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A SEHGAL'S PROBLEM

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Abstract: In this paper we generalize the Krull intersection theorem to group rings and given neccessary and sufficient conditions for the intersection theorem to hold for an arbitrary group ring over a commutative integral domain.

1. Introduction.

Let S be a commutative noetherian ring with unity, and let I be an ideal with $I \neq S$. Let $I^\omega = \cap_{n=1}^{\infty} I^n$. The Krull intersection theorem states that if $x \in I^\omega$ then there exists t in I such that $x = xt$.

The object of this paper is to generalize this result to group rings (see [8], Problem 38).

Let R be a commutative ring with unity, G a group and RG its group ring and let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the ring homomorphism $\phi\colon RG \to R$ which maps the group elements to 1. It is easy to see that as an R-module $A(RG)$ is a free module with the elements $g - 1\,(g \in G)$ as a basis. Let

$$A^\omega(RG) = \bigcap_{i=1}^{\infty} A^i(RG).$$

We shall say that the *intersection theorem* holds for $A(RG)$ if there exists an element $a \in A(RG)$ such that $A^\omega(RG)(1 - a) = 0$.

Sufficient conditions for the intersection theorem to hold for certain RG are given in [2], [9], [6]. In the last paper are given the neccessary and sufficient conditions in the cases when G is finitely generated with a nontrivial torsion element and $R = \mathbf{Z}$ the ring of integers, or if G is finitely genereted and $R = \mathbf{Z_p}$ the ring of p-adic integers.

In this paper we given neccessary and sufficient conditions for the intersection theorem to hold for an arbitrary group ring over a commutative integral domain (Theorem 3.1).

2. Notations and some known facts. If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short when it is obvious from the context what ring R we are working with) denotes the ideal of RG generated by all elements of the form $h - 1$,

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($h \in H$). It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi}: RG \to RG/H$ induced by the group homomorphism ϕ of G onto G/H. It is clear that $I(RG) = A(RG)$.

If $\mathcal{K}$ denotes a class of groups (by which we understand that $\mathcal{K}$ contains all groups of order 1 and, with each $H \in \mathcal{K}$ all isomorphic copies of H) we define the class $R\mathcal{K}$ of residually-$\mathcal{K}$ groups by letting $G \in R\mathcal{K}$ if and only if: whenever $1 \neq g \in G$, there exists a normal subgroup H_g of the group G such that $G/H_g \in \mathcal{K}$ and $g \notin H_g$.

We use the following notations for standard group classes: $\mathcal{N}_o$ — *torsion-free nilpotent groups,* $\overline{\mathcal{N}}_p$ — *nilpotent p-groups of finite exponent,* that is, nilpotent group in which for some $n = n(G)$ every element g satisfies the equation $g^{p^n} = 1$.

Let $\mathcal{K}$ be a class of groups. A group G is said to be *discriminated* by $\mathcal{K}$ if for every finite subset $g_1, g_2, \dots, g_n$ of distinct elements of G, there exists a group $H \in \mathcal{K}$ and a homomorphism ϕ of G into H, such that $\phi(g_i) \neq \phi(g_j)$ for $i \neq j$, $(1 \leq i, j \leq n)$.

The n-th *dimension subgroup* $D_n(RG)$ of G over R is the set of group elements $g \in G$ such that $g - 1$ lies in the n-th power of $A(RG)$. It is well known that for every natural number n the inclusion

$$\gamma_n(G) \subseteq D_n(RG)$$

holds, where $\gamma_n(G)$ is the n-th term of the lower central series of G.

Lemma 2.1. *Let a class $\mathcal{K}$ of groups be closed under the taking of subgroups (that is all subgroups of any member of the class $\mathcal{K}$ are again in the class $\mathcal{K}$) and also finite direct products (that is the direct products of finite member groups of the class $\mathcal{K}$ are again in the class $\mathcal{K}$) and let G be a residually-$\mathcal{K}$ group. Then G is discriminated by $\mathcal{K}$.*

The proof can be obtained immediately.

The ideal $A(RG)$ of the group ring RG is said to be residually nilpotent if $A^\omega(RG) = 0$.

Lemma 2.2. ([1], Proposition 15.1.) *If G is discriminated by a class of groups $\mathcal{K}$ and for each $H \in \mathcal{K}$ the equality $A^\omega(RH) = 0$ holds, then $A^\omega(RG) = 0$.*

Lemma 2.3. ([1], Proposition 1.12.) *The right annihilator L of the left ideal $I(RH)$ is non-zero if and only if H is a finite subgroup of a group G. If H is finite, then $L = (\sum_{h \in H} h)RG$.*

If H, M are two subgroups of G, then we shall denote by $[H, M]$ the subgroup generated by all commutators $[g, h] = g^{-1}h^{-1}gh, g \in H, h \in M$.

A series

$$G = H_1 \supseteq H_2 \supseteq \dots \supseteq H_n \supseteq \dots$$

of normal subgroups of a group G is called an N-*series* if $[H_i, H_j] \subseteq H_{i+j}$ for all $i, j \geq 1$ and also each of the Abelian groups H_i/H_j is a direct product of (possibly infinitely many) cyclic groups which are either infinite or of order p^k, where p is a fixed prime and k is bounded by some integer depending only on G.

The ideal $J_p(R)$ of a ring R is defined by $J_p(R) = \bigcap_{i=1}^{\infty} p^i R$.

In this paper we shall use the following theorems:

Theorem 2.1. ([3], Theorem E.) *Let G be a group with a finite N-series and R be a commutative ring with unity satisfying $J_p(R) = 0$. Then $A^\omega(RG) = 0$.*

We apply Theorem 2.1 for residually-$\overline{\mathcal{N}}_p$ groups. It is clear that the lower central series of a nilpotent p-group of finite exponent is an N-series.

Theorem 2.2. *Let R be a commutative ring with unity satisfying $J_p(R) = 0$. If G is a residually-$\overline{\mathcal{N}}_p$ group, then $A^\omega(RG) = 0$.*

The proof of this theorem follows from Lemmas 2.1 and 2.2 and Theorem 2.1 because the class $\overline{\mathcal{N}}_p$ is closed under the taking of subgroups and also finite direct products.

Theorem 2.3. ([7], VI, Theorem 2.15.) *If G is a residually torsion free nilpotent group and R is a commutative ring with unity such that its additive group is torsion-free, then $A^\omega(RG) = 0$.*

An element g of a group G is called a *generalized torsion element* if for all natural numbers n the order of the element $g\gamma_n(G)$ of the factor group $G/\gamma_n(G)$ is finite.

It is clear that torsion elements of a group G are generalized torsion elements of G.

