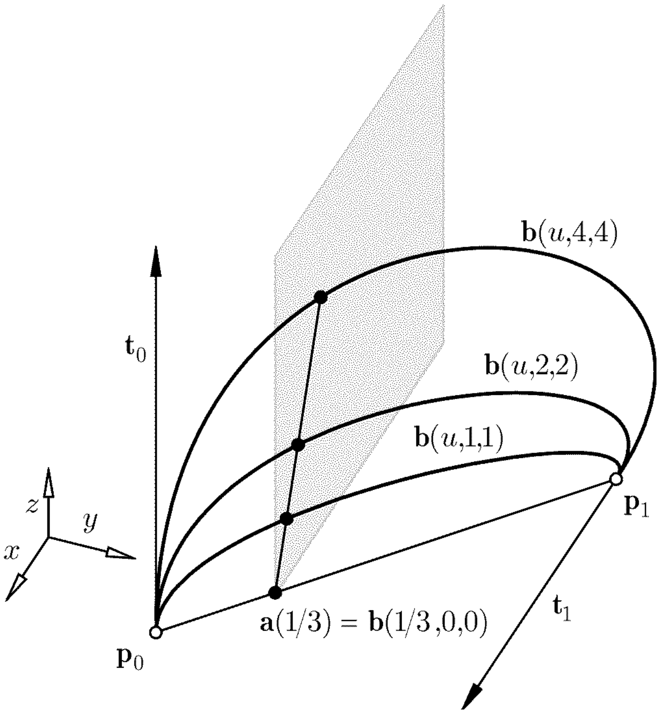

Figure 2: The $\mathbf{b}(u, \lambda, \lambda)$ family of curves

Now, we study the point of the arc that corresponds to an arbitrarily chosen parameter value $\tilde{u} \in (0, 1)$. Based on equation (1), one can see that the points $\mathbf{b}(\tilde{u}, \lambda_0, \lambda_1)$ of the arc are in the plane which is parallel to the end-tangents $\mathbf{t}_0, \mathbf{t}_1$ and intersects the straight line segment $\mathbf{p}_0, \mathbf{p}_1$ at the point $\mathbf{a}(\tilde{u})$ (see Figure 1). In the $\mathbf{t}_0 \parallel \mathbf{t}_1$ special case the points $\mathbf{b}(\tilde{u}, \lambda_0, \lambda_1)$ are on the line that is parallel to the vector $\mathbf{t}_0$ (or $\mathbf{t}_1$) and passes through the point $\mathbf{a}(\tilde{u})$.

If we multiply the end-tangents by the same factor, i.e. $\lambda = \lambda_0 = \lambda_1$ then that part of equation (1) which depends on λ is

$$\frac{\lambda}{3}\left(\mathbf{t}_0 B_1^3(u) - \mathbf{t}_1 B_2^3(u)\right) = \lambda u (1-u)\left((1-u)\mathbf{t}_0 - u\mathbf{t}_1\right),$$

thus the direction of the joining line of $\mathbf{b}(u,\lambda,\lambda)$ and $\mathbf{a}(u)$ is a convex linear combination of the vectors $\mathbf{t}_0$ and $-\mathbf{t}_1$. For an arbitrarily chosen parameter value $\tilde{u} \in (0,1)$ the curve points $\mathbf{b}(\tilde{u},\lambda,\lambda)$ are on a straight line which intersects the straight line segment $\mathbf{p}_0, \mathbf{p}_1$ and parallel to the plane direction determined by $\mathbf{t}_0$ and $\mathbf{t}_1$. (The direction of such lines ranges from $\mathbf{t}_0$ to $\mathbf{t}_1$, cf. Figure 2.) The same effect can be obtained by means of an affine parameter transformation, since the $[0,1] \longrightarrow [a,b]$ transformation results in the uniform scaling of end-tangents by the scaling factor $\lambda = 1/(b-a)$.

3. Prescribed shape modification

Given a parametric cubic arc $\mathbf{r}(u), u \in [a,b]$ by its end-points $\mathbf{p}_0$,$\mathbf{p}_1$ and end-tangents $\mathbf{t}_0$,$\mathbf{t}_1$. By the alteration of the length of end-tangents, we want to modify the shape of the arc in such a way, that its arbitrarily chosen point $\mathbf{r}(\tilde{u}), \tilde{u} \in (a,b)$ should be carried into the predefined point $\mathbf{p}$, i.e. $\mathbf{p} = \mathbf{r}(\tilde{u},\lambda_0,\lambda_1)$ is the expected result.