If $g \in G$ is a generalized torsion element then Ω_g denotes the set of prime divisors of the orders of the elements $g\gamma_n(G) \in G/\gamma_n(G)$ for all $n = 2, 3, \ldots$.

Lemma 2.4. ([4]) *Let g be a generalized torsion element of a group G, Λ an arbitrary subset of Ω_g, $r \in \bigcap_{p\in\Lambda} J_p(R)$ and let*

$$x \in \bigcap_{p\in\Omega_g\setminus\Lambda} \bigcap_{n=1}^{\infty} I(G^{p^n}\gamma_n(G)).$$

Then one of the following statements holds:

1) *if Λ is the proper subset of Ω_g, then $r(g-1)x \in A^\omega(RG)$;*
2) *if $\Lambda = \Omega_g$, then $r(g-1) \in A^\omega(RG)$;*
3) *if $\Lambda = R$, then $(g-1)x \in A^\omega(RG)$.*

We have the following theorem.

Theorem 2.4. ([4]) *Let Ω be a nonempty subset of primes with $\bigcap_{p\in\Omega} J_p(R) = 0$ and suppose that the group G is discriminated by the class of groups $\overline{\mathcal{N}}_\Omega$. If for every proper subset Λ of the set Ω at least one of the conditions*

1) $\bigcap_{p\in\Lambda} J_p(R) = 0$,

2) *G is discriminated by the class of groups $\overline{\mathcal{N}}_{\Omega\setminus\Lambda}$ holds,*

then $A^\omega(RG) = 0$.

3. The intersection theorem. Let R be a commutative ring with unity.

Lemma 3.1. *Let $g \in G$ and $g^{p^n} \in D_t(RG)$ for a prime p and a natural number n. Then there exists a natural number m such that*

$$p^m(g-1) \in A^t(RG).$$

Proof. We prove this by induction on t. For $t = 1$ the statement is obvious. Let $p^s(g-1) \in A^{t-1}(RG)$ for some s. From the decomposition g^{p^m} as $(g-1+1)^{p^m}$ we have that

$$g^{p^m} - 1 = p^m(g-1) + \sum_{i=2}^{t-1} \binom{p^m}{i} (g-1)^i + \sum_{i=t}^{p^m} \binom{p^m}{i} (g-1)^i$$

for every m. If $m \geq n(s+t)$, then p^s divides $\binom{p^m}{i}$ for $i = 1, 2, \ldots, t-1$ and $g^{p^m} \in D_t(RG)$. Therefore we have

$$g^{p^m} - 1 = p^m(g-1) + p^s(g-1)^2 \sum_{i=2}^{t-1} d_i (g-1)^{i-2} + \sum_{i=t}^{p^m} \binom{p^m}{i} (g-1)^i,$$

where $d_i p^s = \binom{p^m}{i}$ for $i = 2, 3, \ldots, t-1$. Since $g^{p^m} - 1 \in A^t(RG)$, by iteration from the preceding identity $p^m(g-1) \in A^t(RG)$ follows. The proof is complete .

Let p be a prime and n a natural number. Denote G^{p^n} is the subgroup of G generated by all elements of the form g^{p^n}, $g \in G$.

Lemma 3.2. *Let $h \in G^{p^n}\gamma_n(G)$ for a natural number n. Then for all natural numbers t and s for which $n \geq t + s$,*

$$h - 1 \equiv p^s F_t(h) \pmod{A^t(RG)}$$

holds, where $F_t(h) \in A(RG)$.

Proof. Writing the element h as $h = h_1^{p^n} h_2^{p^n} \dots h_m^{p^n} y_n \quad (h_i \in G, y_n \in \gamma_n(G))$ and using the identity

$$ab - 1 = (a-1)(b-1) + (a-1) + (b-1) \tag{1}$$

we have

$$h - 1 = (h_1^{p^n} h_2^{p^n} \dots h_m^{p^n} - 1)(y_n - 1) + (h_1^{p^n} h_2^{p^n} \dots h_m^{p^n} - 1) + (y_n - 1).$$

Since $t < n$, it follows that $(y_n - 1) \in A^t(RG)$. It is clear that p^s divides $\binom{p^n}{j}$ for $j = 1, 2, \dots, t-1$. Then from the preceding identity

$$h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv p^s \sum_{i=1}^{m} \sum_{j=1}^{t-1} d_j (h_i - 1)^j b_i \ \equiv p^s F_t(h) \quad (\text{mod } A^t(RG))$$

follows, where $F_t(h) = p^s \sum_{i=1}^{m} \sum_{j=1}^{t-1} d_j (h_i - 1)^j b_i, b_i \in RG$ and $p^s d_j = \binom{p^n}{j}$ for $1 \leq j \leq t-1$. The proof is complete .

Suppose further that R is a commutative integral domain. Let $\mid G \mid$ be the order of the group G.

Lemma 3.3. *Let H be a subgroup of a group G and $I(H)(1-a) = 0$ for a suitable element $a \in A(RG)$. Then H is finite and the order of H is invertible in R.*

Proof. By the condition of our lemma the right annihilator of the left ideal $I(H)$ is non-zero. By Lemma 2.3 H is finite and $1-a$ can be written as $1 - a = (\sum_{h \in H} h)x$ for a suitable element $x \in RG$. If $\phi(y)$ is the sum of the coefficients of the element $y \in RG$, then the map $\phi\colon RG \to R$ is a ring homomorphism of RG onto R, with $\phi(1-a) = \phi((\sum_{h \in H} h)x) = |H|\phi(x) = 1$, that is $|H|$ is invertible in R. The proof is complete .

Let $o(g)$ be the order of the element $g \in G$ and let $D_\omega(RG)$ be the ω-th dimension subgroup of G over R, that is $D_\omega(RG) = \cap_{n=1}^{\infty} D_n(RG)$. It is easy to see that $D_\omega(RG) = \{g \in G \mid g - 1 \in A^\omega(RG)\}$.

Let $R^\star$ denotes the unit group of the ring R.

Lemma 3.4. *Let the intersection theorem hold for $A(RG)$. Then the set $S = \{g \in G \mid o(g) \in R^\star\}$ coincides with $D_\omega(RG)$ and it is the largest finite subgroup of order invertible in R.*

Proof. Let $g \in S$. Then the order $n = o(g)$ of the element g is invertible in R and from the identity

$$0 = g^n - 1 = n(g-1) + \binom{n}{2}(g-1)^2 + \ldots + (g-1)^n$$

we have

$$g-1=-n^{-1}(g-1)\left(\binom{n}{2}(g-1)+\binom{n}{3}(g-1)^2+\ldots+(g-1)^{n-1}\right).$$

Hence, by iteration, we have $g-1\in A^\omega(RG)$. This implies that $S\subseteq D_\omega(RG)$.

Conversely, it is clear that $I(D_\omega(RG))\subseteq A^\omega(RG)$ and from $A^\omega(RG)(1-a)=0$ we have $I(D_\omega(RG))(1-a)=0$. Then by Lemma 3.3 the order of the subgroup $D_\omega(RG)$ is invertible in R. Therefore $D_\omega(RG)\subseteq S$. The proof is complete .

Corollary. *Let the intersection theorem hold for $A(RG)$ and let $\overline{g}\neq\overline{1}$ be an element of finite order n of the group $\overline{G}=G/D_\omega(RG)$. Then the prime divisors of n are not invertible in R.*

Lemma 3.5. *Let G be a group having a p-element g and suppose that the intersection theorem holds for $A(RG)$. If the ideal $J_p(R)$ is non-zero, then $g\in D_\omega(RG)$.*

Proof. Let $A^\omega(RG)(1-a)=0$ for a suitable element $a\in A(RG)$ and let p^n be the order of the element $g\in G$. Therefore for every natural number t we have $g^{p^n}\in\gamma_t(G)\subseteq D_t(RG)$. If $0\neq r\in J_p(R)$ then for every $m\geq 1$ for the element r we have the decomposition $r=p^m r_m(r_m\in R)$. Then by Lemma 3.1

$$r(g-1)=p^m r_m(g-1)\in A^t(RG)$$

for an enough large integer m. Since t is an arbitrary natural number, we conclude that $r(g-1)\in A^\omega(RG)$, and so, $r(g-1)(1-a)=0$. In the group ring over an integral domain this equation implies that $(g-1)(1-a)=0$. Then by Lemma 3.3 the order of the element g is invertible in R. Consequently by Lemma 3.4 $g\in D_\omega(RG)$. The proof is complete .

Let $W_p(G)=\bigcap\limits_{n=1}^{\infty}G^{p^n}\gamma_n(G)$.

Lemma 3.6. *Let $m=p_1^{m_1}p_2^{m_2}\ldots p_s^{m_s}$ be the prime power decomposition of the order of the element $g\in G$. Then for every prime $p\neq p_i,(i=1,2,\ldots,s)$ the element g lies in $W_p(G)$.*

Proof. Since the numbers p and m are coprimes, for an arbitrary n we can be choose the integers k and l with $km+lp^n=1$. Then

$$g=g^{km+lp^n}=(g^m)^k(g^l)^{p^n}=(g^l)^{p^n}\in G^{p^n}\gamma_n(G).$$

Therefore $g\in G^{p^n}\gamma_n(G)$ for all n. Consequently $g\in W_p(G)$.

Let π be the set of those primes p for which the group G contains an element of a prime order p and let $\pi^\star = \{p \in \pi \mid p \in R^\star\}$, where $R^\star$ is the unit group of R. We recall that if the intersection theorem holds for $A(RG)$, then by Lemma 3.4, the set of the prime divisors of $|D_\omega(RG)|$ coincides with $\pi^\star$.

Let $\overline{G} = G/D_\omega(RG)$.

Lemma 3.7. Let the intersection theorem hold for $A(RG)$. Then for all $p \in \pi \setminus \pi^\star$, $W_p(\overline{G})$ is a torsion group with no p-elements and $\bigcap\limits_{p\in\pi\setminus\pi^\star} W_p(\overline{G}) = \langle\overline{1}\rangle$.

Proof. Let $A^\omega(RG)(1-a) = 0$ for a suitable element $a \in A(RG)$. Suppose further that p is a fixed prime in $\pi\setminus\pi^\star$ and $\overline{g} = gD_\omega(RG)$ is an arbitrary element of $W_p(\overline{G})$. We shall prove that the element $\overline{g}$ has a finite order. For every n the element $\overline{g}$ lies in the subgroup $\overline{G}^{p^n}\gamma_n(\overline{G})$. Therefore for the element g we have the decomposition

$$g = g_1^{p^n} g_2^{p^n} \dots g_k^{p^n} h_n d_n,$$

where $h_n \in \gamma_n(G), d_n \in D_\omega(RG), g_i \in G, i = 1, 2, \dots, k$. Since $p \in \pi \setminus \pi^\star$, clearly p is not invertible in R and from Lemma 3.4 and from the definition of the set $\pi\setminus\pi^\star$ it follows that $\overline{G}$ contains a nontrivial p-element $\overline{h} = hD_\omega(RG)$. Let the order of the element $\overline{h}$ be p^s. Then $h^{p^s} \in D_\omega(RG) \subseteq D_t(RG)$ for every natural number t. By Lemma 3.1 then there exists m such that

$$p^m(h-1) \in A^t(RG). \tag{2}$$

By Lemma 3.2 for an enough large n the element $x = g_1^{p^n} g_2^{p^n} \dots g_k^{p^n} h_n \in G^{p^n}\gamma_n(G)$ satisfies the condition

$$x - 1 \equiv p^m F_t(x) \pmod{A^t(RG)}, \quad F_t(x) \in A(RG). \tag{3}$$

Since $d_n - 1 \in A^\omega(RG)$ and $g = xd_n$, from (1) it follows that $(g-1)(h-1) \equiv (x-1)(h-1) \pmod{A^t(RG)}$. Then by (2) and (3) we obtain

$$(g-1)(h-1) \equiv F_t(x)p^m(h-1) \equiv 0 \pmod{A^t(RG)}.$$

Since t is an arbitrary natural number we conclude that

$$(g-1)(h-1) \in A^\omega(RG). \tag{4}$$

By the condition of our lemma it follows, that $(g-1)(h-1)(1-a) = 0$. The order of the element h is not invertible in R, consequently, by Lemma 3.3 $(h-1)(1-a) \neq 0$, and $g-1$ has a non-zero annihilator. Then by Lemma 2.3 the order of the element g is finite. Therefore $\overline{g}$ is an element of finite order. Consequently, $W_p(\overline{G})$ is a torsion subgroup of $\overline{G}$.