i) End-tangents are not parallel

If $\mathbf{t}_0$ is not parallel to $\mathbf{t}_1$ a solution to the problem above exists if and only if the point $\mathbf{p}$ is in the plane which is parallel to $\mathbf{t}_0$ and $\mathbf{t}_1$, and passes through the point $\mathbf{a}(\tilde{u})$. (Certainly, this condition is always fulfilled in case of plane curves.) The corresponding values of λ_0 and λ_1 can be obtained from the coordinates of $\mathbf{p}$ in the affine coordinate system $\mathbf{a}(\tilde{u}), \mathbf{t}_0, \mathbf{t}_1$. We denote these coordinates by μ_0 and μ_1, i.e.

$$\mathbf{p} = \mathbf{a}(\tilde{u}) + \mu_0\mathbf{t}_0 + \mu_1\mathbf{t}_1.$$

Comparing this with equation (1) we obtain

$$\lambda_0 = \frac{3\mu_0}{B_1^3(\tilde{u})} \text{ and } \lambda_1 = \frac{-3\mu_1}{B_2^3(\tilde{u})}.$$

If instead of the constraint $\mathbf{r}(\tilde{u}) \longrightarrow \mathbf{p} = \mathbf{r}(\tilde{u},\lambda_0,\lambda_1)$ we apply a looser condition $\mathbf{r}(u) \longrightarrow \mathbf{p} = \mathbf{r}(u,\lambda_0,\lambda_1)$ for some $u \in (a,b)$, i.e. the modified curve should pass through the point $\mathbf{p}$ at some unknown parameter value u, then we have a solution to the problem if the point $\mathbf{p}$ is between the pair of parallel planes which are parallel to the end-tangents $\mathbf{t}_0$,$\mathbf{t}_1$ and pass through the points $\mathbf{p}_0$ and $\mathbf{p}_1$. In this case we take a plane through the point $\mathbf{p}$ parallel to $\mathbf{t}_0$ and $\mathbf{t}_1$. Let $\tilde{u}$ be the parameter value which corresponds to the intersection point of this plane and the arc. By means of this $\tilde{u}$ the $\mathbf{r}(\tilde{u}) \longrightarrow \mathbf{p} = \mathbf{r}(\tilde{u},\lambda_0,\lambda_1)$ shape modification is always feasible.

In case of plane curves, this looser constraint provides a free parameter, that enables us to fulfill an additional condition, such as tangent direction or curvature.

If $\mathbf{t}_0$ is not parallel to $\mathbf{t}_1$ and we apply the restriction $\lambda = \lambda_0 = \lambda_1$, there can only be a solution to the task $\mathbf{r}(\tilde{u}) \longrightarrow \mathbf{p} = \mathbf{r}(\tilde{u}, \lambda, \lambda)$ if the point $\mathbf{p}$ is on the joining line of $\mathbf{a}(\tilde{u})$ and $\mathbf{r}(\tilde{u})$. This shape modification can also be obtained by an affine parameter transformation.

The $\mathbf{r}(u) \longrightarrow \mathbf{p} = \mathbf{r}(u, \lambda, \lambda), u \in (a, b)$ shape modification can have a solution if the points $\mathbf{a}_0, \mathbf{q}, \mathbf{p}$ are collinear, where $\mathbf{a}_0$ and $\mathbf{q}$ are the intersection points of the plane determined by $\mathbf{p}, \mathbf{t}_0, \mathbf{t}_1$ with the straight line segment $\mathbf{p}_0, \mathbf{p}_1$ and the arc $\mathbf{r}(u)$ respectively. In the planar case we have to find a u value for which the points $\mathbf{a}_0, \mathbf{q}, \mathbf{p}$ are collinear.

ii) Parallel end-tangents

In the case $\mathbf{t}_0 \parallel \mathbf{t}_1$, the $\mathbf{r}(\tilde{u}) \longrightarrow \mathbf{p} = \mathbf{r}(\tilde{u}, \lambda_0, \lambda_1)$ shape modification is feasible if $\mathbf{p} - \mathbf{r}(\tilde{u}) \parallel \mathbf{t}_0$ (or $\mathbf{t}_1$). The $\mathbf{r}(u) \longrightarrow \mathbf{p} = \mathbf{r}(u, \lambda_0, \lambda_1)$ and $\mathbf{p} = \mathbf{r}(u, \lambda, \lambda), u \in (a, b)$ modifications can always be performed provided, the point $\mathbf{p}$ is between the parallel end-tangent lines.

From the discussions above, we can obtain those special cases as well when only one of the end-tangents can be scaled.

4. Conclusions

In this paper we have examined how the scaling of end-tangents effects the shape of Hermite arcs. Based on these observations, we have detailed the shape modifications of parametric cubic arcs which can be obtained by scaling the end-tangents and by affine parameter transformations.

Note, that the results of section 3 also provide a constructive solution and a solvability criterion to the following interpolation problem:
Given the end-points, end-tangent directions and an additional point (either in plane or in space).
Find a parametric cubic arc that passes through the point and fulfills the end-conditions.

References

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