Now suppose that the element $\overline{g} = gD_\omega(RG)$ is a non-trivial p-element. (Note that $p \in \pi \setminus \pi^\star$.) Since (4) is true for every p-element from the group $\overline{G}$, it follows that

$$(g-1)^2 \in A^\omega(RG). \tag{5}$$

By Lemma 3.4 $D_\omega(RG)$ is finite and therefore the order of the element g is finite. Let $o(g) = l$. From the identity

$$0 = g^l - 1 = l(g-1) + \binom{l}{2}(g-1)^2 + \ldots + (g-1)^l$$

and from (5) we conclude that $l(g-1) \in A^\omega(RG)$. Hence $l(g-1)(1-a) = 0$, because the intersection theorem holds for $A(RG)$. Since R is an integral domain, it follows that $(g-1)(1-a) = 0$, which is impossible by Lemma 3.3 because p divides $o(g)$ and therefore $o(g)$ is not invertible in R. Consequently, for every $p \in \pi \setminus \pi^\star$ the subgroup $W_p(\overline{G})$ contains no p-elements.

Now we prove the equation $\bigcap\limits_{p\in\pi\setminus\pi^\star} W_p(\overline{G}) = \langle\overline{1}\rangle$. Let $\overline{v} \in \bigcap\limits_{p\in\pi\setminus\pi^\star} W_p(\overline{G})$. Then the order $o(\overline{v})$ of the element $\overline{v}$ is finite and by Corollary of Lemma 3.4. the prime divisors of $o(\overline{v})$ are not invertible in R, that is are lies in the set $\pi \setminus \pi^\star$. This is impossible since by above facts the subgroup $W_p(\overline{G})$ with no p-elements for all $p \in \pi \setminus \pi^\star$. Consequently $\overline{v} = \overline{1}$ and $\cap_{p\in\pi\setminus\pi^\star} W_p(\overline{G}) = \langle\overline{1}\rangle$. The proof is complete.

From Lemma 5.2 of [6] it we have

Lemma 3.8. *Let H_1, H_2 be normal subgroups of a group G with $H_1 \cap H_2 = \langle 1\rangle$. Then $I(H_1) \cap I(H_2) = I(H_1)I(H_2)$.*

Lemma 3.9. *Let the set of elements of finite order of a group G form a finite nilpotent group $T(G)$ and let $A^\omega(RG/T(G)) = 0$. Suppose that for all $p \in \pi \setminus \pi^\star$ the group $W_p(G)$ is finite with no p-elements and $J_p(R) = 0$. Then the intersection theorem holds for $A(RG)$.*

Proof. Note that in this case $\pi = \pi^\star$ and it is the set of prime divisors of the order $|T(G)|$ of the group $T(G)$.

We prove lemma by induction on the order of $T(G)$. If $|T(G)| = 1$, then $A^\omega(RG) = 0$ and in this case the proof is complete.

Suppose first that $T(G)$ is a p-group. If $p \in \pi \setminus \pi^\star$, then p is not invertible in R and from the conditions of our lemma it follows that $W_p(G) = \langle 1\rangle$ and $J_p(R) = 0$. The group $G/G^{p^n}\gamma_n(G)$ is a nilpotent p-group of finite exponent and by Theorem 2.1 $A^\omega(RG/G^{p^n}\gamma_n(G)) = \overline{0}$ for all n. Since $W_p(G) = \langle 1\rangle$ it follows, that G is a residually-$\overline{\mathcal{N}}_p$ group. Therefore by Theorem 2.2 $A^\omega(RG) = 0$ and the intersection theorem holds for $A(RG)$.

Now let $p \in \pi^\star$, that is p is invertible in R. From $A^\omega(RG/T(G)) = 0$ it follows that $A^\omega(RG) \subseteq I(T(G))$. If p^n is the order of $T(G)$ then the element $1 - (p^n)^{-1} \sum_{g\in T(G)} g$ in ideal $A(RG)$ and by Lemma 2.3

$$A^\omega(RG)(1-a) \subseteq I(T(G))(1-a) = 0.$$

Now assume that there exist at least two different primes dividing $|T(G)|$. Then for the finite nilpotent group $T(G)$ we have the direct product decomposition

$$T(G) = S_{p_1} \otimes S_{p_2} \otimes \ldots \otimes S_{p_k},$$

of its Sylow p-subgroups S_{p_i}, $(i = 1, 2, \ldots, k)$ with $k \geq 2$.

If $\pi^\star \neq \emptyset$, that is among the primes p_i $(i = 1, 2, \ldots, k)$ there exists p_j which is invertible in R, then by Lemma 2.3, the element $b = 1 - \mid S_{p_j} \mid^{-1} \sum_{g\in S_{p_j}} g$ satisfies the equation

$$I(S_{p_j})(1-b) = 0, \quad b \in A(RG). \tag{6}$$

By the induction hypothesis there exists an element $\overline{c} \in A(RG/S_{p_j})$ such that $A^\omega(RG/S_{p_j})(1-\overline{c}) = \overline{0}$. If $c \in A(RG)$ is an element from the inverse image of $\overline{c}$ by the homomorphism $\phi\colon RG \to RG/S_{p_j}$, then

$$A^\omega(RG)(1-c) \subseteq I(S_{p_j}).$$

Let $1 - a = (1-c)(1-b)$. Then from the above inclusion and from (6) we obtain the equality $A^\omega(RG)(1-a) = 0$.

Suppose that $\pi^\star = \emptyset$, that is all p_i $(i = 1, 2, \ldots, k)$ are not invertible in R. By the induction there exist $\overline{c}_1 \in A^\omega(RG/S_{p_l})$ and $\widetilde{c}_2 \in A^\omega(RG/S_{p_t})$ such that

$$A^\omega(RG/S_{p_l})(1-\overline{c}_1) = \overline{0} \quad \text{and} \quad A^\omega(RG/S_{p_t})(1-\widetilde{c}_2) = \widetilde{0}.$$

Then $A^\omega(RG)(1-c_1) \subseteq I(S_{p_l})$ and $A^\omega(RG)(1-c_2) \subseteq I(S_{p_t})$ for suitable elements $c_1, c_2 \in A(RG)$. Hence by Lemma 3.8 we have

$$A^\omega(RG)(1-c_1)(1-c_2) \subseteq I(S_{p_l}) \cap I(S_{p_t}) = I(S_{p_l})I(S_{p_t}). \tag{7}$$

We can be choose the integers n and m such that $n|S_{p_l}| + m|S_{p_t}| = 1$. It is easy to see that the sum of the coefficients of the element $1 - d = n\sum_{g\in S_{p_1}} g + m\sum_{g\in S_{p_2}} g$ equals to 1 and therefore $d \in A(RG)$. Since $T(G)$ is a finite group, by Lemma 2.3 it follows, that $I(S_{p_l})I(S_{p_t})(1-d) = 0$. Then by (7) the element $1 - a = (1-c_1)(1-c_2)(1-d)$ satisfies the condition $A^\omega(RG)(1-a) = 0$. The proof is complete .

We shall say that G is a *generalized nilpotent* group if it is discriminated by the class of the nilpotent group. This is equivalent to the equality $\bigcap_{n=1}^{\infty} \gamma_n(G) = \langle 1 \rangle$.

Lemma 3.10. ([5]) *Let g, h are an elements of a nilpotent group G. Suppose that $\gamma_{t+1}(G) = \langle 1 \rangle$ and $h^{n^s} = 1$. Then the element h commute with $g^{n^{s(t-1)}}$.*

We generalize this Lemma.

Lemma 3.11. *Let G be a generalized nilpotent group, Ω a subset of the primes and let $g \in \cap_{p\in\Omega} W_p(G)$. If the prime divisors of the order $o(h)$ of the element h are in Ω, then $gh = hg$. If the orders of the elements g and h are coprimes, then $gh = hg$.*

Proof. Let $g \in \cap_{p\in\Omega} W_p(G)$ and let $c = g^{-1}h^{-1}gh \neq 1$ be the commutator of g and h. Since G is a generalized nilpotent group, there exists an integer $t \geq 2$ such that $c \notin \gamma_{t+1}(G)$. Let $\overline{g}$ and $\overline{h}$ be the image of the elements g and h in $\overline{G} = G/\gamma_{t+1}(G)$.

First we suppose that $\overline{h}$ is a p-element ($p \in \Omega$) of $\overline{G}$ and $o(h) = p^s$. Since the element $g \in \cap_{p\in\Omega} W_p(G)$, that for g we have

$$g = g_1^{p^{2s(t-1)}} g_2^{p^{2s(t-1)}} \dots g_k^{p^{2s(t-1)}} x_{2s(t-1)},$$

where $x_{2s(t-1)} \in \gamma_{2s(t-1)}(G), g_i \in G, i = 1, 2, \dots, k$. Then

$$\overline{g} = \overline{g}_1^{p^{2s(t-1)}} \dots \overline{g}_2^{p^{2s(t-1)}} \dots \overline{g}_k^{p^{2s(t-1)}} \dots \overline{x}_{2s(t-1)}.$$

From Lemma 3.10 $\overline{h}\overline{g}_i^{p^{2s(t-1)}} = \overline{g}_i^{p^{2s(t-1)}}\overline{h}$ follow for all $i = 1, 2, \dots, k$. Since $t \geq 2$, we have $2s(t-1) \geq t$ and therefore $\overline{x}_{2s(t-1)}$ is a central element of $\overline{G}$. Consequently, $\overline{g}\overline{h} = \overline{h}\overline{g}$.

Let h be a torsion element of G and let the prime divisors of the order of h be in Ω. Then the element $\overline{h}$ of the nilpotent group $\overline{G}$ has the decomposition $\overline{h} = \overline{h}_1\overline{h}_2 \dots \overline{h}_l$, where $l \geq 1, \overline{h}_i^{{p_i}^{n_i}} = \overline{1}, p_i \in \Omega, i = 1, 2, \dots, l$. From the above fact we have that $\overline{g}\overline{h}_i = \overline{h}_i\overline{g}$ for all i. Therefore $\overline{g}\overline{h} = \overline{h}\overline{g}$. Consequently $c \in \gamma_{t+1}(G)$, which is a contradiction.

Let $g^n = h^m = 1$. Suppose that n and m are coprimes. If the set Ω is the set of the prime divisors of m then by Lemma 3.6 $g \in \cap_{p\in\Omega} W_p(G)$ and the by above argument $gh = hg$. The proof is complete.

Lemma 3.12. *Let $\{H_\alpha\}_{\alpha\in K}$ be an arbitrary set of normal subgroups of a group G. Suppose that H is a subgroup of G of finite exponent k. If $g \in \cap_{\alpha\in K}(H_\alpha H)$, then $g^k \in \cap_{\alpha\in K} H_\alpha$.*

Proof. Let $g \in \cap_{\alpha\in K}(H_\alpha H)$. Then for all $\alpha \in K$ the element g lies in $H_\alpha H$ and $gh \in H_\alpha$ for a suitable element $h \in H$. We shall show that for every $\alpha \in K$ we have $g^s h^s \in H_\alpha$ by induction on s.

For $s = 1$ the proof is similar as above. Suppose that $g^{s-1}h^{s-1} \in H_\alpha$. Then $hghh^{-1} = hg \in H$ and $hgg^{s-1}h^{s-1} = hg^s h^{s-1} \in H_\alpha$. Then $h^{-1}hg^s h^{s-1}h = h^s g^s \in H_\alpha$ since H_α is a normal subgroup of G. If $s = k$ then $h^s = 1$ and therefore $g^k \in H_\alpha$. Consequently, $g^k \in \cap_{\alpha\in K} H_\alpha$.

Let $\overline{G} = G/D_\omega(RG)$.

Lemma 3.13. *Let the intersection theorem hold for $A(RG)$. Then the following assertionare satisfied:*

1) *If the set $\pi\backslash\pi^\star$ contains more than one element then the set $T(G)$ of the torsion elements of G form a finite normal subgroup of G, and for all $p \in \pi \setminus \pi^\star$ the subgroup $W_p(\overline{G})$ is finite with no p-elements and $J_p(R) = 0$.*
2) *If $\pi \setminus \pi^\star = \{p\}$ then $\overline{G}$ is a residually-$\overline{\mathcal{N}}_p$ group and $J_p(R) = 0$.*
3) *If $\pi = \pi^\star$ then either $\overline{G}$ is discriminated by the torsion free nilpotent groups, or there exists a nonempty subset Ω of the set of primes such that $\cap_{p\in\Omega} J_p(R) = 0$, the group $\overline{G}$ is discriminated by the class of groups $\overline{\mathcal{N}}_\Omega$ and for every proper subset Λ of the set Ω at least one of the conditions*
   1) *$\cap_{p\in\Lambda} J_p(R) = 0$*
   2) *$\overline{G}$ is discriminated by the class of groups $\overline{\mathcal{N}}_{\Omega\setminus\Lambda}$*

holds.

Proof. Let $A^\omega(RG)(1-a) = 0$ for a suitable element $a \in A(RG)$.

Case 1. Suppose that the set $\pi \setminus \pi^\star$ contains more than one element. First we prove that the elements of finite order of the group $\overline{G}$ form a normal subgroup.

Let $\overline{g}, \overline{h} \in \overline{G}$ and $o(\overline{g}) = n, o(\overline{h}) = m$. It is evident that the order of $\overline{g}^{-1}$ is finite. Therefore it is enough to show that the order of the element $\overline{g}\overline{h}$ is finite.

By Corollary of Lemma 3.4 it follows that the prime divisors of the integers n and m are in $\pi \setminus \pi^\star$. Let us denote by $d = d(n, m)$ the greatest common divisor of n and m. Then $n = n' d$ and $m = m' d$ for a suitable n' and m' with $d(n', m') = 1$.

Put $k = n'm'd$. If $d = 1$ then by Lemma 3.7 $\overline{g}\overline{h} = \overline{h}\overline{g}$ and therefore $(\overline{g}\overline{h})^{nm} = \overline{1}$.

Let now $d \neq 1$. Suppose that k is a prime power p. Then $\overline{g}$ and $\overline{h}$ are p-elements of $\overline{G}$. Since $\pi \setminus \pi^\star$ contains more than one element, there exists $q \in \pi \setminus \pi^\star$ such that $q \neq p$. By Lemma 3.6 $\overline{g}$, $\overline{h}$ are in $W_q(\overline{G})$, which by Lemma 3.7 is a torsion group, and therefore the order of the element $\overline{g}\overline{h}$ is finite.

Let k be a composed number. Then for k we have the decomposition $k = p^\alpha l$ for some prime $p \in \pi \setminus \pi^\star$ and some natural number l where $(l, p) = 1$. Then there exist integers s and r such that $sp^\alpha + rl = 1$. Hence $\overline{g}\overline{h} = \overline{g}^{sp^\alpha}\overline{g}^{rl}\overline{h}^{sp^\alpha}\overline{h}^{rl}$. Since

$d(o(\overline{g}^{rl}), o(\overline{h}^{sp^\alpha})) = d(o(\overline{g}^{sp^\alpha}), o(\overline{h}^{rl})) = 1$ then, Lemma 3.11 $\overline{g}^{rl}\overline{h}^{sp^\alpha} = \overline{h}^{sp^\alpha}\overline{g}^{rl}$ and $\overline{g}^{sp^\alpha}\overline{h}^{rl} = \overline{h}^{rl}\overline{g}^{sp^\alpha}$. By the induction we have

$$(\overline{g}\overline{h})^t = (\overline{g}^{sp^\alpha}\overline{h}^{sp^\alpha})^t(\overline{g}^{rl}\overline{h}^{rl})^t \tag{8}$$

for every t. The orders of the elements $\overline{g}^{sp^\alpha}$ and $\overline{h}^{sp^\alpha}$ are coprimes with p and by Lemma 3.6 $\overline{g}^{sp^\alpha}, \overline{h}^{sp^\alpha}$ are in $W_p(\overline{G})$ which by Lemma 3.7 is a torsion group. Therefore $(\overline{g}^{sp^\alpha}\overline{h}^{sp^\alpha})^{t_1} = \overline{1}$ for a suitable t_1. By similar arguments we obtain that $(\overline{g}^{rl}\overline{h}^{rl})^{t_2} = \overline{1}$ for a suitable t_2. Then by (8) the integer $t = t_1t_2$ satisfies the equality $(gh)^t = 1$. Consequently $T(\overline{G})$ is a torsion subgroup of $\overline{G}$ and clearly it is normal in $\overline{G}$.

Now we show that $T(G)$ is a torsion normal subgroup of G. It is clear that $T(\overline{G})$ is a generalized nilpotent group, because $\overline{G}$ is a generalized nilpotent group. By Lemma 3.11. for $T(\overline{G})$ we have the direct product decomposition

$$T(\overline{G}) = \prod_{p\in\pi\setminus\pi^\star} \overline{S}_p \tag{9}$$

of its Sylow p-subgroups $\overline{S}_p$.

Let S_p be the inverse image of $\overline{S}_p$ in G. We shall show that $I(S_p)I(S_q) \subseteq A^\omega(RG)$ for $p \neq q$ and $p, q \in S_p, p \in \pi \setminus \pi^\star$. It will be sufficient to show that $(g-1)(h-1) \in A^\omega(RG)$ for all $g \in S_p$, and $h \in S_q$. By (9) for the elements g and h we have the decompositions $g = vx$ and $h = wy$ where the elements x, y, v^{p^i}, w^{q^j} are in $D_\omega(RG)$ for suitable i and j. Applying the identity (5) to the elements $g-1, h-1$ we have

$$(g-1)(h-1) \equiv (v-1)(w-1) \pmod{A^\omega(RG)}, \tag{10}$$

because $x-1$ and $y-1$ in $A^\omega(RG)$. For the elements $v^{p^i}-1$ and $w^{q^j}-1$ we have

$$v^{p^i} - 1 = p^i(v-1) + \binom{p^i}{2}(v-1)^2 + \ldots + (v-1)^{p^i},$$

$$w^{q^j} - 1 = q^j(w-1) + \binom{q^j}{2}(v-1)^2 + \ldots + (w-1)^{q^j}.$$

Choose the integers s and l such that $sp^i + lq^j = 1$. Multiplying these equations by $s(w-1)$ and $l(v-1)$ respectively and adding we obtain

$$\begin{aligned}(v-1)(w-1) = (v-1)(w-1)b + c(v-1)(w-1)+\\ + s(v^{p^i}-1)(w-1) + l(v-1)(w^{q^j}-1),\end{aligned}$$

where

$$b = -l(\binom{q^j}{2}(w-1) + \binom{q^j}{3}(w-1)^2 + \ldots + (w-1)^{q^j-1}),$$

$$c = -s(\binom{p^i}{2}(v-1) + \binom{p^i}{3}(v-1)^2 + \ldots + (v-1)^{p^i-1}).$$

Since the elements $b, c \in A(RG)$ and $v^{p^i} - 1$ and $w^{q^j} - 1$ are in $A^\omega(RG)$, from the above identity we conclude that $(v-1)(w-1) \in A^t(RG)$ for all integers $t \geq 1$. Therefore $(v-1)(w-1) \in A^\omega(RG)$ and by (10), $(g-1)(h-1) \in A^\omega(RG)$. Consequently $I(S_p)I(S_q) \subseteq A^\omega(RG)$.

Now we show that $T(G)$ is a finite subgroup. Let q be an arbitrary element of $\pi \setminus \pi^\star$, $H_q = S_{p_1}S_{p_2}\ldots$, where $q,\ p_i, \in \pi \setminus \pi^\star$ and $p_i \neq q$ for all i. Then from the above argument it follows that

$$I(H_q)I(S_q) \subseteq A^\omega(RG) \text{ and } I(S_q)I(H_q) \subseteq A^\omega(RG).$$

Since $A^\omega(RG)(1-a) = 0$ we have that

$$I(H_q)I(S_q)(1-a) = 0 \text{ and } I(S_q)I(H_q)(1-a) = 0. \tag{11}$$

The prime q is not invertible in R and so, by Lemma 3.1, $I(S_q)(1-a) \neq 0$. Therefore by (11) the ideal $I(H_q)$ has a non-zero right annihilator. Consequently H_q is a finite subgroup of G and therefore the set $\pi \setminus \pi^\star$ is finite. Furthermore $I(H_q)(1-a) \neq 0$ because the order of H_q is not invertible in R. It follows that S_p is finite for all $p \in \pi \setminus \pi^\star$. Then by (9) we obtain that $T(\overline{G})$ is finite. By Lemma 3.4 $D_\omega(RG)$ is a finite subgroup of G and from the isomorphism $T(\overline{G}) \cong T(G)/D_\omega(RG)$ it follows that $T(G)$ is a finite subgroup of G.

Let $p \in \pi \setminus \pi^\star$. Then in G there exists a p-element g such that $g \in D_\omega(RG)$. Therefore by Lemma 3.5 $J_p(R) = 0$. From Lemma 3.7 we have that $W_p(\overline{G}) \subseteq T(\overline{G})$, and $W_p(\overline{G})$ contains no p-elements. Since $T(\overline{G})$ is finite, it follows that $W_p(\overline{G})$ is also a finite subgroup of G with no p-elements.

Case 2. Let $\pi \setminus \pi^\star = \{p\}$. Then from the Corollary of Lemma 3.4 it follows that the elements of finite order of $\overline{G}$ are p-elements. By Lemma 3.7 $W_p(\overline{G})$ is a torsion group with no p-elements. Consequently $W_p(\overline{G}) = \langle \overline{1} \rangle$ that is $\overline{G}$ is a residually nilpotent p-group of finite exponent.

Case 3. Let $\pi = \pi^\star$. By Lemma 3.2 $T(G) = D_\omega(RG)$ and it is a finite group.

Assume G contains no generalized torsion element of infinite order, and let $\sqrt{\gamma_n(G)}$ be the isolator of $\gamma_n(G)$ in G, that is

$$\sqrt{\gamma_n(G)} = \{g \in G \mid g^m \in \gamma_n(G) \text{ for some integer } m \geq 1\}.$$

Then $\cap_{n=1}^{\infty}\sqrt{\gamma_n(G)} = D_\omega(RG)$ and therefore for every element $\overline{g} = gD_\omega(RG)$ there exists an integer n such that $g \in \sqrt{\gamma_n(G)}$. If ϕ is the homomorphism of $\overline{G}$ onto the torsion free nilpotent group $G/\gamma_n(G)$ then $\phi(\overline{g}) \neq \widetilde{1}$, that is, $\overline{G}$ is a residually torsion free nilpotent group.

Let now g be a generalized torsion element of G of infinite order. Since $\cap_{n=1}^{\infty}\gamma_n(G) \subseteq D_\omega(RG)$ and the order of $D_\omega(RG)$ is finite, it follows that $g \in \cap_{n=1}^{\infty}\gamma_n(G)$.

Let Ω denote the set of prime divisors of the orders of the elements $g\gamma_n(G) \in G/\gamma_n(G)$ for all $n = 2, 3, \ldots$. It is obvious that Ω is non-empty.

Let $r \in \cap_{p\in\Omega}J_p(R)$. Then by Lemma 3.3 it follows that $r(g-1) \in A^\omega(RG)$ and therefore $r(g-1)(1-a) = 0$ because $A(RG)$ satisfies the intersection theorem. Since R is an integral domain and the order of the element g is infinite, from the above equality it follows that $r = 0$. Consequently $\cap_{p\in\Omega}J_p(R) = 0$.

Now we show that if Λ is a subset of Ω such that Λ is either empty, or $\cap_{p\in\Lambda}J_p(R) \neq 0$ then the group $\overline{G}$ is discriminated by the class of groups $\overline{\mathcal{N}}_{\Omega\setminus\Lambda}$.

Let $\overline{h}_1 = h_1D_\omega(RG), \overline{h}_2 = h_2D_\omega(RG), \ldots, \overline{h}_m = h_mD_\omega(RG)$, $m \geq 2$ be an arbitrary set of a distinct elements of $\overline{G}$. Note that if $\overline{h}_i = \overline{1}$ for a some i then we write $h_iD_\omega(RG) = D_\omega(RG)$ and $h_i = 1$. Suppose further

$$K = \left\{g_n \mid g_n = h_i,\ \text{or}\ g_n = h_ih_j^{-1}, i \geq j, i = 1, 2, \ldots, m, j = 2, 3, \ldots, m\right\}.$$

Note that $h_ih_j{}^{-1} \in D_\omega(RG)$ for all $i \neq j$. Since $\pi = \pi^\star$, from the construction of the set K it follows that the elements $1 \neq g_i \in K$ are of infinite order.

Suppose there exists an element $g_i \neq 1$ in K such that

$$g_i \in \bigcap_{n=1}^{\infty} G^{p^n}\gamma_n(G)D_\omega(RG)$$

for all $p \in \Omega\setminus\Lambda$. Then by Lemma 3.12 $g_i{}^t \in \cap_{n=1}^{\infty}G^{p^n}\gamma_n(G)$ for every $p \in \Omega\setminus\Lambda$, where $t = |D_\omega(RG)|$. Therefore

$$g_i^t - 1 \in \bigcap_{p\in\Omega\setminus\Lambda}\bigcap_{n=1}^{\infty} I(G^{p^n}\gamma_n(G)).$$

For a non-zero element $r \in \cap_{p\in\Lambda}J_p(R)$ the element $r(g-1)(g_i^t-1)$ is in $A^\omega(RG)$ by Lemma 3.3 (if $\Lambda = \emptyset$, then $(g-1)(g_i^t-1) \in A^\omega(RG)$). Therefore $r(g-1)(g_i^t-1)(1-a) = 0$ (respectively $(g-1)(g_i^t-1)(1-a) = 0$). The right annihilator of the element g is zero, because g is an element of infinite order. Therefore $(g_i^t-1)(1-a)r = 0$ (respectively $(g_i^t-1)(1-a) = 0$). Similarly, since g_i is an element of infinite order, we conclude that the preceding equality implies $(1-a)r = 0$. This is a contradiction. Consequently, there exists a prime $p_\circ \in \Omega \setminus \Lambda$ such that

$$\bigcap_{n=1}^{\infty} G^{{p_\circ}^n}\gamma_n(G)D_\omega(RG)\bigcap K = M,$$

where either M is the empty set or $M = \{1\}$. Then for all $g_i \in K$ there exists n such that

$$g_i \in G^{{p_\circ}^n}\gamma_n(G)D_\omega(RG).$$

Therefore

$$h_iG^{{p_\circ}^n}\gamma_n(G)D_\omega(RG) \neq h_jG^{{p_\circ}^n}\gamma_n(G)D_\omega(RG)$$

whenever $i \neq j$. Then by the homomorphism

$$\phi\colon G/D_\omega(RG) \to G/G^{{p_\circ}^n}\gamma_n(G)D_\omega(RG)$$

we obtain that

$$\phi(h_iD_\omega(RG)) \neq \phi(h_jD_\omega(RG))$$

whenever $i \neq j$. Consequently $\overline{G}$ is discriminated by the class of groups $\overline{\mathcal{N}}_{\Omega\setminus\Lambda}$. The proof is complete.

Theorem 3.1. *Let R be a commutative integral domain. The intersection theorem holds for $A(RG)$ if and only if $D_\omega(RG)$ is the largest finite subgroup of G of order invertible in R and at least one of the following conditions holds:*

1) $G/D_\omega(RG)$ is a residually torsion free nilpotent group;

2) there exists a subset Ω of primes such that $G/D_\omega(RG)$ is discriminated by the class of groups $\overline{\mathcal{N}}_\Omega$, $\cap_{p\in\Omega}J_p(R) = 0$ and for an arbitrary subset Λ of Ω, $\cap_{p\in\Lambda}J_p(R) = 0$ or $G/D_\omega(RG)$ is discriminated by the class of groups $\overline{\mathcal{N}}_{\Omega\setminus\Lambda}$;

3) the set of the elements of finite order of G forms a finite normal subgroup $T(G)$, and for every prime divisor p of $|T(G)|$, which is not invertible in R, the group $W_p(G/D_\omega(RG))$ is finite with no p-elements and $J_p(R) = 0$.

Proof. Let the conditions 1) or 2) be satisfied. Then by Theorems 2.3 and 3.1 $A^\omega(R\overline{G}) = 0$ and therefore $A^\omega(RG) \subseteq I(D_\omega(RG))$. The order $t = |D_\omega(RG)|$ is invertible in R and the element $a = 1 - t^{-1}\sum_{g\in D_\omega(RG)} g$ is in $A(RG)$. By Lemma 2.3 the element $1 - a$ satisfies the equality

$$A^\omega(RG)(1-a) \subseteq I(D_\omega(RG)(1-a) = 0,$$

that is in these cases the intersection theorem holds for $A(RG)$.

Case 3. Let $\overline{G} = G/D_\omega(RG)$. By Lemma 3.2 $T(G) \supseteq D_\omega(RG)$ and because

$$D_\omega(RG) \supseteq \cap_{n=1}^{\infty}\gamma_n(G),$$

by the isomorphism $T(\overline{G}) \cong T(G)/D_\omega(RG)$ we conclude that $T(\overline{G})$ is a finite nilpotent group.

Let $\overline{g} \in T(\overline{G})$ and let the prime p divides $|T(G)|$ and let p be not invertible in R. Then $\overline{g}$ is an element of infinite order. By the conditions of our theorem $W_p(\overline{G})$ and $T(\overline{G})$ are finite subgroups of $\overline{G}$. It is clear that

$$\overline{g} \in \cap_{n=1}^{\infty} \overline{G}^{p^n} \gamma_n(\overline{G}) T(\overline{G})$$

because in the antipodal case by Lemma 3.12

$$\overline{g}^{|T(\overline{G})|} \in \cap_{n=1}^{\infty} \overline{G}^{p^n} \gamma_n(\overline{G}) = W_p(\overline{G}),$$

which is a contradiction, since the order of the element $\overline{g}$ is infinite. Therefore there exists an integer n such that $\overline{g} \in \overline{G}^{p^n} \gamma_n(\overline{G}) T(\overline{G})$. It is easy to see that $\widetilde{H} = \overline{G}/\overline{G}^{p^n} \gamma_n(\overline{G}) T(\overline{G})$ is a nilpotent p-group of finite exponent and $\overline{g}\overline{G}^{p^n} \gamma_n(\overline{G}) T(\overline{G})$ is a nontrivial element of $\widetilde{H}$. Therefore $\overline{G}/T(\overline{G})$ is a residually nilpotent p-group of finite exponent. Since $J_p(R) = 0$ and the class of nilpotent p-groups of finite exponent is closed under taking subgroup and finite direct product, from Lemma 2.1 and Theorem 2.2 it follows that $A^\omega(R\overline{G}/T(\overline{G})) = \overline{0}$. Since $R\overline{G})$ satisfies the conditions of Lemma 3.5, there exists $\overline{b} \in A(R\overline{G})$ with $A^\omega(R\overline{G})(1-\overline{b}) = 0$. Then $A^\omega(RG)(1-b) \subseteq I(D_\omega(RG))$ for a suitable element $b \in A(RG)$. If $c = 1 - t^{-1} \sum_{g \in D_\omega(RG)} g$ ($t = |D_\omega(RG)|$) then $c \in A(RG)$ and the element $1 - a = (1-b)(1-c)$ satisfies the equality $A^\omega(RG)(1-a) = 0$.

Sufficiency is proved in Lemmas 3.2 and 3.10. The proof is complete.

References

[1] Bovdi, A. A., Group rings, UMK VO, KIEV, 1988.

[2] Nouazè, Y. and Gabriel, P., Indéaux primiers de l'algebra anveloppante d'une algebra de Lie nilpotente, *J. Algebra*, **6**, (1967), 77–99.

[3] Hartley, B., The residual nilpotence of wreath products, *Proc. London Math. Soc.*, **20** (3), (1970), 365–392.

[4] Király, B., The residual nilpotency of the augmentation ideal, *Publ. Math. Debrecen*, **45** 1–2, (1994), 133–144.

[5] Malcev, A. I., Generalized nilpotent algebras and their adjoint, groups, *Mat. Sbornik*, **25** (67), (1949), 347–366 (*Amer. Math. Soc. Transl.* **69** (2), (1968), 1–121).

[6] Parmenter, M .M. and Sehgal, S. K., Idempotent elements and ideals in group ring and the intersection theorem, *Arc. Math.*, **24**, (1973), 586–600.

[7] Passi, I. B., Group ring and their augmentation ideals, Lecture notes in Math., 715, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[8] Sehgal, S. K., Topics in group rings, Marcel Dekker, Inc., New York and Basel, 1978.

[9] Smith, P. F., On the intersection theorem, *London Math. Soc.*, **21**, (1970), 22–27.

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